# A COVARIANCE EQUIVALENT DISCRETISATION OF THE CIR MODEL

GRISELDA DEELSTRA
VRIJE UNIVERSITEIT BRUSSEL
DEPARTEMENT WISKUNDE
PLEINLAAN 2
B-1050 BRUSSEL - BELGIUM
TELEPHONE: 32-2-629 34 73
FAX: 32-2-629 34 95
E-MAIL: GDEELSTR@VNET3.VUB.AC.BE

GARY PARKER
SIMON FRASER UNIVERSITY
DEPARTMENT OF MATHEMATICS AND STATISTICS
BURNABY, BC, CANADA V5A 1S6
TELEPHONE: (604) 291 4981
FAX: (604) 291 4947
E-MAIL: GPARKER@CS.SFU.CA

## *Abstract*

Estimating the parameters of the Cox-Ingersoll-Ross square root model from discrete data requires a representation of this system sampled at uniform intervals. In this paper, we use the principle of covariance equivalence to find such a discrete model. The parametric relations between the continuous and the discrete representations are established. The least squares method is used to estimate the parameters from past rates on Canadian Government 3-month T-Bills. We compare our approach with another discrete model proposed by others.

## *Keywords*

Interest rates, Cox-Ingersoll-Ross square root model, Discrete representation, Covariance equivalence, Least squares estimates.

# A COVARIANCE EQUIVALENT DISCRETISATION OF THE CIR MODEL

GRISELDA DEELSTRA
VRIJE UNIVERSITEIT BRUSSEL
DEPARTEMENT WISKUNDE
PLEINLAAN 2
B-1050 BRUSSEL - BELGIUM
TELEPHONE: 32-2-629 34 73
FAX: 32-2-629 34 95
E-MAIL: GDEELSTR@VNET3.VUB.AC.BE

GARY PARKER
SIMON FRASER UNIVERSITY
DEPARTMENT OF MATHEMATICS AND STATISTICS
BURNABY, BC, CANADA V5A 1S6
TELEPHONE: (604) 291 4981
FAX: (604) 291 4947
E-MAIL: GPARKER@CS.SFU.CA

***Résumé***

L'estimation des paramètres du modèle de Cox-Ingersoll-Ross sur base de données discrètes demande une représentation du système discrétisé à des intervalles uniformes. Dans cet article, nous utilisons le principe de l'équivalence de la covariance pour trouver un modèle discret. Les relations entre les paramètres du modèle discret et du modèle continu sont déterminées. La méthode des moindres carrés est utilisée pour estimer les paramètres sur base des taux passés des bons du Trésor à 3 mois du gouvernement canadien. Nous comparons notre méthode avec un autre modèle proposé par des auteurs différents.

## *1. Introduction.*

The model of Cox, Ingersoll and Ross (1985) for the short interest rate has been studied and used in many papers. Some papers present results about the process and/or about some of its applications assuming that its parameters are known. One example is the paper by Deelstra and Delbaen (1994).

Other papers are concerned with the estimation of the parameters from past data and the comparison of different models based on how well they capture the dynamics of the short-term interest rate. Chan et al. (1992) is a paper about the empirical comparison of models.

When using discrete data for estimating the parameters of a continuous model, one needs a discrete representation of the process. Some authors use what we will call a simple discretisation of the Cox-Ingersoll-Ross model. For references, see Chan et al. (1992).

An alternative and useful way of looking at the problem is to consider that the continuous process is being sampled at, say uniform intervals. From this point of view, the observed discrete process should have the same expected value, variance and distribution as the continuous process at all sampling times.

In this paper, we will use the principle of covariance equivalence to find a discrete representation of the Cox-Ingersoll-Ross model. This principle requires that the first two moments of both the discrete and the continuous models be equal. It is presented in Pandit and Wu (1983) and used for Gaussian processes by Parker (1995).

The layout of the paper is as follows. In section 2, we recall the Cox-Ingersoll-Ross square root model. For convenience, we also introduce a centered version of it. The expected value and autocovariance function of the process are obtained.

In section 3, we consider two discrete models as possible sampled representations of the Cox-Ingersoll-Ross model. One is the simple discretisation and the other is the covariance equivalent discretisation. We also establish the parametric relations between the continuous and the discrete models.

Three methods of estimation of the parameters of the Cox-Ingersoll-Ross model are presented in section 4. We use least squares estimates of the parameters of the discrete models.

In section 5, past monthly rates on Canadian Government 3-month T-Bills over a period of 165 months are used to illustrate these three methods of estimation.

Section 6 contains a few remarks and our conclusion.

## *2. The centered Cox-Ingersoll-Ross model.*

Cox, Ingersoll and Ross (1985) represented the dynamics of the short interest rate $r_t^*$ by the stochastic differential equation:

(1) $dr_t^* = \kappa(\gamma - r_t^*)dt + \sigma\sqrt{r_t^*}dB_t$,

where $\kappa, \gamma$ and $\sigma$ are positive constants and where $(B_t)_{t\geq 0}$ is a Brownian motion.

This model, referred to as the Cox-Ingersoll-Ross (CIR) square root model, has some good properties. First, the interest rates are continuous and remain positive. Moreover, for $2\kappa\gamma \geq \sigma^2$ the interest rates do not reach zero. Second, the volatility is proportional to the short interest rates. Third, the short interest rates are elastically pulled to the constant long-term value $\gamma$.

For convenience, we introduce the shifted short interest rate $r_t = r_t^* - \gamma$. The process $r_t$ is then centered around $\gamma$ and the long-term return converges almost everywhere to zero (see Deelstra and Delbaen, 1994). For the remaining of this paper we will study $r_t$ instead of $r_t^*$. Their expected values simply differ by $\gamma$ and their autocovariance functions are the same.

In this section, we derive the first two moments of the centered short interest rate $r_t$. The process $r_t$ satisfies the stochastic differential equation

(2) $dr_t = -\kappa r_t dt + \sigma\sqrt{r_t + \gamma}\,dB_t$ ,

with initial condition $r_0$.

Its solution is given by:

$$r_t = e^{-\kappa t}r_0 + \sigma e^{-\kappa t}\int_0^t e^{\kappa u}\sqrt{r_u + \gamma}\,dB_u .$$

The expected value immediately follows:

(3) $E[r_t] = e^{-\kappa t}r_0$.

The covariance between $r_t$ and $r_s$ can be obtained from

$$(4) \quad \operatorname{cov}(r_t, r_s) = \mathrm{E}\left[(r_t - \mathrm{E}[r_t])(r_s - \mathrm{E}[r_s])\right].$$

Substituting the solution of the centered stochastic differential equation minus its expected value gives

$$\operatorname{cov}(r_t, r_s) = \mathrm{E}\left[\left(\sigma e^{-\kappa t}\int_0^t e^{\kappa u}\sqrt{r_u+\gamma}\,dB_u\right)\left(\sigma e^{-\kappa s}\int_0^s e^{\kappa v}\sqrt{r_v+\gamma}\,dB_v\right)\right].$$

By stochastic calculus we obtain, for $0 \le s \le t$,

$$\operatorname{cov}(r_t, r_s) = \sigma^2 e^{-\kappa(s+t)}\int_0^s e^{2\kappa v}\,\mathrm{E}[r_v + \gamma]\,dv.$$

Using the expression for the expected value, this becomes

$$\operatorname{cov}(r_t, r_s) = \sigma^2 e^{-\kappa(s+t)}\int_0^s e^{2\kappa v}\left(e^{-\kappa v} r_0 + \gamma\right)dv.$$

Integrating, we find

$$(5) \quad \operatorname{cov}(r_t, r_s) = \sigma^2 \frac{e^{-\kappa t} - e^{-\kappa(s+t)}}{\kappa} r_0 + \sigma^2 \frac{e^{-\kappa(t-s)} - e^{-\kappa(t+s)}}{2\kappa}\gamma, \quad s \le t.$$

The variance immediately follows:

$$(6) \quad \operatorname{Var}[r_t] = \frac{-\sigma^2}{2\kappa}(2r_0 + \gamma)e^{-2\kappa t} + \frac{\sigma^2}{\kappa} r_0 e^{-\kappa t} + \frac{\gamma\sigma^2}{2\kappa}.$$

The stationary variance is found by taking the limit for t going to infinity in the expression of the variance, i.e.

$$(7) \quad \lim_{t\to\infty} \operatorname{Var}[r_t] = \frac{\gamma\sigma^2}{2\kappa}.$$

### *3. Discretisation of the Cox-Ingersoll-Ross model.*

In this section, we consider two models as candidates for the discrete representation of the CIR model sampled at uniform intervals. Our analysis is based on the principle of covariance equivalence as described by Pandit and Wu (1983) and used for Gaussian first and second order stochastic differential equations by Parker (1995). By this principle, we try to match the first two moments of the CIR model with those of a discrete model at all sampling times.

We first consider a simple discretisation of the continuous CIR model found in the literature. Secondly, we propose another discrete model which has the same expected value and autocovariance function as the continuous

CIR model. We also establish the parametric relations between the continuous and discrete representations.

### *3.1 A simple discretisation.*

Since the continuous centered interest rate is defined by the stochastic differential equation

$$dr_t = -\kappa r_t dt + \sigma\sqrt{r_t + \gamma}\, dB_t,$$

we propose the following simple discretisation:

$$(8) \qquad r_t = \phi r_{t-1} + \sigma_a \sqrt{r_{t-1} + \gamma}\; a_t, \qquad t = 1,2,3,\ldots$$

with for all t, $a_t$ a standard Normal variable (N(0,1)), independent of $\{a_i | i \le t-1\}$ and with $\phi$ and $\sigma_a$ positive constants. $\phi$ represents a kind of drift rate and $\sigma_a$ is a diffusion parameter. This model is consistent with the non-centered model used by others. See, for example, Chan et al. (1992).

By recursion, (8) is equivalent to

$$r_t = \phi^t r_0 + \sum_{i=0}^{t-1} \phi^i \sigma_a \sqrt{r_{t-i-1} + \gamma}\; a_{t-i} \qquad t = 1,2,3,\ldots.$$

The expected value easily follows:

$$(9) \quad E[r_t] = \phi^t r_0.$$

Substituting the simple discretised process minus its expected value in (4) gives us the expression:

$$\begin{aligned} \operatorname{cov}(r_t, r_s) &= E\left[\sum_{i=0}^{t-1} \phi^i \sigma_a \sqrt{r_{t-i-1} + \gamma}\; a_{t-i} \sum_{j=0}^{s-1} \phi^j \sigma_a \sqrt{r_{s-j-1} + \gamma}\; a_{s-j}\right] \\ &= \sum_{i=0}^{t-1}\sum_{j=0}^{s-1} \phi^{i+j}\sigma_a^2 E\left[\sqrt{r_{t-i-1} + \gamma}\; a_{t-i} \sqrt{r_{s-j-1} + \gamma}\; a_{s-j}\right]. \end{aligned}$$

Since the $a_t$'s are independently and identically distributed as standard Normal variables, the expected value is only different from zero if $t-i = s-j$ or $i = t-s+j$. Therefore, we are left with only one summation, and for $s \le t$, we have

$$\operatorname{cov}(r_t, r_s) = \sum_{j=0}^{s-1} \phi^{t-s+2j} \sigma_a^2 E\left[r_{s-j-1} + \gamma\right].$$

Substituting the expression for the expected value, we get

$$\operatorname{cov}(r_t,r_s)=\sum_{j=0}^{s-1}\phi^{t-s+2j}\sigma_a^2\left(\phi^{s-j-1}r_0+\gamma\right)$$

$$=\sum_{j=0}^{s-1}\phi^{t+j-1}\sigma_a^2 r_0+\sum_{j=0}^{s-1}\phi^{t-s+2j}\sigma_a^2\gamma\,.$$

And expanding these geometric sums gives

(10) $$\operatorname{cov}(r_t,r_s)=\phi^{t-1}\sigma_a^2 r_0\frac{1-\phi^s}{1-\phi}+\phi^{t-s}\sigma_a^2\gamma\frac{1-\phi^{2s}}{1-\phi^2},\quad s\le t\,.$$

The variance is then,

(11) $$\operatorname{Var}[r_t]=\phi^{t-1}\sigma_a^2 r_0\frac{1-\phi^t}{1-\phi}+\sigma_a^2\gamma\frac{1-\phi^{2t}}{1-\phi^2},$$

and the stationary variance is, for $\phi<1$,

$$\lim_{t\to\infty}\operatorname{Var}[r_t]=\frac{\sigma_a^2\gamma}{1-\phi^2}\,.$$

In order to obtain parametric relations between the continuous CIR model and the discrete model defined by (8), we equate the expected values and the stationary variances.

For the expected value of the CIR model (2) to be equal to the one of the discrete model (8) for all t, we must have (see (3) and (9)):

(12) $\phi=e^{-\kappa}$.

The stationary variance for the continuous CIR process is

$$\lim_{t\to\infty}\operatorname{Var}[r_t]=\frac{\gamma\sigma^2}{2\kappa}$$

and that of the discrete model (8) is

$$\lim_{t\to\infty}\operatorname{Var}[r_t]=\frac{\sigma_a^2\gamma}{1-\phi^2}\,.$$

Consequently, we require that

(13) $$\sigma_a^2=\sigma^2\frac{1-\phi^2}{2\kappa}=\sigma^2\frac{1-e^{-2\kappa}}{2\kappa}\,.$$

Although we have obtained explicit parametric relations, the result is not satisfactory. A substitution of these relations in the covariances of the continuous and simple discretisation models reveals that these covariances are not equal for all (s,t).

### *3.2 A covariance equivalent discretisation.*

We now consider the following discrete model:

$$(14) \qquad r_t = \phi r_{t-1} + \sigma_a \sqrt{\frac{2\phi}{1+\phi} r_{t-1} + \gamma}\ a_t \qquad t = 1,2,3,\ldots$$

again with for all $t, a_t$ a standard Normal variable (N(0,1)), independent of $\{a_i | i \le t-1\}$ and with $\phi$ and $\sigma_a$ positive constants.

This discrete process can also be written in a recursive way:

$$r_t = \phi^t r_0 + \sum_{i=0}^{t-1} \phi^i \sigma_a \sqrt{\frac{2\phi}{1+\phi} r_{t-i-1} + \gamma}\ a_{t-i} \qquad t = 1,2,3,\ldots.$$

The expected value is the same as above, i.e.

$\mathbf{E}[r_t] = \phi^t r_0$.

Therefore, the same parametric relation, $\phi = e^{-\kappa}$, must hold for the continuous and the discrete processes to have the same expected value.

To obtain the covariance between $r_t$ and $r_s$, we substitute the discretised process and its expected value in (4), as follows

$$\mathrm{cov}(r_t, r_s) = \mathbf{E}[(r_t - \mathbf{E}[r_t])(r_s - \mathbf{E}[r_s])]$$

$$= \mathbf{E}\left[\sum_{i=0}^{t-1} \phi^i \sigma_a \sqrt{\frac{2\phi}{1+\phi} r_{t-i-1} + \gamma}\ a_{t-i} \sum_{j=0}^{s-1} \phi^j \sigma_a \sqrt{\frac{2\phi}{1+\phi} r_{s-j-1} + \gamma}\ a_{s-j}\right]$$

$$= \sum_{i=0}^{t-1}\sum_{j=0}^{s-1} \phi^{i+j} \sigma_a^2 \mathbf{E}\left[\sqrt{\frac{2\phi}{1+\phi} r_{t-i-1} + \gamma}\ a_{t-i} \sqrt{\frac{2\phi}{1+\phi} r_{s-j-1} + \gamma}\ a_{s-j}\right].$$

As above, most terms equal zero, by the independence of the N(0,1) distributed $a_t$'s and this double summation simplifies to:

$$\mathrm{cov}(r_t, r_s) = \sum_{i=0}^{s-1} \phi^{t-s+2i} \sigma_a^2 \mathbf{E}\left[\frac{2\phi}{1+\phi} r_{s-i-1} + \gamma\right], \qquad s \le t$$

Substituting the expected value, we find

$$\mathrm{cov}(r_t, r_s) = \sum_{i=0}^{s-1} \phi^{t-s+2i} \sigma_a^2 \left(\frac{2\phi^{s-i}}{1+\phi} r_0 + \gamma\right)$$

$$= \sum_{i=0}^{s-1} \frac{2\phi^{t+i}}{1+\phi} \sigma_a^2 r_0 + \sum_{i=0}^{s-1} \phi^{t-s+2i} \sigma_a^2 \gamma.$$

Finally, we obtain the following expression for the autocovariance function:

$$\text{(15)} \quad \operatorname{cov}(r_t, r_s) = 2\phi^t \sigma_a^2 r_0 \frac{1-\phi^s}{1-\phi^2} + \phi^{t-s}\sigma_a^2 \gamma \frac{1-\phi^{2s}}{1-\phi^2}, \qquad s \le t$$

and the variance is found by substituting $t = s$,

$$\text{(16)} \quad \operatorname{Var}[r_t] = 2\phi^t \sigma_a^2 r_0 \frac{1-\phi^t}{1-\phi^2} + \sigma_a^2 \gamma \frac{1-\phi^{2t}}{1-\phi^2}.$$

Note that this variance is different than (11), the variance for the model defined in (8).

Taking the limit of (16) as t tends to infinity, we have,

$$\lim_{t\to\infty} \operatorname{Var}[r_t] = \frac{\sigma_a^2 \gamma}{1-\phi^2}.$$

Thus, the stationary variance of (14) is the same as that of the simple discrete model (8) (see (12)).

So the second parametric relation is unchanged (see (13)) and it is:

$$\sigma_a^2 = \sigma^2 \frac{1-\phi^2}{2\kappa} = \sigma^2 \frac{1-e^{-2\kappa}}{2\kappa}.$$

We will now check that the covariance of this discrete model (14) equals the covariance of the continuous CIR model for all $t$ and $s$. Let us start with (15), the covariance in this discretised case:

$$2\phi^t \sigma_a^2 r_0 \frac{1-\phi^s}{1-\phi^2} + \phi^{t-s}\sigma_a^2 \gamma \frac{1-\phi^{2s}}{1-\phi^2}.$$

If we substitute our parametric relations, $\phi = e^{-\kappa}$ and $\sigma_a^2 = \sigma^2 \frac{1-e^{-2\kappa}}{2\kappa}$, we find

$$2e^{-\kappa t}\sigma^2 \frac{1-e^{-2\kappa}}{2\kappa} r_0 \frac{1-e^{-\kappa s}}{1-e^{-2\kappa}} + e^{-\kappa(t-s)}\sigma^2 \frac{1-e^{-2\kappa}}{2\kappa}\gamma \frac{1-e^{-2\kappa s}}{1-e^{-2\kappa}}.$$

Simplifying this expression, gives us:

$$\sigma^2 \frac{e^{-\kappa t} - e^{-\kappa(s+t)}}{\kappa} r_0 + \sigma^2 \frac{e^{-\kappa(t-s)} - e^{-\kappa(t+s)}}{2\kappa}\gamma,$$

which is the covariance of the continuous CIR model, see (5).

We then conclude that we found a discretisation of the CIR model so that the first moment and the autocovariance function are equal for all sampling times. Further, we found the parametric relations between the continuous model and the discrete one. For this reason, we suggest that model (14) be referred to as the covariance equivalent discrete representation of the CIR model.

It should be acknowledged, unfortunately, that the CIR model is distributed as a non-central $\chi^2$ variable and that matching the first two moments does not necessarily match the distributions (as in the Gaussian case). So the distribution of the covariance equivalent discrete process defined in (14) might be different than the non-central $\chi^2$ distribution of the CIR model but at least they have the same expected value and autocovariance function at all sampling times.

An intuitive and admittedly rather imprecise explanation for the presence of the coefficient, $2\phi/(1+\phi)$, multiplying $r_0$ in the square root term of (14) is the following. For the continuous model, the instantaneous variance of the Noise process is $\sigma^2(r_0+\gamma)$ at time 0 and it changes gradually to $\sigma^2(r_1+\gamma)$ at time 1. In the discrete model presented in (14) this variance term is constant at $\sigma_a^2\,(r_0+\gamma)$ between time 0 and time 1. The parametric relation (13) captures the global impact of the volatility term of the continuous model over the interval and translates it into a one time impact at the end of the interval for the discrete model. This relation was obtained for a constant volatility term (see chapter 6 of Pandit and Wu 1983), so it applies to the $\gamma$ portion. For the $r_0$ portion, the fact that the volatility changes in the continuous model must be reflected in the discrete representation. And this cannot be accomplished by parametric relations alone. Note that if the process is stationary and $r_0>0$, the volatility is expected to decrease during the interval, so the adjustment should be less than one. For a stationary process, we have $\phi<1$. This implies that $2\phi/(1+\phi)$ is less than 1 as expected. A similar reasoning applies when $-\gamma< r_0 <0$.

Since the models defined by (8) and (14) have different dynamics, they will produce different results when used as approximations of the CIR model. In practice, the difference is even compounded by the fact that a given series of data will produce different parameter estimates depending on which of (8) or (14) is used. These aspects and others are addressed briefly in the next sections.

## *4. Estimating the parameters of the Cox-Ingersoll-Ross model.*

Suppose we want to estimate the parameters of the CIR model from discrete data and use those parameters in the bond price formula of Pitman and Yor (1982).

The first step is naturally to estimate the parameters of a discrete representation of the CIR model. We chose to use the least squares method to estimate the parameters of the discrete model. Note that one could use a different estimation method such as the method of maximum likelihood or the generalized method of moments.

Our least squares estimate of $\phi$ is the value that minimizes the residual sum of squares,

$$(17) \qquad RSS = \sum_{t=1}^{N} \sigma_a^2 a_t^2 ,$$

where N is the number of observations and the residuals, $\sigma_a^2 a_t^2$, are calculated from the data according to the specific model used.

For the simple discretisation (8), we have

$$(18) \qquad \sigma_a^2 a_t^2 = \frac{\left(r_t - \phi\, r_{t-1}\right)^2}{r_t + \gamma} .$$

For the covariance equivalent discretisation (14), we have

$$(19) \qquad \sigma_a^2 a_t^2 = \frac{\left(r_t - \phi\, r_{t-1}\right)^2}{\frac{2\phi}{1+\phi} r_t + \gamma} .$$

Finally, the least squares estimate of $\sigma_a^2$ is given by RSS/(N-1).

We now consider three ways of obtaining some parameter estimates for the CIR model.

### *4.1 Simple discretisation and same parameters (Method 1).*

Chan et al. (1992) use the discrete model (8) while keeping the same parameters in both the discrete and continuous processes. This suggests that, in our notation, they find an estimate for $\phi$ and then use $\kappa=1-\phi$. Also, they use $\sigma_a = \sigma$. We will use this as our first method.

On page 1213 of their paper they acknowledge that this is "only an approximation of the continuous-time specification" and that the error is "of second-order importance if changes in r are measured over short periods of time".

If the time unit of t is small, the approximation will be good, even if we use annualized rates. In this case, $\kappa$ will be relatively small and from $\phi = e^{-\kappa}$ which preserves the expected value in the continuous model, we will have $\phi \approx 1-\kappa$. Also, if $\kappa$ is small, (13) will give $\sigma_a \approx \sigma$.

### *4.2 Simple discretisation and parametric relations (Method 2).*

For our second method, we will use (17) and (18) to estimate the parameters $\phi$ and $\sigma_a$ of the model defined by (8) and use the parametric relations (12) and (13) to find $\kappa$ and $\sigma$.

### *4.3 Covariance equivalent discretisation (Method 3).*

As a third method, we will consider using (17) and (19) to estimate the parameters $\phi$ and $\sigma_a$ of the covariance equivalent discrete model defined in (14) and then use the parametric relations (12) and (13) to find $\kappa$ and $\sigma$.

## *5. Results.*

We will illustrate the three methods presented in the previous section by applying them to data consisting of (annualized) rates on Canadian Government 3-month T-Bills for 165 months covering the period of January 1981 until September 1994. The rates can be found in Table F1 of the *Bank of Canada Review.*

The results for the three methods of estimation of the parameters of the CIR model are presented in Table 1. The least squares estimates of the parameters of the discrete model used are also given.

**Table 1.** Parameter Estimates for the CIR Model.
Canadian 3-month T-Bills, 01/1981-09/1994.

| | Method 1 | Method 2 | Method 3 |
|---|---|---|---|
| Discrete Model Used | (8) | (8) | (14) |
| $\phi$ | .975654 | .975654 | .985590 |
| $\sigma_a$ | .022772 | .022772 | .022884 |
| $\kappa$ | .024346 | .024648 | .014514 |
| $\sigma$ | .022772 | .023053 | .023050 |

We can see from Table 1 that the choice of a discrete representation is important. The estimate for $\kappa$ is quite different when model (14) is used instead of model (8). Note that those parameters are expressed in monthly units since we used monthly data. For example, with method 3, the process will tend towards its long-term mean, $\gamma$, at a rate of .014 times its distance from $\gamma$ each month.

In Table 2, we present the expected value and standard deviation of the CIR process for each of the three sets of parameters given in Table 1. The expected value is given by (3) plus $\gamma$ and the variance is given by (6). We chose the starting value $r_0$=.052, the last rate in our data set (September 1994). For $\gamma$ we use the average of the 165 monthly values, i.e. .09948.

**Table 2.** Expected Value and Standard Deviation (sd) of the CIR model with $r_0$=.052 and $\gamma$=.09948.

| Time (in Months) | Method 1 | | Method 2 | | Method 3 | |
|---|---|---|---|---|---|---|
| | $\mathrm{E}[r_t]$ | $\mathrm{sd}[r_t]$ | $\mathrm{E}[r_t]$ | $\mathrm{sd}[r_t]$ | $\mathrm{E}[r_t]$ | $\mathrm{sd}[r_t]$ |
| 1 | .053142 | .005159 | .053156 | .005222 | .052684 | .005236 |
| 2 | .054257 | .007248 | .054284 | .007336 | .053358 | .007375 |
| 6 | .058453 | .012233 | .058527 | .012378 | .055960 | .012577 |
| 12 | .064029 | .016659 | .064157 | .016849 | .059589 | .017383 |
| 24 | .073010 | .021925 | .073201 | .022157 | .065966 | .023511 |
| 60 | .088462 | .028674 | .088660 | .028925 | .079605 | .032844 |
| 120 | .096923 | .031699 | .097014 | .031924 | .091160 | .038906 |
| 240 | .099342 | .032504 | .099352 | .032706 | .098022 | .042040 |
| 600 | .099480 | .032549 | .099480 | .032748 | .099472 | .042667 |

As expected, we can see that methods 1 and 2 produce similar results. Recall that method 1 is a good approximation of method 2 when $\kappa$ is small. Method 3 produces expected future short-term rates that are slower to approach $\gamma$. And it produces larger standard deviations although the volatility parameter $\sigma$ is very comparable to the one of method 2. This is caused by the presence of a smaller $\kappa$ in (6).

Next we will evaluate the bond price formula of Pitman and Yor (1982):

$$\text{(20)} \quad \mathrm{E}\left[e^{-\int_0^t r_u du}\right] = \frac{\exp\left\{-\frac{x}{\sigma^2} w \frac{1+\frac{\kappa}{w}\coth(wt/2)}{\coth(wt/2)+\frac{\kappa}{w}}\right\} e^{\frac{\kappa x}{\sigma^2}} e^{\kappa^2 \gamma t/\sigma^2}}{\left(\cosh(wt/2)+(\kappa/w)\sinh(wt/2)\right)^{\frac{2\kappa\gamma}{\sigma^2}}},$$

where $x = r_0$ and $w = \sqrt{\kappa^2 + 2\sigma^2}$ .

Since this formula assumes annual rates and annualized parameters, we need to adapt our parameter values before evaluating it. This is accomplished by multiplying $\kappa$ by 12 and by multiplying $\sigma$ by $\sqrt{12}$. Note that the adjustments in the parameters of the discrete models would be different. In fact, $\phi$ should be raised to the power 12 and $\sigma_a$ should be such that the stationary variance is unchanged after the change in the time unit. That is, $\sigma_a$ should be multiplied by $\left(1-\phi^{24} \big/ 1-\phi^2\right)^{\frac{1}{2}}$ using the unadjusted value of $\phi$.

The bond prices for different maturities are presented in Table 3.

**Table 3.** Bond Prices
using the CIR model with $r_0$=.052 and $\gamma$=.09948.

| Time (in years) | Method 1 | Method 2 | Method 3 |
|---|---|---|---|
| 1 | 0.943404 | 0.943338 | 0.945677 |
| 2 | 0.880976 | 0.880776 | 0.888335 |
| 3 | 0.816577 | 0.816233 | 0.829974 |
| 4 | 0.752782 | 0.752316 | 0.772074 |
| 5 | 0.691222 | 0.690666 | 0.715692 |
| 10 | 0.437543 | 0.436923 | 0.473859 |
| 20 | 0.168356 | 0.168069 | 0.194982 |
| 50 | 0.009426 | 0.009410 | 0.012839 |

This shows the importance of the discrete model used. The bond prices of method 3 are larger than those of the other two methods. This is caused by the smaller value of $\kappa$ and the fact that $r_0$ is smaller than $\gamma$.

Had $r_0$ been larger than $\gamma$, method 3 (always with $\kappa$ smaller than in methods 1 and 2) would have produced smaller bond prices for short maturities and larger bond prices for large maturities. This is illustrated in Table 4 where $r_0$ was set at .152.

**Table 4.** Bond Prices
using the CIR model with $r_0$=.152 and $\gamma$=.09948.

| Time (in years) | Method 1 | Method 2 | Method 3 |
|---|---|---|---|
| 1 | 0.865109 | 0.865179 | 0.862827 |
| 2 | 0.757509 | 0.757717 | 0.750746 |
| 3 | 0.669489 | 0.669839 | 0.658155 |
| 4 | 0.595888 | 0.596355 | 0.580802 |
| 5 | 0.533199 | 0.533751 | 0.515471 |
| 10 | 0.319097 | 0.319700 | 0.301178 |
| 20 | 0.121037 | 0.121314 | 0.116507 |
| 50 | 0.006772 | 0.006788 | 0.007603 |

## *6. Remarks and conclusion.*

In this contribution, we compared a simple discretisation method with a covariance equivalent discretisation of the CIR square root model. The parametric relations were established. The simple model (8) has the same expected value and stationary variance as the continuous model (2). The covariance equivalent model (14) has the same first two moments as model (2) at all sampling times. For this reason, we suggest that model (14) be used as a discrete representation of the continuous model (2).

An interesting question not addressed here is whether the two models, (2) and (14), have the same distribution at all sampling times. This will be the subject of further investigation by the authors. Since it is known that model (2) has a non-central $\chi^2$ distribution, the big question is whether (14) has the same distribution or not.

We also illustrated that although models (8) and (14) look alike, they can produce significantly different estimates for the parameters of the CIR model. Consequently, they can lead to different conclusions when used with known results about the CIR model such as the bond price formula (20).

We point out that we do not claim that model (14) necessarily captures the dynamics of short interest rates better than model (8). This is likely to depend on the data used. For example, data for different Treasury bill yields and/or different periods may call for a different model if goodness-of-fit is our goal.

We merely argue that when the short-term interest is assumed to follow a CIR square root model, the covariance equivalent discretisation is more appropriate then the simple discretisation.

### *Acknowledgements*

Financial support by the project SPES-CT91-0089 and by the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

### *References*

Chan K.C., Karolyi G.A., Longstaff F.A. and Sanders A.B. (1992) An Empirical Comparison of Alternative Models of the Short-Term Interest Rate. *The Journal of Finance Vol **XLVII**, 1209-1227.*

Cox J.C., Ingersoll J.E. and Ross S.A. (1985) A Theory of the Term Structure of Interest Rates. *Econometrica **53***, 385-407.

Deelstra G. and Delbaen F. (1994) Long-Term Returns in Stochastic Interest Rates Models. *Forthcoming.*

Pandit S.M. and Wu S.M. (1983) *Time Series and System Analysis with Applications.* John Wiley and Sons. 586p.

Parker G. (1995) Stochastic Rates of Return: Practical Applications of First and Second Order SDE. *25th International Congress of Actuaries.* Brussels.

Pitman J. and Yor M. (1982) A Decomposition of Bessel Bridges. *Zeitschrift für Warhrscheinlichkeitstheorie und Verwandte Gebiete **59***, 425-457.