REFORMULATION OF THE THEOREM OF IMMUNISATION OF THE YIELD OF A FIXED INCOME PORTFOLIO

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Abstract

In all demonstrations known by the author about the theorem of immunization of the accumulated value or of the obtained yield in fixed investment portfolios, we consider that all the funds i are invested in zero coupon bonds and we hold until maturity of the securities, such investment is immunized against variations of the market rate. Since a zero coupon bond has a duration equal to maturity, we infer that if the investment horizon of a fixed income portfolio coincides with the portfolio duration (calculated at the market rate), the obtained yield of the portfolio will be at least equal to the predicted yield which is equal to the initial market rate.

From this we conclude that the immunization theorem of the obtained yield is based on the calculation of the minimum of a function with one variable. In the present work I demonstrate that we could determine the behavior of the obtained yield, calculating the minimum of a function of two variables: the investment horizon and the market rate. The conclusion reached in the demonstration is that such minimum exists only in a certain direction, since the second order condition leads to an indefinite quadratic form, and therefore, a saddle point.

Keywords

Fixed income portfolio, duration, immunisation, market yield, portfolio yield

1. Determination of the optimum yield of a portfolio

Let us consider a portfolio formed by two hybrids with the following features:

- Hybrid A: coupon 12% (quarterly); term to maturity, 7 years at par.
- Hybrid B: coupon 12% (quarterly); term to maturity, 20 years at
- Yield to maturity : 12% (quarterly)
- Duration and investment horizon of the portfolio: *6* years (or *24* quarters)
- Proportions invested in each hybrid in order to achieve the immunisation of the portfolio:

Hybrid *A: 60,357%* Hybrid *B: 39,643%*

- Initial value of the investment in the portfolio: 100

We suppose that the initial market yield 12% (quarterly), changes according to the range and values set up in the first row of the table 1; in the second row, we have the equivalent annual rates, and in the third row we have the present values of the portfolio calculated according to the interest rates which heads the corresponding column.

The yield per period is calculated from the equation:

$$V_0(i_0)(1+r)^h = V_0(i)(1+i)^h \qquad (1)$$

which gives:

$$r = \left[\frac{V_0(i)}{V_0(i_0)}\right]^{\frac{1}{h}}(1+i) - 1 \qquad (2)$$

where:

- $V_0(i_0)$ is the present value of the portfolio calculated at the initial rate i_0 (12% in this case).
- $V_0(i)$ is the present value of the portfolio calculated at the new rate i.
- h is the portfolio holding period.

Thus, for example, if the market rate becomes 10,40%, the obtained yield is 15.355% if the holding period of the portfolio is 2 years; on the other hand, if the market yield rises to 13.60% the yield of the portfolio would become 8.943% quarterly during the same period. We can see that for rates lower than 12% the yields are falling if we allow for only one variation during the holding period; on the contrary these yields are increasing if the market rates are increasing, again if we allow for only one variation during the holding period.

If we move through a row, the characteristics to highlight about the yields are:

- The yield is always decreasing if the investment horizon is less than or equal to 5 years, and
- It shows a minimum yield of 12% annual if the investment horizon is 6 years (which it is precisely the duration of the portfolio)

Likewise we have a minimum when we consider an investment horizon of 7 years. This minimum yield is 11.682% quarterly and occurs when the market rate becomes 8% (quarterly).

But the behaviour of the yield is really anomalous; if instead of analysing in terms of years, we make the same analysis in terms of quarters we get the output which appear in the table 2; although this is not yet a complete analysis, we can see more implicit minimums than in the previous case. Thus, starting from the quarter 21 and until the 28th, one could sense intuitively the presence of a minimum in each one of the rows, that is to say, for each investment horizon.

Why could we not have found a row minimum for each and every market rate? The reason is very simple: we have taken into account only complete quarters which is equivalent to saying that we only contemplate the possibility of the variable, time, taking the values 1, 2,..., that is to say, the group of the integers. But it is evident that time is a continuous variable and that in order to reach whatever point it should pass unavoidably through all the preceding. Moreover, Bierwag (1987) demonstrates that for each rate of interest the function of a single variable:

$$r = f\left[i; h = D(i)\right] \qquad (3)$$

has one minimum because it is strictly convex.

Consequently, if we calculated the duration for each interest rate (of the market) that we have considered we get the values detailed in the first column of the table 3; if we now calculated the yields achieved for each of these investment horizons the result is that in each row there is a minimum in the intersection of the duration *D(i)* with the interest rate. Thus for the market rate of 10.40%, the yield minimum is 11.956% which corresponds to an investment horizon of 25.376 quarters which is the duration of the portfolio calculated at rate mentioned: 10.40%. The same happens for every market rate: there exists one minimum (marked in bold) for each investment horizon which coincides with the duration calculated at the respective rate, that is to say, in each row of the table there is a minimum .

Now, fix our attention in the row for 24 quarters, which corresponds to the portfolio duration calculated at initial interest rate. The minimum yield is 12%, but this 12% is a maximum with regard to all the other "minimums"; thus, as the rates move away from 12% the minimums are lower. Now, we must ask the following question: can this result be generalised for whatever type of fixed income portfolio and for whatever interest rate scenario ?

2. New formulation of the theorem of immunisation

Consider a fixed income portfolio whose securities belong to the same class of risk. The present value of this portfolio at the initial market rate, is:

$$V_0(i_0) = \sum_{t=1}^{n} F_t (1+i_0)^{-t} \qquad (4)$$

where $F_t = \{F_1, F_2, \ldots, F_n\}$ is the portfolio net flow of income; we suppose that immediately after making the valuation at the rate i_0 the rate changes from i_0 to $i \neq i_0$. The accumulated value of the portfolio at the end of *h* years will be $V_0(i) \cdot (1+i)^h$, but since the actual quantity invested is $V_0(i_0)$, the obtained yield *r* will be the solution of the equation:

$$V_0(i_0) \cdot (1+r)^h = V_0(i) \cdot (1+i)^h \qquad (5)$$

which when resolved gives:

$$r = \left[\frac{V_0(i)}{V_0(i_0)} \right]^{\frac{1}{h}} \cdot (1+i) - 1 \qquad (6)$$

As we could see, r is function of h and i:

$$r = f(h,i) \tag{7}$$

In order to know what is the behaviour of the yield when we change both h and i, we will calculate the partial derivatives:

$$\frac{\partial r}{\partial i} = \frac{1}{h}\left[\frac{V_0(i)}{V_0(i_0)}\right]^{\frac{1}{h}-1} \cdot \frac{V_0'(i)}{V_0(i_0)} \cdot (1+i) + \left[\frac{V_0(i)}{V_0(i_0)}\right]^{\frac{1}{h}}$$

$$\frac{\partial r}{\partial h} = (1+i) \cdot \left[\frac{V_0(i)}{V_0(i_0)}\right]^{\frac{1}{h}} \cdot \ln\frac{V_0(i)}{V_0(i_0)} \cdot \left[-\frac{1}{h^2}\right] \tag{8}$$

In order to find the critical points, we equate to zero these derivatives and then solve the system. From the second of equation (8), we deduce that it disappears when $i = i_0$ since it then results:

$$\ln\left[\frac{V_0(i_0)}{V_0(i_0)}\right] = \ln 1 = 0 \tag{9}$$

Substituting i for i_0 in the first equation of (8), gives:

$$\frac{1}{h}\left[\frac{V_0'(i_0)}{V_0(i_0)}\right]\cdot(1+i_0)+1=0 \qquad (10)$$

From where we deduce h:

$$h^* = -\frac{V_0'(i_0)}{V_0(i_0)}\cdot(1+i_0) \qquad (11)$$

which as we know is the duration of the portfolio $D(i_0)$ computed at the initial interest rate i_0. Therefore, the system of equations formed by the partial derivatives of r is cancelled at point of co-ordinates:

$$i=i_0 \qquad h=D(i_0) \qquad (12)$$

3. Conditions of second order

In order to know what kind of point it is, we calculate the second derivatives of (8) and determine the sign of the quadratic form. The associate matrix to such a quadratic form is:

$$H[i=i_0,h=D(i_0)]=\begin{bmatrix} \dfrac{\partial^2 r}{\partial i^2} & \dfrac{\partial^2 r}{\partial i\partial h} \\ \dfrac{\partial^2 r}{\partial i\partial h} & \dfrac{\partial^2 r}{\partial h^2} \end{bmatrix} \qquad (13)$$

it has as determinant:

$$|H| = \begin{vmatrix} -\dfrac{D(i_0)+1}{1+i_0} - \dfrac{V_0''(i_o)}{V_0'(i_0)} & \dfrac{1}{D(i_0)} \\ \dfrac{1}{D(i_0)} & 0 \end{vmatrix} = -\frac{1}{[D(i_0)]^2} < 0 \quad \forall i_0 \tag{14}$$

which as we can see is negative, which means that the quadratic form is indefinite. Therefore, the function $r = f(h,i)$ has a saddle point at $i = i_0$, $h = D(i_0)$, such as we have just demonstrated and we could verify (see figure) in the example referred to.

4 Consequences of the theorem

Corollary 1. The function $r = f[h = D(i_0); i]$ has a minimum at point $i = i_0$ being the obtained value $r = i_0$, with which the classical theorem of immunisation continues being valid if we take the investment horizon of the portfolio equal to the duration computed at initial interest rate i_0. The demonstration of this corollary could be found in the Appendix 1 and the graphical representation referred to the example in the figure (3).

Corollary 2. The functions $r = f[h > D(i_0); i]$, $r = f[h < D(i_0); i]$, cut the function $r = f[h = D(i_0); i]$ at point of co-ordinates: $h = D(i_0)$, $i = i_0$, that is to say, pass through the minimum of such function. The function $r = f[h > D(i_0); i]$ is increasing at minimum point of $r = f[h = D(i_0); i]$, while the function $r = f[h < D(i_0); i]$ is decreasing at such point. The demonstration of this corollary could be found in Appendix 2 and the graphical representation of the example of reference in figure (3).

Corollary 3. The function $r = f\left[h; i_1 > i_0\right]$ has a minimum at $h = D(i_1) < D(i_0)$, while the function $r = f\left[h; i_2 < i_0\right]$ has a minimum at $h = D(i_2) > D(i_0)$. In both cases the yield obtained in the respective "minimums" will always be lower than $r = f\left[h = D(i_0); i_0\right]$, and therefore this will be a comparative maximum with regard to the rest of the "minimums" to the functions of the kind $r = f\left[h, i \neq i_0\right]$.

Demonstration that $r = f(h, i_1 > i_0)$ has a minimum at $h = D(i_1)$ with $D(i_1) < D(i_0)$ and the obtained yield $r_1 = f\left[h_1 = D(i_1); i_1 > i_0\right]$ is less than $r_0 = f\left[h_0 = D(i_0); i_0\right]$ with $r_0 \equiv i_0$.

From corollary 1, we know that whatever is the market rate, the function $r = f(h, i_1)$, has a minimum at $\left[h = D(i_1); i_1\right]$, and if $i_1 > i_0$ we found that $D(i_1) < D(i_0)$, since the duration is a decreasing function of the market rate, and if this rate has changed from i_0 to $i_1 > i_0$. (and it doesn't suffer another modification) the accumulated value of the portfolio calculated at the new rate i_1 is less than the value got at rate i_0 whenever the lapsed time is less than the duration calculated at the initial rate. Consequently, if $D(i_1) < D(i_0)$ the accumulated value, and therefore, the obtained yield will be lower than the accumulated value and at the rate initially calculated.

With an identical reasoning, we could demonstrate the second assertion of Corollary 3.

APPENDIX 1

Demonstration that function:

$$r = f\left[h = D(i_0); i\right] \qquad \text{(A1.1)}$$

has a minimum at the point $i = i_0$.

We must know in what direction the function has a local minimum, that is to say, if we cut the surface of function $r = f(h,i)$ through plan $h = D(i_0)$, we know that in that direction the function has a critical point; now then, if the sign of the second partial derivative of r with respect to i at the point of coordinates $\left[h = D(i_0), i = i_0\right]$, that is to say, the sign of the first element of the hessian matrix results be positive, the function will have a minimum in the indicated direction.

Therefore we must check:

$$\frac{-D-1}{1+i_0} - \frac{V_0''(i_0)}{V_0'(i_0)} > 0 \quad \Rightarrow \quad D+1 < \frac{V_0''(i_0)}{V_0'(i_0)} \cdot (1+i_0) \qquad \text{(A1.2)}$$

In order to demonstrate this last inequality we must take into account the following equations:

$$D = \frac{\sum t \cdot F_t \cdot (1+i_0)^{-t}}{\sum F_t \cdot (1+i_0)^{-t}} \qquad \text{(A1.3)}$$

$$V_0'(i_0) = \sum (-t) \cdot F_t \cdot (1+i_0)^{-t-1} \qquad \text{(A1.4)}$$

$$V_0''(i_0) = \sum t \cdot (t+1) \cdot F_t \cdot (1+i_0)^{-t-2} \qquad \text{(A1.5)}$$

Substitute for D, $V_0'(i_0)$ and, $V_0''(i_0)$, from eq. (A1.3) to (A1.5) in eq. (A1.2) yields:

$$\left[\sum t \cdot F_t \cdot (1+i_0)^{-t}\right]^2 < \left[\sum t^2 \cdot F_t \cdot (1+i_0)^{-t}\right] \cdot \left[\sum F_t \cdot (1+i_0)^{-t}\right] \qquad \text{(A1.6)}$$

Developing the left side of inequality (A1.6):

$$\left[\sum t\cdot F_t\cdot(1+i_0)^{-t}\right]^2=\sum t^2\cdot(F_t)^2\cdot(1+i_0)^{-2t}+2\sum_{t<u} t\cdot u\cdot(1+i_0)^{-t-u} \qquad \text{(A1.7)}$$

Developing the right side now of (A1.6), gives:

$$\left[\sum t^2 F_t(1+i_0)^{-t}\right]\left[\sum F_t(1+i_0)^{-t}\right]=\sum t^2(F_t)^2(1+i_0)^{-2t}+\sum_{t<u}\left(t^2+u^2\right)F_t\cdot F_u\cdot(1+i_0)^{-t-u} \qquad \text{(A1.8)}$$

Substituting from eq. (A1.7) and (A1.8) in (A1.6), clearly we check:

$$\sum_{t<u} 2\cdot t\cdot u\cdot(1+i_0)^{-t-u}<\sum_{t<u}(t^2+u^2)\cdot F_t\cdot F_u\cdot(1+i_0)^{-t-u} \text{ if } t>1 \qquad \text{(A1.9)}$$

In the case $t=1$, the duration would likewise be equal to 1 and the obtained yield would be equal to the planned yield since it is a zero coupon bond.

We have demonstrated that *function* $r=f(h,i)$ *has a minimum local at point* $[i=i_0]$, *in the direction of the plan* $[h=D(i_0)]$.

APPENDIX 2

Demonstration that function $r=f[i;h<D(i_0)]$, is decreasing with respect to market interest rate at point $[i=i_0;h=D(i_0)]$, and it passes through the minimum of function $r=f[i;h=D(i_0)]$ as $i=i_0$; the function

$r = f[i;h > D(i_0)]$ is increasing with the market yield at point $[i = i_0;h = D(i_0)]$, and it passes through the minimum of the function $r = f[i;h = D(i_0)]$ as $i = i_0$, (Corollary 2).

Lets us consider function r, obtained in (6):

$$r = \left[\frac{V_0(i)}{V_0(i_0)}\right]^{\frac{1}{h}} \cdot (1+i) - 1 \qquad \text{(A2.1)}$$

Deriving with respect to *i*:

$$\frac{dr}{di} = \left[\frac{V_0(i)}{V_0(i_0)}\right]^{\frac{1}{h}} \cdot \left[\frac{1}{h} \cdot \frac{V_0'(i_0)}{V_0(i)} \cdot (1+i) + 1\right] \qquad \text{(A2.2)}$$

The value of this derivative at point $i = i_0$, as $h < D(i_0)$, gives:

$$\frac{dr}{di}[i_0;h < D(i_0)] = \frac{1}{h}\frac{V_0'(i_0)}{V_0(i_0)}(1+i_0) + 1 \qquad \text{(A2.3)}$$

Taking into account that duration $D(i_0)$ is:

$$D(i_0) = -\frac{V_0'(i_0)}{V_0(i_0)}(1+i_0) \qquad \text{(A2.4)}$$

substituting $\frac{V_0'(i_0)}{V_0(i_0)}(1+i_0)$ from (A2.4) in (A2.3) gives:

$$\frac{dr}{di}\left[i_0;h<D(i_0)\right]=\frac{1}{h}\left[-D(i_0)\right]+1 \qquad \text{(A2.5)}$$

derivative that gives be negative, since $h<D(i_0)$, and therefore function $r=f\left[i;h<D(i_0)\right]$ is decreasing at $i=i_0$, with the one which is demonstrated the first part of the proposition.

As for the second $r=f\left[i;h>D(i_0)\right]$ is an increasing function at the mentioned point, we have that the derivative of this function at point $i=i_0$ with $h>D(i_0)$, is:

$$\frac{dr}{di}\left[i_0;h>D(i_0)\right]=\frac{1}{h}\left[-D(i_0)\right]+1 \qquad \text{(A2.6)}$$

is always positive, and, therefore the obtained yield is an increasing function at saying point when the horizon of investment is greater than the duration $D(i_0)$ computed at the initial interest rate.

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TABLE 1

MARKET YIELD													
(QUARTERLY)	7,20%	8,00%	8,80%	9,60%	10,40%	11,20%	12,00%	12,80%	13,60%	14,40%	15,20%	16,00%	16,80%
ANNUAL	7,40%	8,24%	9,09%	9,95%	10,81%	11,68%	12,55%	13,43%	14,31%	15,20%	16,09%	16,99%	17,89%
Present value of the portfolio	135,91	128,61	121,91	115,75	110,08	104,84	100,00	95,51	91,34	87,46	83,84	80,46	77,30

MARKET YIELD														
(QUARTERLY)		7,20%	8,00%	8,80%	9,60%	10,40%	11,20%	12,00%	12,80%	13,60%	14,40%	15,20%	16,00%	16,80%
ANNUAL		7,40%	8,24%	9,09%	9,95%	10,81%	11,68%	12,55%	13,43%	14,31%	15,20%	16,09%	16,99%	17,89%
HOLDING TIME														
		OBTAINED YIELD (QUARTERLY) FOR EVERY HOLDING PERIOD AND FOR EVERY MARKET YIELD												
Years	Quarters													
1	4	39,666%	34,485%	29,553%	24,853%	20,371%	16,091%	12,000%	8,087%	4,339%	0,747%	-2,698%	-6,007%	-9,187%
2	8	23,122%	21,034%	19,048%	17,157%	15,355%	13,638%	12,000%	10,437%	8,943%	7,517%	6,152%	4,847%	3,597%
3	12	17,747%	16,644%	15,604%	14,623%	13,697%	12,824%	12,000%	11,223%	10,490%	9,798%	9,146%	8,531%	7,951%
4	16	15,085%	14,466%	13,892%	13,361%	12,870%	12,417%	12,000%	11,617%	11,265%	10,944%	10,651%	10,386%	10,146%
5	20	13,496%	13,165%	12,869%	12,606%	12,375%	12,173%	12,000%	11,853%	11,731%	11,633%	11,557%	11,502%	11,468%
6	24	12,440%	12,299%	12,188%	12,104%	12,045%	12,011%	12,000%	12,011%	12,042%	12,093%	12,162%	12,249%	12,352%
7	28	11,687%	11,682%	11,702%	11,745%	11,810%	11,895%	12,000%	12,123%	12,264%	12,422%	12,595%	12,782%	12,984%
8	32	11,123%	11,220%	11,338%	11,476%	11,633%	11,808%	12,000%	12,208%	12,431%	12,668%	12,919%	13,183%	13,459%
9	36	10,686%	10,861%	11,055%	11,267%	11,496%	11,741%	12,000%	12,274%	12,561%	12,860%	13,172%	13,495%	13,829%
10	40	10,336%	10,574%	10,829%	11,100%	11,386%	11,686%	12,000%	12,326%	12,664%	13,014%	13,374%	13,745%	14,125%

TABLE 2

MARKET YIELD													
(QUARTERLY)	7,20%	8,00%	8,80%	9,60%	10,40%	11,20%	12,00%	12,80%	13,60%	14,40%	15,20%	16,00%	16,80%
ANNUALISED	7,40%	8,24%	9,09%	9,95%	10,81%	11,68%	12,55%	13,43%	14,31%	15,20%	16,09%	16,99%	17,89%
HOLDING PERIOD Quarters	OBTAINED YIELD (QUARTERLY) FOR EVERY HOLDING PERIOD AND FOR EVERY MARKET YIELD												
20	13,50%	13,16%	12,87%	12,61%	12,38%	12,17%	12%	11,85%	11,73%	11,63%	11,56%	11,50%	11,47%
21	13,19%	12,92%	12,67%	12,46%	12,28%	12,13%	12%	11,90%	11,82%	11,76%	11,73%	11,72%	11,72%
22	12,92%	12,69%	12,50%	12,33%	12,20%	12,08%	12%	11,94%	11,90%	11,88%	11,89%	11,91%	11,95%
23	12,67%	12,49%	12,34%	12,21%	12,12%	12,05%	12%	11,98%	11,97%	11,99%	12,03%	12,09%	12,16%
24	12,44%	12,30%	12,19%	12,10%	12,05%	12,01%	12%	12,01%	12,04%	12,09%	12,16%	12,25%	12,35%
25	12,23%	12,13%	12,05%	12,00%	11,98%	11,98%	12%	12,04%	12,10%	12,18%	12,28%	12,40%	12,53%
26	12,03%	11,97%	11,93%	11,91%	11,92%	11,95%	12%	12,07%	12,16%	12,27%	12,39%	12,54%	12,69%
27	11,85%	11,82%	11,81%	11,82%	11,86%	11,92%	12%	12,10%	12,21%	12,35%	12,50%	12,66%	12,84%
28	11,69%	11,68%	11,70%	11,75%	11,81%	11,90%	12%	12,12%	12,26%	12,42%	12,59%	12,78%	12,98%

TABLE 3

MARKET YIELD (QUARTERLY)	7,20%	8,00%	8,80%	9,60%	10,40%	11,20%	12,00%	12,80%	13,60%	14,40%	15,20%	16,00%	16,80%
ANNUALISED	7,40%	8,24%	9,09%	9,95%	10,81%	11,68%	12,55%	13,43%	14,31%	15,20%	16,09%	16,99%	17,89%
Duration in quarters	OBTAINED YIELD (QUARTERLY) FOR EVERY HOLDING PERIOD AND FOR EVERY MARKET YIELD												
20,626	13,303%	13,007%	12,745%	12,515%	12,315%	12,144%	12%	11,882%	11,788%	11,716%	11,667%	11,638%	**11,629%**
21,121	13,159%	12,889%	12,652%	12,446%	12,270%	12,122%	12%	11,903%	11,830%	11,779%	11,749%	**11,740%**	11,749%
21,64	13,015%	12,771%	12,559%	12,378%	12,225%	12,100%	12%	11,925%	11,872%	11,842%	**11,832%**	11,842%	11,870%
22,186	12,871%	12,653%	12,466%	12,309%	12,180%	12,077%	12%	11,946%	11,915%	**11,905%**	11,914%	11,943%	11,990%
22,76	12,727%	12,535%	12,373%	12,241%	12,135%	12,055%	12%	11,968%	**11,957%**	11,967%	11,997%	12,045%	12,111%
23,364	12,583%	12,417%	12,280%	12,172%	12,090%	12,033%	12%	**11,989%**	12,000%	12,030%	12,080%	12,147%	12,231%
24	12,440%	12,299%	12,188%	12,104%	12,045%	12,011%	**12%**	12,011%	12,042%	12,093%	12,162%	12,249%	12,352%
24,67	12,296%	12,182%	12,096%	12,035%	12,000%	**11,989%**	12%	12,032%	12,084%	12,155%	12,244%	12,350%	12,472%
25,376	12,154%	12,065%	12,003%	11,968%	**11,956%**	11,967%	12%	12,053%	12,126%	12,217%	12,326%	12,451%	12,592%
26,12	12,012%	11,949%	11,912%	**11,900%**	11,911%	11,945%	12%	12,075%	12,168%	12,279%	12,408%	12,552%	12,711%
26,904	11,871%	11,833%	**11,821%**	11,833%	11,867%	11,923%	12%	12,096%	12,210%	12,341%	12,489%	12,652%	12,830%
27,729	11,731%	**11,719%**	11,731%	11,766%	11,824%	11,902%	12%	12,117%	12,251%	12,402%	12,569%	12,751%	12,947%
28,596	**11,593%**	11,605%	11,641%	11,700%	11,780%	11,881%	12%	12,137%	12,292%	12,463%	12,649%	12,849%	13,064%

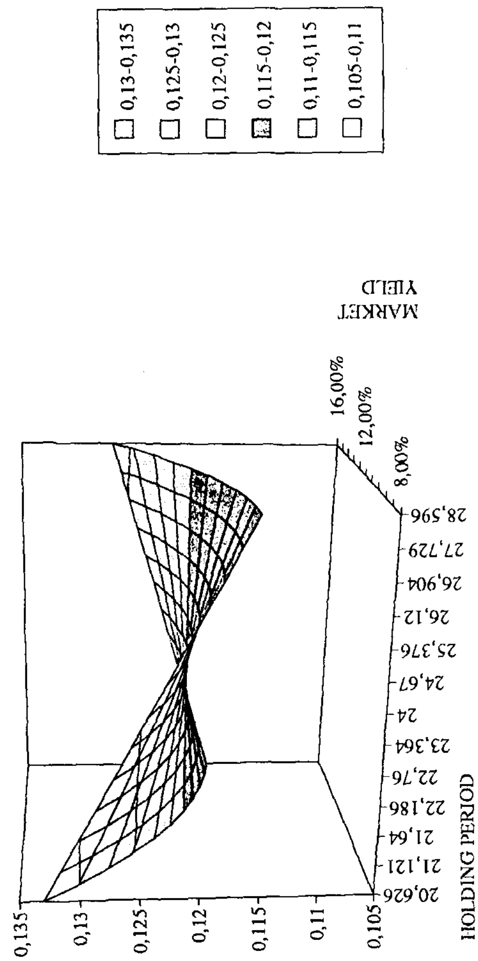

0,13-0,135
0,125-0,13
0,12-0,125
0,115-0,12
0,11-0,115
0,105-0,11
MARKET YIELD
16,00%
12,00%
8,00%
OBTAINED YIELD
0,135
0,13
0,125
0,12
0,115
0,11
0,105
HOLDING PERIOD
20,626
21,121
21,64
22,186
22,76
23,364
24
24,67
25,376
26,12
26,904
27,729
28,596

FIGURE 1

FIGURE 2

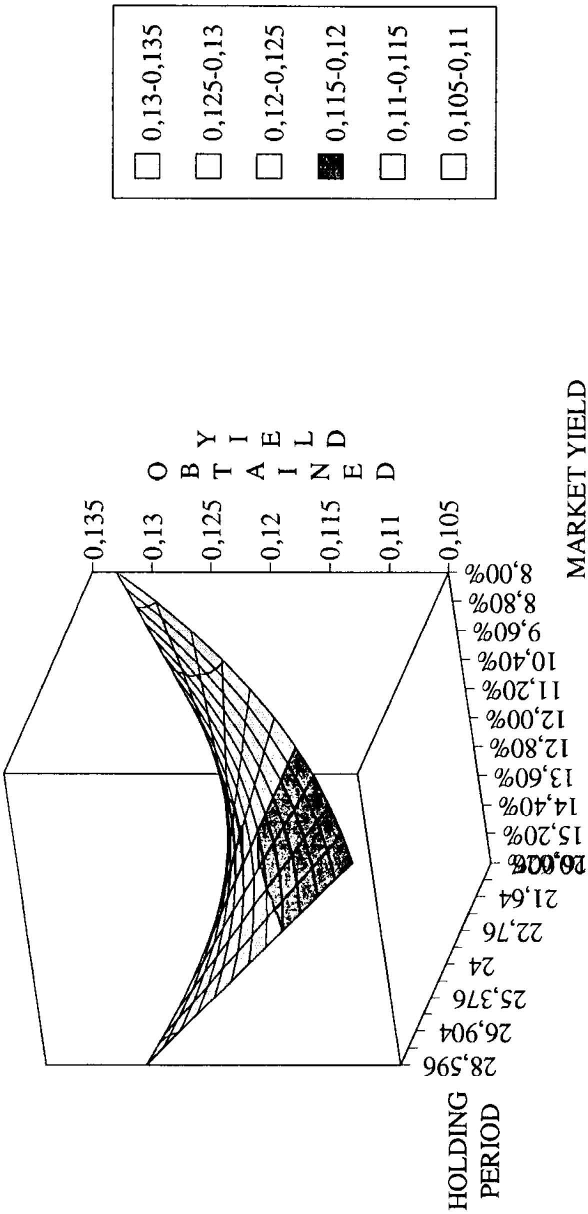

0,13-0,135
0,125-0,13
0,12-0,125
0,115-0,12
0,11-0,115
0,105-0,11
OBTAINED YIELD
0,135
0,13
0,125
0,12
0,115
0,11
0,105
MARKET YIELD
8,00%
8,80%
9,60%
10,40%
11,20%
12,00%
12,80%
13,60%
14,40%
15,20%
HOLDING PERIOD
21,64
22,76
24
25,376
26,904
28,596

FIGURE 3

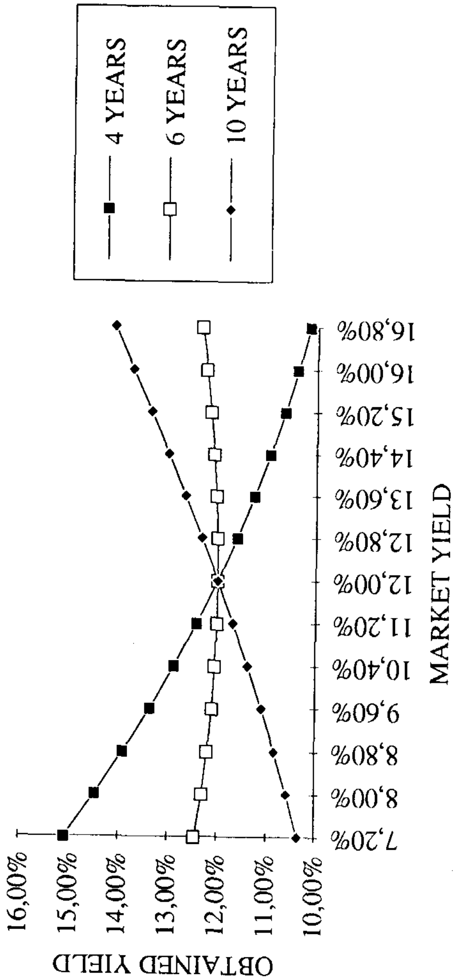
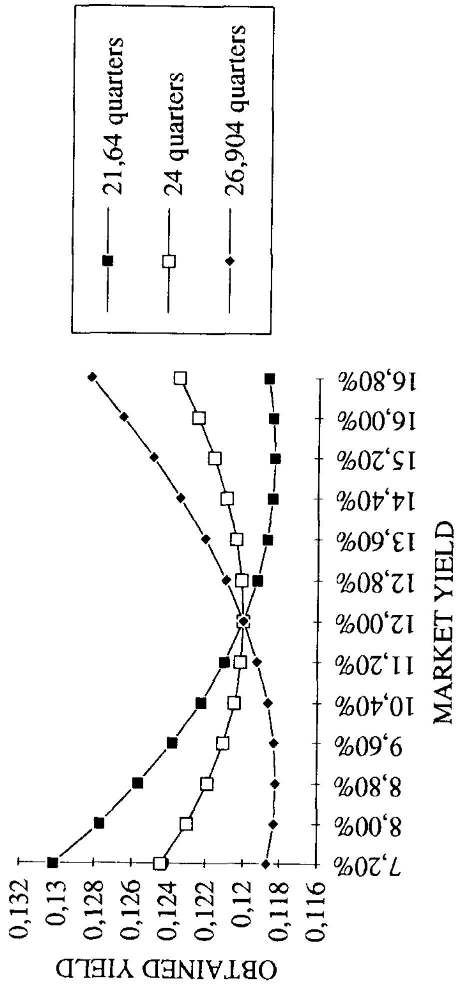

4 YEARS
6 YEARS
10 YEARS
OBTAINED YIELD
16,00%
15,00%
14,00%
13,00%
12,00%
11,00%
10,00%
MARKET YIELD
7,20%
8,00%
8,80%
9,60%
10,40%
11,20%
12,00%
12,80%
13,60%
14,40%
15,20%
16,00%
16,80%

FIGURE 4


21,64 quarters
24 quarters
26,904 quarters
OBTAINED YIELD
0,132
0,13
0,128
0,126
0,124
0,122
0,12
0,118
0,116
7,20%
8,00%
8,80%
9,60%
10,40%
11,20%
12,00%
12,80%
13,60%
14,40%
15,20%
16,00%
16,80%
MARKET YIELD