PRICING STRIKE PRICE AVERAGE OPTIONS

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Abstract

This paper addresses the pricing of the European strike price geometric average option. The paper shows that this option can be priced using a modifed binomial lattice which generates the probabilities of all possible outcomes on the expiration date recusively through time in a manner analagous to generating the lattice of prices.

Keywords

Options, binomial lattice, strike price, geometric average

1. Introduction

This paper addresses the exact pricing of the discrete European strike price option when the price of the underlying security follows a standard recombining binomial lattice. The call version of the option has a payoff function that is the difference (if positive) of the price of the security price over the geometric average calculated over the entire option period. This option hedges the volatility in price on the expiration date.

The average used in this paper is the geometric average of a discrete number of equally spaced time points, often called reset points. The traditional difficulty with pricing this option exactly is that the joint distribution of the security value and the average value are required. Neave (1993) addresses the pricing of this option, among others. We simplify and generalize his method. Ritchken, Sankarasubramanain and Vijh (1992) use approximating functions to get approximate prices of the options. In this paper, we obtain the exact joint probabilities of all possible outcomes on the maturity date by obtaining the probability generating function.

The method generates the probability using a two stage recursive procedure the require $O(n^3)$ calculations rather than $O(2^n)$ using the standard lattice approach, when there are n equally spaced reset points.

Since option prices using the binomial lattice are well known to converge to corresponding prices based on a continuous time price behaviour model, the method in this paper can be used to obtain (with an arbitrary degree of accuracy) the price of the corresponding option based on the continuous price movements. In addition, it can be extendeded to discrete time average options using continuous price movements. Here the term "discrete" refers to the fact the average is computed over a fixed number of reset points even though prices movements are modelled using shorter time intervals.

2. The Model

Suppose that the price movement of the underlying security $S(t)$ at time t is represented by Brownian motion with a stationary drift given by the stochastic diffential equation

$$\frac{1}{S(t)} \frac{d}{dt} S(t) = \mu \, dt + \sigma \, dz, \quad 0 \le t \le T \quad (1)$$

where z represents a standard Wiener process and T is the expiration date. This process is generally approximated by a recombining binomial lattice with time interval Δt with $S(t_k)$ represented as

$$S(t_k) = S(0) \, u^j \, d^{k-j} \quad (2)$$

where

$$u = d^{-1} = \exp(\sigma\sqrt{\Delta t}) \quad (3)$$

The probabilities of up-jumps (and down-jumps) in the lattice under the usual equivalent martingale measure are p and $1\text{-}p$ are given by

$$p = \frac{\exp(r\Delta t) - d}{u - d} \quad (4)$$

Let X_j be a Benoulli random variable taking on value 1 for an up-jump and 0 for a down-jump at the j-th reset point $t_j = jT/n$. Then

$$Y_j = X_1 + X_2 + \ldots + X_j \quad (5)$$

represents the number of up-jumps in the first j set points. The value of the
underlying security at the j-th set point can be obtained directly from Y_j as

$$S(t_j) = S(0)\, u^{2Y_j - j}, \qquad j = 1, 2, \ldots, n \quad (6)$$

The geometric average at the j-th reset point is defined as

$$A(t_j) = \prod_{i=1}^{j} S(t_i)^{1/j} \quad (7)$$

The geometric average can be rewritten as

$$A(t_j) = S(0)\, u^{2\frac{Z_j}{j} - \frac{j}{2}} \quad (8)$$

where

$$Z_j = Y_1 + Y_2 + \ldots + Y_j = j\, X_1 + (j-1)\, X_2 + \ldots + X_j. \quad (9)$$

The price of the strike price option depends on the joint distribution of the security price on the maturity date and the average over the entire option, period. Since Y_n and Z_n can be mapped isomorphically into $S(t_n)$ and $A(t_n)$, it is sufficient to obtain the joint distribution of Y_n and Z_n. Since both of these variables are defined on the non-negative integers, we develop the probability generating function (*pgf*) of the joint distribution.

The pgf of (Y_j, Z_j) at the j-th reset point is given by

$$\begin{aligned} P^{(j)}(s,t) &= E\left[s^{Y_j} t^{Z_j}\right] \\ &= E\left[s^{\sum_{i=1}^{j} X_i} t^{\sum_{i=1}^{j} (j-i+1) X_i}\right] \\ &= E\left[\prod_{i=1}^{j} (st^{j-i+1})^{X_i}\right] \\ &= \prod_{i=1}^{j} E\left[(st^{j-i+1})^{X_i}\right] \\ &= \prod_{i=1}^{j} P\left(st^{j-i+1}\right) \\ &= \prod_{i=1}^{j} P\left(st^{i}\right) \end{aligned} \tag{10}$$

where P( ) is the common *pgf* of the X_i's. Hence,

$$P^{(j)}(s,t) = \prod_{i=1}^{j} \left(1-p+pst^{i}\right) \tag{11}$$

From the pgf, the probabilities for all combinations of Y_j and Z_j can be obtained . Once they have been obtained at theexpiration date, the option price can be evaluated exactly directly.

3. Recursive Evaluation

By examining the *pgf* (11), it is easy to see that the following recursive relationship holds:

$$P^{(j)}(s,t) = P^{(j-1)}(s,t)\left(1-p+p\ s\ t^{j}\right) \tag{12}$$

Let $p_j(a,b)$ denote the probability that $Y_j = a$ and $Z_j = b$. Then, by equating coefficients on each side of (12) the following recursive relationship holds:

$$p_j(a,b) = (1-p)\ p_{j-1}(a,b) + p\ p_{j-1}(a-1,b-j) \tag{13}$$

for $a = 1,2,...,j$ and $b = a(a+1)/2,...,(a+1)(a+2)/2-1$ with $p_j(0,0) = (1-p)^j$ and $p_j(a,b) = 0$ otherwise.

This shows that the probabilities at successive reset points can be easily calculated like a binomial lattice, but with jumps of a number of units that depends on the set point.

4. Computing the Option Price

From (6) and (8), the option price can be computed directly as

$$E\left[\, S(t_n) - A(t_n) \,\right] = \sum \left[\square \; u^{2a-n} - u^{2\frac{b}{n} - \frac{n}{2}} \; \square \right] p_n(a,b). \tag{14}$$

The sum in (14) is taken only over values of a and b where the security price exceeds the average.

References

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