

A FREQUENCY DISTRIBUTION METHOD FOR VALUING AVERAGE OPTIONS

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Abstract :

This paper finds payoff frequency distributions for valuing European and American average spot price options on a discrete time, recombining multiplicative binomial asset price process. The distributions, obtained analytically using a generating function, greatly reduce valuation calculations. Less data are needed to value geometric than arithmetic averages, but the magnitude of calculation is similar for both instruments. Calculations of order T^3 are used to value European instruments, of order T^4 to value their American counterparts. A frequency distribution of a quantity called path sums is used to value geometric average options, and a joint distribution of path sums and realized prices is used to value arithmetic average options. The frequency distributions give an exact value for geometric average instruments, an approximate value for arithmetic average instruments. Additional detail from the generating function permits both estimating approximation errors and reducing them if significant. Error reduction can be carried out selectively to minimize additional calculation.

Keywords: Average Options, Binomial Models, Generating Functions, Frequency Distributions, Martingale Valuation.

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1. Introduction

The problems of valuing European geometric average options on a continuous time process have known analytic solutions for both continuously and discretely determined averages (cf., e.g., Goldman, Sosin, and Gatto [1979]). Valuing European arithmetic average options on a continuous time process is more difficult, mainly because the conventional choice of process is a geometric diffusion for which the distribution of prices' arithmetic averages is not lognormal. Nevertheless, analytic solutions for European arithmetic average options have been found for continuously determined averages by Reiner [1991], Yor [1992], and Geman and Yor [1993]. There are no similar formulae for American average options, geometric or arithmetic.

Nor are there analytic solutions for continuous time models with discrete averaging, although Turnbull and Wakeman [1991], Levy [1991], and Curran [1992] offer approximate solutions. Neither have discrete time models been studied analytically. Hull and White [1993] approximately value arithmetic average instruments using a binomial process. Neave [1993] and Neave and Turnbull [1994] use a binomial model to calculate exact values for European and American arithmetic average options for T up to about 20, approximate values for T up to about 60, where T is the expiry date. Ho [1992] and Tilley [1993] propose simulation with bundling techniques for reducing the calculations to value path dependent instruments; Tilley values both European and American average options with this technique.

Quicker, easier valuation methods for discrete averaging are still needed. Existing approximation methods are sufficiently accurate for processes with an annual volatility of .40 or less, but some price processes (e.g., aluminum and crude oil) exhibit higher volatilities. Neave [1993] and Neave and Turnbull [1994] find solutions for any volatility, but their exact solutions use large amounts of computer memory if $T > 20$, as do their approximations if $T > 60$. The solutions converge at a decreasing rate as the number of time periods increases, and the convergence rate slows as volatility increases while T is held constant.

This paper reduces computational tasks by using frequency distributions derived from a generating function. While less data are needed to value geometric average options than their arithmetic average counterparts,

the magnitude of calculations used is of order T^3 for European instruments of either type and of order T^4 for American instruments.

Frequency distributions are derived for sets of paths called bundles. Distributions describing sums of path price indices, henceforth called path sums, are used to value European geometric average options exactly. They are also used recursively to value American geometric average options. Joint frequency distributions of path sums and of realized prices are used to value arithmetic average options in an approach that provides good approximate values for both European and American arithmetic average instruments. The approximation error can be estimated using additional detail from the generating function, and if the error is significant it can be reduced selectively with a minimum of additional calculations.

The method can be applied to a variety of path dependent options, but only average spot price calls are valued here. The paper is organized as follows. Section 2 specifies the asset price process and defines the options. Section 3 describes the problem structure, defines the generating function, and shows how it is used to obtain frequency distributions. Section 4 values a European and an American geometric average call. Section 5 values the corresponding arithmetic average calls; Section 6 concludes.

2. The Price Process and the Options

This section defines the price process and formulates the valuation problems for European and American average calls.

2.1 The Process and its Averages

Let $S_0 = 1$ and let $\{ S_t \}$, the asset price process, be:

$$S_t = US_{t-1}; \quad (2.1)$$

$t \in \{1, 2, \dots, T\}$, where U is a random variable whose realized values are either $u > 1$ with probability p or u^{-1} with probability $1 - p \equiv q$.

The realized price cannot become negative, and remains finite for finite values of T and u . For later use, it is helpful to rewrite (2.1) as

$$S_t = u^{J_t} J_t = \sum_{s=0}^t X_s; t=1, \dots, T \quad (2.2)$$

where $X_0 \equiv 0$ and the X_s , $s > 0$ are independent, identically distributed random variables whose realized values are 1 with probability p and -1 with probability q . The relation

$$V_t \equiv \sum_{s=0}^t J_s = \sum_{s=0}^t (t-s)X_{s+1} \quad (2.3)$$

will also aid subsequent analysis. Define geometric and arithmetic averages respectively by

$$\begin{aligned} G_t &\equiv \left[\prod_{s=0}^t S_s \right]^{1/(t+1)} \\ &= \prod_{s=0}^t [u^{J_s}]^{1/(t+1)} \\ &= u^{V/(t+1)}, \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} H_t &\equiv \left[\sum_{s=0}^t S_s \right] / (t+1) \\ &= \left[\sum_{s=0}^t u^{J_s} \right] / (t+1). \end{aligned} \quad (2.5)$$

2.2 European Spot Price Calls

The payoff to a European average spot price call with exercise date T is $C_T \equiv (A_T - K)^+$ where A_T is a random variable representing the average in question and X^+ means $\max\{X, 0\}$. The geometric average call is defined by $A_T \equiv G_T$; the arithmetic average call by $A_T \equiv H_T$. Given a probability measure¹ p , the time zero values of the European options are

$$C_0 \equiv R^{-T} E((A_T - K)^+) \quad (2.6)$$

where E denotes expectation under p and $R^t \equiv (1 + r)^t$ indicates the t - period accumulation of \$1 at the risk free interest rate r .

Recursive methods are not needed to value European options, but help to relate European and American option valuation methods. Let Z be a state variable whose realized values $w \in \{0, 1, \dots, 2^T - 1\}$ denote individual paths.² Each number w is defined as a sum of t terms, one for each of the associated path's first differences. If the first difference between times t and $t+1$ is positive the term in the sum is 2^{T-t} ; if negative, the term is zero. For example, 0 -1 -2 -1 -2 -1 0, where the indices refer to times 0 through 6, has positive first differences between times 2 and 3, 4 and 5, 5 and 6. Hence $w = 2^{6-3} + 2^{6-5} + 2^{6-6} = 11$. Next, rewrite (2.6) as:

$$C_0(w) \equiv R^{-T} E\{ (A(w) - K)^+ \}, \quad (2.7)$$

taking the expectation over w . The notation $A(w)$ refers to a path average whose realized value depends on w . Since $H(w) \geq G(w)$ for all w , (2.7) shows the value of a European arithmetic average call is never less than the value of the corresponding geometric average call.

¹The recursive approaches developed below can be used with either a martingale or other probability measures; cf. Dixit and Pindyck [1994]. This paper assumes the martingale is used for valuation.

²For large T , Z has too many values for computational purposes, but describing individual paths will lead to defining a new state variable whose distinct values are only of order T^3 .

By ordering paths to be increasing in w , (2.7) can be written recursively as

$$\begin{aligned}
 C_T(w) &\equiv (A_T(w) - K)^+ ; w \in \{0, 1, \dots, 2^T - 1\} \equiv Z_T; \\
 C_{T-t}(w) &\equiv R^{-t} \{ pC_{T-t+1}(w + 2^{t-1}) + qC_{T-t+1}(w) \}; \\
 w &\in \{ j \cdot 2^t : j = 0, 1, \dots, 2^{T-t} - 1 \} \equiv Z_{T,t}; t \in \{1, \dots, T\};
 \end{aligned}
 \tag{2.8}$$

where $C_t(w)$ is the value of the European call at time t if the realized path is w . When $t = T$, (2.8) equals C_0 , the initial value of the call.

2.3 American Spot Price Calls

The recursion for the American call combines the methods of (2.8) with the early exercise feature.

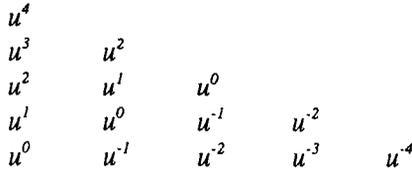
$$\begin{aligned}
 D_T(w) &\equiv C_T(w); w \in Z_T; \\
 D_{T-t}(w) &\equiv \max \{ (A_{T-t}(w) - K)^+, \\
 R^{-t} [pD_{T-t+1}(w + 2^{t-1}) + qD_{T-t+1}(w)] \}; w &\in Z_{T,t}; t \in \{1, \dots, T\};
 \end{aligned}
 \tag{2.9}$$

where $D_t(w)$ is the value of an American call at time t . Equations (2.9) show immediately $D_t(w) \geq C_t(w)$ for all feasible values of w and t ; the value of an American call is never less than that of its European counterpart.

2.4 Standard Indices

The possible outcomes of (2.1) can be indexed and arranged as in Figure 1, which illustrates $T = 4$. Successive periods' outcomes are arrayed along the main diagonals, starting with $t = 0$ in the lower left hand corner. Price increases are represented by upward moves, decreases by horizontal moves to the right. Realized prices (t, j) are usually referred to just by their indices. For example, the time 3 price of u^1 , denoted by $(3, 1)$, is usually referred to as index 1 (at time 3).

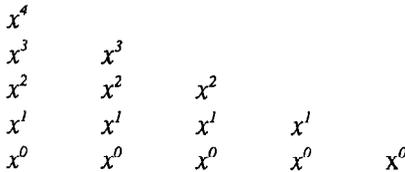
Figure 1. Original Indexing



This paper's generating function derives frequency distributions over bundles, where a bundle $B(t, j)$ is the set of all paths ending at (t, j) . A sub-bundle $B(t, j, V)$ is the set of all paths in $B(t, j)$ whose indices sum to V . Some features of a bundle are easily determined by reference to Figure 1. For example, there are $4!/2!2! = 6$ paths ending at $(4, 0)$, and the feasible set of realized indices for paths in $B(4, 0)$ is defined by the rectangle whose lower left-hand corner is $(0, 0)$ and whose upper right-hand corner is $(4, j)$.

The generating function defines price indices relative to their lowest attainable value at each time t , as shown in Figure 2.

Figure 2: Standard Indexing



The standardized process illustrated in Figure 2 is obtained by replacing S_t in (2.1) with S_t^* , where

$$S_t^* \equiv u^{J_t^*}; J_t^* \equiv \sum_{s=0}^t X_s^*; \tag{2.10}$$

$X_0^* \equiv 0, X_s^* = 1$ if $X_s = 1$ and $X_s^* = 0$ if $X_s = -1$. The processes (2.1) and

(2.10) have a one-to-one correspondence in the timing and sign³ of their first differences. It is not difficult to establish:

$$J_i^* = (J_i + i)/2; V_i^* = [V_i + i(t+1)/2]/2, \quad (2.11)$$

where the values of J_i^* are integers from 0 to t , while the values of V_i^* are non-negative integers between⁴ 0 and $t(t+1)/2$. Let $V_{i,j}^*$ be the values of V_i^* attained by paths in $B(t, j^*)$. The minimal and maximal values of $V_{i,j}^*$ are : $j^*(j^* + 1)/2$ and $(t-j^*)j^* + j^*(j^* + 1)/2$.

3. Problem Data and the Generating Function

This section specifies the options to be valued, organizes problem data, defines the generating function and derives frequency distributions.

3.1 Process Parameters; Option Specifications

The paper's four options are all defined on (2.1) with $T = 6$ quarterly time intervals and an annual volatility $\sigma = 0.40$. The risk free rate r is assumed to be a constant 0.10 per annum, all options expire at time T . All averages are defined using $t + 1$ points, $0, \dots, t$; the options are exercised at time t . For European options $t = T$, for American options $t \leq T$. The initial asset price $S_0 = 1.00$, and the options' common strike price $K = 1.00$. Finally, define k as the solution to $u^k = K$. These values imply $u = 1.221403$, $p = 0.510051$, and $q = 0.489949$.

³A first difference of zero in a path defined by (2.10) is identified with a negative first difference in a path defined by (2.1).

⁴The result is an extension of a theorem due to Gauss; cf. Berman and Fryer [1972].

3.2 Ordering and Bundling Paths

Valuing an average spot price call involves finding a probability distribution for paths whose averages exceed K , a task which can be simplified by ordering path averages. Figure 3 suggests that bundles can be used to reduce the number of distinct states. The bars in Figure 3 show the geometric averages of paths in $B(8, 0)$, the line above the bars the same paths' arithmetic averages.

Bars of equal height indicate the sub-bundles of $B(8, 0)$. Paths are ordered by sub-bundle geometric averages, and by arithmetic averages within each sub-bundle. For volatilities of less than

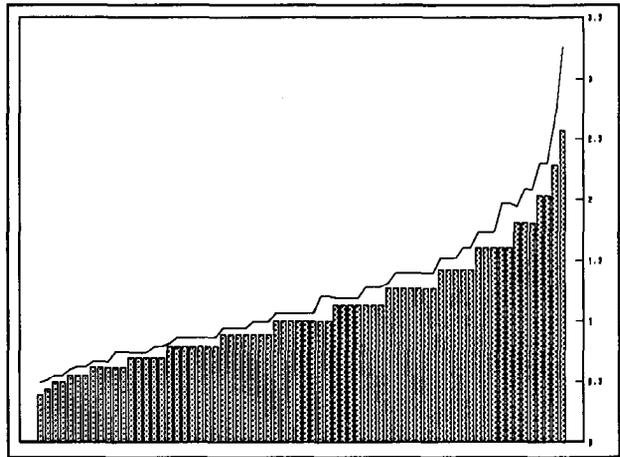


Figure 3. Geometric and Arithmetic Averages, $B(8, 0)$

approximately 1.00, this ordering means arithmetic averages increase monotonically between sub-bundles. For the annual volatility of 1.06 used in Figure 3 monotonicity is not preserved.

Using sub-bundles to define states reduces computation. At time t there are $t+1$ bundles, and from (2.12) each bundle $B(t, j^*)$ contains $1+j^*(t-j^*)$ sub-bundles. Thus the methods illustrated by Figure 3 use a total of $(t^3 + 5t + 6) / 6$ sub-bundles. Each line of Table 3 describes a sub-bundle of $B(6, 0)$, and corresponds to one of the values of the geometric averages shown in Figure 3. The first eleven⁵ columns of Table 3 indicate the frequencies indices attained by all paths in each sub-bundle. Column g indicates the numbers of paths in each sub-bundle, V the path sum defining

⁵In practice Table 4 can be organized to have at most $T + 1$ columns.

each sub-bundle. Column *M* is calculated from the index frequency section of Table 3. For example, the value of *M* for *B*(6, 0, -5), 12.253023, equals $2u^{-2} + 6u^{-1} + 6u^0$ with $u = 1.221403$.

Table 3. Frequency Distributions for B(6, 0)

Indices											g	V	M
-5	-4	-3	-2	-1	0	1	2	3	4	5			
0	0	1	2	2	2	0	0	0	0	0	1	-9	5.526912
0	0	0	2	3	2	0	0	0	0	0	1	-7	5.796831
0	0	0	2	6	6	0	0	0	0	0	2	-5	12.253023
0	0	0	2	7	10	2	0	0	0	0	3	-3	19.514560
0	0	0	0	6	12	3	0	0	0	0	3	-1	20.576593
0	0	0	0	3	12	6	0	0	0	0	3	1	21.784610
0	0	0	0	2	10	7	2	0	0	0	3	3	23.170933
0	0	0	0	0	6	6	2	0	0	0	2	5	16.312069
0	0	0	0	0	2	3	2	0	0	0	1	7	8.647860
0	0	0	0	0	2	2	2	1	0	0	1	9	9.248576

3.3 The Generating Function

All the data in Table 3, and all data underlying Figure 3, can be derived from the generating function. Let

$$f_i(x, z, v) = \prod_{j=1}^i (1 + xz^{alog(j-1)}v^j), \tag{3.1}$$

where $alog(j) \equiv 2^j$. While (3.1) has too many terms to be written explicitly for large *T*, it can still be used to derive the frequency distributions of Table 3. The powers of *x* count the number of positive first differences in a price path and thus generate the distribution of *J**. The powers of *v* cumulate the effects of positive first differences, generating the distributions of *V**. Finally the powers of *z* summarize both the occurrence and the timing of each positive first difference.

The paper uses the explicit form of (3.1) to reduce approximation

errors in valuing arithmetic average options. However, the rest of the paper uses a specialized form of (3.1) determined by setting $z = 1$:

$$f_t(x, v) = \prod_{j^*=1}^t (1 + xv^{j^*}) = \sum_{j^*=0}^t g_{t, j^*}(v) v^{j^*(t+1)/2} x^{j^*}, \tag{3.2}$$

where

$$g_{t, j^*}(v) = \prod_{k=1}^{j^*} \frac{v^{t+1-k}-1}{v^k-1}; \tag{3.3}$$

$1 \leq j^* \leq t$ and $g_{t, 0}(v) \equiv 1$. The $g_{t, j^*}(v)$ define the so-called Gaussian binomial coefficients, which describe the distribution of V^* conditional on $J^* = j^*$. To see this, expand (3.3) to obtain polynomials of degree $j^*(t-j^*)$:

$$g_{t, j^*}(v) = \sum_{k=0}^{j^*(t-j^*)} a_{j^*, k} v^k \tag{3.4}$$

where $a_{j^*, 0} \equiv 1$. Taking partial derivatives and evaluating at $v = 0$ gives

$$g_{t, j^*} \equiv (a_{j^*, 0}, \dots, a_{j^*, j^*(t-j^*)})'. \tag{3.5}$$

The vectors (3.5) describe the frequency distributions of path sums for both $B(t, j^*, V^*)$ and $B(t, j, V)$.

3.4 Frequency Distributions

All data in Table 3 are derived from the generating function.

Columns g and V are obtained using $g_{i,j}(v)$ and (2.1). The index frequency data are derived from two-fold convolutions of (3.2). Consider each in turn.

The function (3.2) defines columns g and V on a bundle-by-bundle basis. Consider $B(6, 0)$; i.e. $B(6, 3^*)$ in standardized notation. The range of values of $V^*_{6,3^*}$, determined from (2.12), is from 6^* to 15^* . The frequencies of values obtained by expanding $g_{6,3^*}(v)$ as in (3.3) are reported in Table 3, column g .

The index frequency columns in Table 3 are derived on a bundle-by-bundle basis using selected terms of two-fold convolutions of (3.1). Consider any price attained by one or more paths in the bundle. Next, consider each time at which that price can be attained. Then, consider the convolution describing any such time-index combination. A term from this convolution gives a time T frequency distribution of path sums for the paths attaining the given time - price combination. The sum of these terms across the different times the price is attained gives a frequency distribution of price index and time T path sums. Calculating these distributions for all attainable prices gives the joint frequency distribution for the bundle.

For example, Table 3 shows that of the paths in $B(6, 0)$, eight pass through the index -2, that they attain four values of V^* equalling 6, 7, 8, and 9, and that each of these values is attained by 2 paths. In standard notation the nodes at which the index -2 is attained are $(2, 0^*)$ and $(4, 1^*)$, the bundle is $B(6, 3^*)$.

Consider $(2, 0^*)$. The appropriate two-fold convolution is $f^2(x, v) * f^4(x, v)$. The product terms are those associated with x^0 for the movement from $(0, 0^*)$ to $(2, 0^*)$ and with x^3 for the movement from $(2, 0^*)$ to $(6, 3^*)$. From (3.1), the product is $g_{2,0^*}(v) \cdot g_{4,3^*}(v) \cdot v^6$. The vector $g_{2,0^*}$ has only the single element 1 because the four paths can only reach $(2, 0^*)$ in one way. Up to time 2 the increase in V^* is 0 for each of the four paths. From $(2, 0^*)$ to $(6, 3^*)$ takes four steps, three up and one across, and the increases in V^* are 6, 7, 8, or 9 according to how $(6, 3^*)$ is reached. The coefficients $g_{4,3^*}$ show there is one path for each value. Thus four paths reach $(2, 0^*)$ and have time T values of V^* equal to 6, 7, 8, and 9 respectively.

The frequency distribution for $(4, 1^*)$ is found in exactly the same

way. Adding the terms for $(2, 0^*)$ and $(4, 1^*)$ gives the column of values in Table 3 associated with -2. While the Table 3 frequency data are generated column by column, its rows are used to calculate sub-bundle means of arithmetic averages.⁶

4. Valuing Geometric Average Calls

This section values geometric average calls, first the European and then the American.

4.1 European Geometric Average Call

European geometric average calls can be valued using just columns g and V of Table 3. Sub-bundle frequency distributions and the parameters of Table 2 are used to calculate call payoffs. For example, the payoff to $B(6, 0, 5)$ is: $2 \cdot (1.221403^{5/7} - 1, 0)^+ = .307130$.

The 2 is the number of paths in $B(6, 0, 5)$, 1.221403 is the value of u , 5/7 is the index of the geometric average over the periods 0 through 6, and 1 is the exercise price. The payoffs are summed and multiplied by the appropriate probabilities, the resulting products summed and discounted to obtain the time 0 call value of .121869.

European Geometric Average Call	0.121869
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⁶Effectively, this method circumvents the analytical difficulty that the (discrete analogue to the) sum of lognormal variables is not lognormal.

4.2 American Geometric Average Call: Recursions

The American geometric average call is valued using a specialized version of (2.8) that defines states in terms of sub-bundles, as suggested by Figure 3:

$$\begin{aligned}
 D^{T-t}(j, w) &\equiv g_{T-t,j,w} \max \{ (G_{T-t}(j, w) - K)^+, \\
 &R^{-1} [pD_{T-t+1}(j+1, w+j+1) / g_{T-t+1,j+1,w+j+1} \\
 &+ qD_{T-t+1}(j-1, w+j-1) / g_{T-t+1,j-1,w+j-1}] \};
 \end{aligned}
 \tag{4.1}$$

$$j \in \{-(T-t), -(T-t)+2, \dots, T-t\}; w \in \{V_{t,j}\}; t \in \{0, \dots, T\}; D_{T+1}(\cdot) \equiv 0,$$

where $g_{T-t,j,w}$ is the number of paths in $B(T-t, j, w)$. The recursion relations are obtained using (3.1). For example, (4.2) relates time 6 frequency distributions to time 5:

$$\begin{aligned}
 f_6(x, v) &= \\
 (1 + xv^6) \cdot \sum_{j^*=0}^5 g_{5,j^*}(v) v^{j^*} v^{(j^*+1)} x^{j^*} .
 \end{aligned}
 \tag{4.2}$$

It is simplest to explain (4.2) using standard indexing, then give an example using conventional indexing. A bundle at time 6 is determined by combining paths from adjacent end points at time 5, while a backward recursion to a bundle at time 5 uses frequencies from adjacent end points at time 6. That is, $B(6, j^*)$ has a conditional distribution of path sums g_{6,j^*} derived by summing the generating function terms $v^6 g_{5,(j-1)^*}$ and g_{5,j^*} referring to two separate time 5 nodes. However to carry out the backward induction the time 5 payoffs having frequencies g_{5,j^*} must be compared with the time 6 payoffs having frequencies described by $v^6 g_{5,j^*}$ and g_{5,j^*} ; i.e., by all paths emanating from the time 5 node in question.

To interpret (4.2) using conventional indices, consider the following example. The single path in $B(5, -5, -15)$ extends to the single path in $B(6, -6, -21)$ if the price decreases between times 5 and 6, and to a single path in

$B(6, -4, -19)$ if the price increases. The expected value of not exercising associated with $B(5, -5, -15)$ is thus the expected value of payoffs to the single path in $B(6, -6, -21)$ and to a single path in $B(6, -4, -19)$. Since the time 6 payoffs are .720918 and .822120 respectively, the expected discounted payoff at time 5 is .754346. The immediate payoff to $B(5, -5, -15)$ is .648722 < .754346. Since the optimal policy for this sub-bundle is not to exercise, the value of .754346 is recorded as the optimal time 5 payoff. Remaining paths in $B(6, -4, -19)$ are reached from $B(5, -3)$, and form a part of the calculation of expected payoffs for $B(5, -3)$.

The backward induction proceeds from time 5 to time 4 much as it did from time 6 to time 5, but now comparing the discounted expected value of the optimal payoffs at time 5 with the payoffs to immediate exercise at time 4. Continuing the backward induction procedure until time zero is reached gives an American call value of 0.126932.

European Geometric Average Call	0.121869
American Geometric Average Call	0.126932

While sub-bundles can contain many paths, the payoffs to all paths in a sub-bundle are equal for a geometric average option, as suggested by Figure 3 and as can be established formally by examining (4.1) for $T, T-1, \dots, 0$. This permits dividing the payoffs by path frequencies, according to (4.2), without affecting the optimality of the result. Thus apart from using the recursions defined by (4.1), valuing the American option is similar to valuing the European option.

5. Valuing Arithmetic Average Calls

Arithmetic average calls can be valued approximately using the kind of joint frequency distributions illustrated in Table 3. The approximation replaces the arithmetic averages of the individual paths in a sub-bundle with the sub-bundle mean of their arithmetic averages.

5.1 Initial Approximate Value, European Arithmetic Average Call

To assess the approximation, consider the Table 3 data for the three paths of $B(6, 3^*, 12^*)$:

Indices	-1	0	1	2	g	V*	M
Frequencies	2	10	7	2	3	12	23.170933

The mean of the path arithmetic averages in $B(6, 3^*, 12^*)$ is: $M/3 \cdot 7 = (2u^1 + 10u^0 + 7u + 2u^2)/3 \cdot 7$, and $M = 23.170933$ when $u = 1.221403$.

To see how M is related to individual path averages, consider the three paths of $B(6, 3^*, 12^*)$, which are defined by (3.1) with $j = 3$ and powers of v equal 12. Equation (2.3) shows that the task of finding the paths is equivalent to the task of finding how many samples of size j , with members adding to the power of v , can be drawn without replacement from the first six integers. In this case the samples can be written 651, 642, and 543. The associated values of Z and the paths themselves can then be determined from (3.1) as:

Paths							Z	V	N
0	1	0	1	0	1	0	42	3	7.664209
0	1	2	1	0	-1	0	49	3	7.753362
0	-1	0	1	2	1	0	28	3	7.753362.

The values in column N equal 7 times the individual path arithmetic averages, and show that $M/21$ gives exact payoffs to the sub-bundle if either $K \leq 7.664209/7$ or $7.753362/7 \leq K$. However if $7.664209/7 < K < 7.753362/7$ the individual path averages are needed for exact valuation.

The example generalizes: M determines the exact payoff for any sub-bundle whose arithmetic averages either all exceed or all fall below K . Moreover, M determines approximate payoffs to bundles having averages above and below K . Since there are relatively few such bundles, it is possible to estimate and if necessary reduce approximation error with little supplemental calculation.

The option is valued using calculations similar to those for the geometric mean, but payoffs are now estimated using the values of M . The resulting approximate value of the European arithmetic average spot price call, 0.136520, exceeds the value 0.121869 found for the European geometric average call.

5.2 Assessing and Reducing Approximation Error

To assess and reduce any significant approximation error, it is necessary to find which sub-bundles contain paths with arithmetic averages both above and below K . For given j , it is possible there is more than one such sub-bundle; cf. Figure 3. On the other hand, the parameters of Figure 3 are chosen to reflect what are likely to be pathological conditions and the number of sub-bundles to be examined will usually be just one for each j .

For each j , elimination is used to find those values of V which define sub-bundles to be examined in detail. For any value V such that $V/(T+1) > k$, both the geometric and the arithmetic averages⁷ of paths in $B(T, j, V)$ exceed K , and hence such sub-bundles need no further examination. Second, if for some V the maximal arithmetic average over paths in $B(T, j, V)$ is less than K , then any $B(T, j, W)$, $W < V$, contains only paths whose arithmetic averages are less than K . The foregoing tests define the few remaining sub-bundles for which detailed examination is necessary.

In the present example, any approximation error can only arise from sub-bundles whose path sum is -1 ; i.e., $B(6, -2, -1)$, $B(6, 0, -1)$, and $B(6, 2, -1)$. Since $B(6, -2, -1)$ contains only one path with an arithmetic average $.990265 < K$, the 5.1 valuation of its payoff is exact. Similarly, $B(6, 2, -1)$ contains only one path with an average $1.00301 > K$, and its payoff was also assessed exactly. It remains to examine $B(6, 0, -1)$, whose three paths are:

$$\begin{array}{ccccccc} 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0. \end{array}$$

Since each path has the same arithmetic average $6.858864/7 = 0.979838 < K$, the valuation of 5.1 is exact in these instances also, and hence the approximately determined value $.136520$ is an exact time zero value for the arithmetic average option.

⁷Any path's arithmetic average is at least as great as its geometric average.

European Geometric Average Call	0.121869
American Geometric Average Call	0.126932
European Arithmetic Average Call	0.136520

5.3 Initial Approximate Value, American Arithmetic Average Call

The approximate value of the American arithmetic average call is determined recursively using the methods of 5.2. A recursion similar to (4.1), but with $H_{T,t}$ replacing $G_{T,t}$, is used. Assuming that payoffs are equal across all paths in a given sub-bundle, backward induction calculations can be performed as in 3.4 to obtain an approximate time zero value of 0.141093 for the arithmetic average call.

5.4 Assessing and Reducing Approximation Error

Assuming all paths in a sub-bundle have arithmetic averages equal to the sub-bundle determines a sub-optimal option value. But, the approximation error can be assessed and reduced just as for the European option. For example, suppose it is desired to find the two paths in $B(5, 1, 3)$ to assess the effect of dividing payoffs in a 1:1 ratio. Using the appropriate term of (3.1) shows the paths to be 010101 and 0 -1 0 1 2 1. Since their arithmetic averages are 1.110702 and 1.125660 respectively, the respective payoffs to immediate exercise are 0.110702 and 0.125660 respectively.

If the two paths are extended to time 6, their expected values are respectively $(.165418p + .094887q)/R = .127646$ and $(.177884p + .107623q)/R = .140082$. The better policy for $B(5, 1, 3)$ is not to exercise at time 5. Moreover, the optimal sub-bundle payoff of $.267728 = .127646 + .140082$ should be divided on the basis of time 5 expected values rather than by numbers of paths as in 5.3. In the present example, this is the only refinement to the approximation needed to determine an optimum; a single modification suffices to obtain the exact American arithmetic average option value of 0.141269.

European Geometric Average Call	0.121869
American Geometric Average Call	0.126932
European Arithmetic Average Call	0.136520
American Arithmetic Average Call (*Approximate Value)	0.141093* 0.141269

The literature does not stress the importance of assessing approximation error relative to a model determined optimum, but the method indicates the importance of assessing approximation errors, especially for American options. In practice it may be useful to find an exact solution for a set of typical parameter values and use that value to help calibrate approximation errors for other instruments valued by the approximations of Section 5.3.

6. Extensions and Conclusions

This paper valued European and American average spot price options on a discrete time, recombining multiplicative binomial asset price process. Using generating functions to find frequency distributions of option payoffs, the paper showed how to eliminate much of the calculation needed to value path dependent options. The procedures value European geometric average options exactly, and European arithmetic average options approximately. Approximation errors can be estimated and selectively reduced using relatively few additional computations. The approach organizes and reduces computational tasks in new ways that permit comparing approximations and known optima within the same model.

The paper's methods can also be used to value instruments whose averages are computed on a subset of the time points, and those with time weighted averages. For example, if arithmetic averages are computed on a subset of time points, the joint frequency distributions need only be modified to record the frequencies at chosen reset points. The approach can be extended to average strike options by determining joint distributions of the averages and path ends, readily available from the information developed in this paper.

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