LONG-TERM RETURNS IN STOCHASTIC INTEREST RATE MODELS : APPLICATIONS

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Abstract

We extend the Cox-Ingersoll-Ross model of the short interest rate by assuming a stochastic reversion level, which better reflects the time dependence caused by the cyclical nature of the economy or by expectations concerning the future impact of monetary policies. In this framework, we have studied the convergence of the long-term return by using the theory of generalized Besselsquare processes.

We emphasize on the applications of the convergence results. We propose an approximation of bond prices in very general situations. These approximating formula can be used in insurance. During the evaluation, we use a comparison with the bond price obtained in the paper of Pitman-Yor (1982). Different simulations illustrate the results.

Keywords : Interest rates, Cox-Ingersoll-Ross model, Stochastic reversion level, Generalised Besselsquare processes, Convergence, Bond prices.

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Résumé

Nous généralisons le modèle de Cox-Ingersoll-Ross du taux d'intérêt instantané en supposant un réversion stochastique. Dans cette situation, nous avons étudié la convergence du retour à long terme en utilisant la théorie du processus de Bessel généralisé. Cet article est consacré aux applications des résultats de la convergence. Nous proposons des approximations des prix des obligations dans des situations très générales. Les formules approximatives peuvent être utilisées en assurance. Afin d'évaluer nos résultats, nous les comparons avec les prix des obligations obtenus dans l'article de Pitman-Yor (1982) et nous faisons plusieurs simulations.

1 Introduction

Interest rate models are indispensable tools in modern theoretical and empirical research in finance and macro-economics. In the working paper "Do Interest Rates Converge?" [1], which is a preliminary version of "Long Forward Rates can Never Fall." [2], Dybvig, Ingersoll and Ross show that nearly all such models have the surprising implication that long run forward rates and zero coupon rates converge to a constant, which is independent of the current state of the economy.

They particularly focus in on forward rates and they recognize two emerging issues. They define the "strong convergence property" (SCP) as the characteristic that the rates converge to a constant as the maturity is increased, where the constant is independent of the earlier shape of the term structure and, indeed, of the current state of the economic environment. If the rates converge to a constant, that will generally depend upon the current economic environment and that may change in a stochastic fashion over time, they say the "weak convergence property" (WCP) holds.

Dybvig, Ingersoll and Ross conjecture that an important class of cases where the weak convergence property holds, but the strong convergence property may not, arises when there are two or more stochastic state variables whose movements determine the term structure.

In this paper, we concentrate on the convergence of the long-term return $\frac{1}{t}\int_0^t r_u du$, using a very general two-factor model, which is an extension of the Cox-Ingersoll-Ross model [3]. Cox, Ingersoll and Ross express the short interest rate dynamics as

$$dr_t = \kappa(\gamma - r_t)dt + \sigma\sqrt{r_t}dB_t.$$

with $(B_t)_{t\geq 0}$ a Brownian motion and κ, γ and σ positive constants. This model has some realistic properties. First, negative interest rates are precluded. Second, the absolute variance of the interest rate increases when the interest rate itself increases. Third, the interest rates are elastically pulled to the long-term value γ, where κ determines the speed of adjustment.

As Dybvig, Ingersoll and Ross remarked, it is very surprising that this long-term value is a constant independent of the current state of the economy, and that the SCP holds. This is our primary motivation for assuming a two-factor model of the short interest rate with a stochastic reversion level, which better reflects the time dependence caused by the cyclical nature of the economy or by expectations concerning the future impact of monetary policies.

Since it is inherent in a single-factor model that price changes in bonds of all maturities are perfectly correlated, a lot of authors already have suggested two-factor models. For example, Brennan and Schwartz [4] use the short-term interest rates and the console rates; Schaefer and Schwartz [5] look at the long-term interest rates and the spread between the long-term and short-term rates; Longstaff and Schwartz [6] study the short interest rates and the instantaneous variance of changes in the short-term interest rates. Cox, Ingersoll and Ross [3] allow the central tendency parameter γ itself to vary randomly according to the equation

$$d\gamma_t = \tilde{\kappa}(Y_t - \gamma_t)dt,$$

where $\tilde{\kappa}$ is a positive constant and where Y denotes the state variable which describes the change in production opportunities.

In this paper, we assume that the short interest rate X is governed by the stochastic differential equation

$$dX_s = (2\beta X_s + \delta_s)ds + v\sqrt{X_s}dB_s$$

with $\beta < 0$ the drift rate parameter, v a constant and δ a nonnegative predictable stochastic process such that $\int_0^t \delta_u du < \infty$ a.e. for all $t \in I\!R^+$. In "Existence of Solutions of Stochastic Differential Equations related to the Bessel process." [7], we have shown that this stochastic differential equation has a unique (non-negative) strong solution.

Remark that the stochastic process $(\delta_s)_{s\geq 0}$ determines a reversion level. If it is chosen to be a constant and if $v = 2$, the process $(X_s)_{s\geq 0}$ is a Besselsquare process with drift, a process which is studied in great detail by Yor [8,9]. As the model is a generalisation of Besselsquare processes with drift, it is fairly easy to treat. Nevertheless, the model is very general and it includes various models, for example the single-factor and the two-factor Cox-Ingersoll-Ross models mentioned above.

Using the theory of Bessel processes, we proved a theorem in "Long-term returns in stochastic interest rate models." [10], which is very useful in deducing the convergence almost everywhere of the long-term return in general situations. In section 2, we use this theorem to affirm the conjecture of Dybvig, Ingersoll and Ross: it is easy to build two-factor models in which the long-term return fulfils the weak converging property, but not necessarily the strong convergence property. However, we also give some examples of two-factor models in which the long-term return has the strong convergence property. In this case, the average of the accumulating factor (also called return) $\left(\exp(\int_0^t r_u du)\right)^{1/t}$ converges almost everywhere to a constant independent of the current market, as the observing period tends to infinity.

In "Long-term returns in stochastic interest rate models: Convergence in law." [11], we found conditions necessary to prove the convergence in law of a transformation of the long-term return to a Brownian motion. In section 3, we propose a slightly generalised theorem and we recall the idea of approximating $\int_0^t r_u du$ for t large enough. If the objective is to approximate the distribution of the long-term return of an investment made at time 0, it is appropriate to approximate $\int_0^t r_u du$ by a scaled Brownian motion with drift for t going to infinity. In the past, a lot of authors have proposed Wiener models since on long-term, the Central Limit Theorems are applicable. We present some simulation results which show that t has to be fairly large, namely more than 40 years, to obtain good approximations. For practical reasons, we are interested in an approximation of $\int_0^t r_u du$ for all values of t in order to be able to calculate bond prices, accumulation and discounting factors ... Therefore, we suggest an improved approximation, which will be discussed and evaluated. The obtained results are very satisfactory.

In section 4, we give some applications in insurance. This section is based on chapter 4 and 5 of the Ph. D. of Parker, namely "An application of stochastic interest rates models in life assurance."[12]. Like in Bowers et al. [13], the present values of an n-year temporary life assurance, an whole-life assurance and an endowment assurance are defined. We approximate the expected values, the variances and the skewness of these present values by the approximation proposed in section 3. An evaluation will be given in case of the Cox-Ingersoll-Ross model. We remark that our results are very useful in finding the expected value, the variance and the skewness of the total present value of all benefits to be paid within a portfolio. This is very interesting with regard to the problems of setting contingency reserves and assessing the solvency of life assurance companies.

Without further notice we assume that a probability space $\left(\mathbf{A}, (\mathcal{F}_t)_{0\leq t}, I\!P\right)$ is given and that the filtration $(\mathcal{F}_t)_{0\leq t}$ satisfies the usual assumptions with respect to $I\!P$, a fixed probability on the sigma-algebra $\mathcal{F}_\infty = \bigvee_{t\geq 0}\mathcal{F}_t$. Also B is a continuous process that is a Brownian motion with respect to $(\mathcal{F}_t)_{0\leq t}$.

2 Various two-factor models with SCP.

In this section, we use a theorem obtained in [10] to show that it is easy to build two-factor models in which the long-term return owns the weak convergence property, but not the strong convergence property. However, two-factor models do not necessarily imply that the strong convergence property does not hold. Various two-factor models are presented, in which the long-term return has the strong convergence property.

We remark that the almost everywhere convergence limit of $\frac{1}{t}\int_0^t r_u du$ is very interesting to study since economists and actuaries work with the multiplicative accumulating factor (return) over t years, namely $\exp(\int_0^t r_u du)$. The average return in one year, where the average is taken over t years, is denoted by $\left(\exp(\int_0^t r_u du)\right)^{1/t}$. If the observing period goes to infinity, it converges to the exponent of the almost everywhere limit of $\frac{1}{t}\int_0^t r_u du$.

We recall from [10] that if X is defined by

$$dX_s = (2\beta X_s + \delta_s)ds + v\sqrt{X_s}dB_s$$

with $(B_s)_{s\geq 0}$ a Brownian motion, $\beta < 0$, v a constant and δ a positive, predictable stochastic process such that $\frac{1}{s}\int_0^s \delta_u du \xrightarrow{a.e.} \overline{\delta}$ with $\overline{\delta} : \mathbf{A} \longrightarrow$

$I\!R$, then the following convergence almost everywhere holds:

$$\frac{1}{s}\int_0^s X_u du \xrightarrow{a.e.} \frac{-\overline{\delta}}{2\beta}.$$

It is easy to show that for $r_t = \frac{\sigma^2}{4}X_t$, $v = 2$, $\beta = -\kappa/2$ and $\delta_t = \frac{4\kappa\gamma_t}{\sigma^2}$, we obtain a generalised two-factor Cox-Ingersoll-Ross model

$$dr_t = \kappa(\gamma_t - r_t)dt + \sigma\sqrt{r_t}dB_t$$

with $(\gamma_s)_{s\geq 0}$ a stochastic reversion level process. This process $(\gamma_s)_{s\geq 0}$ can be chosen completely arbitrarily as long as it remains positive and $\frac{1}{t}\int_0^t \gamma_s ds$ converges almost everywhere to a random variable $\overline{\delta} : \mathbf{A} \longrightarrow I\!R^+$. The central tendency process $(\gamma_s)_{s\geq 0}$ may be dependent or independent to the short interest rate process.

For example, let us describe the stochastic reversion level process $(\gamma_s)_{s\geq 0}$ by a Cox-Ingersoll-Ross square root process [3]

$$d\gamma_t = \tilde{\kappa}(\gamma^* - \gamma_t)dt + \tilde{\sigma}\sqrt{\gamma_t}d\tilde{B}_t$$

or by a Courtadon process [14]

$$d\gamma_t = \tilde{\kappa}(\gamma^* - \gamma_t)dt + \tilde{\sigma}\gamma_t d\tilde{B}_t$$

or by a cubic variance process [15,16]

$$d\gamma_t = \tilde{\kappa}(\gamma^* - \gamma_t)dt + \tilde{\sigma}\sqrt{\gamma_t^3}d\tilde{B}_t,$$

with $(\tilde{B}_s)_{s\geq 0}$ a Brownian motion and with $\tilde{\kappa}, \gamma^*$ and $\tilde{\sigma}$ positive constants. The Brownian motion $(\tilde{B}_s)_{s\geq 0}$ may be correlated with the Brownian motion $(B_s)_{s\geq 0}$ of the short rate process and this correlation may be in a random way. In contrast with most papers, we do not need the technical assumption of fixed correlation or independence between the two factors of the model.

The three proposed reversion level processes are from the same family. They all remain positive for $\tilde{\kappa}, \gamma^* \geq 0$, a property which is necessary if one wants to work with nominal interest rates. For $\tilde{\kappa}, \gamma^* > 0$, these processes are mean-reverting to the long-term constant value γ^*, where $\tilde{\kappa}$ represents the speed of adjustment. The volatility increases in

all three cases with the reversion level.
For this class of stochastic reversion levels, $\frac{1}{t}\int_0^t \gamma_s ds \xrightarrow{a.e.} \gamma^*$ and since $\delta_s = \frac{4\kappa\gamma_s}{\sigma^2}$, $\frac{1}{t}\int_0^t \delta_s ds \xrightarrow{a.e.} \frac{4\kappa\gamma^*}{\sigma^2}$. By the theorem mentioned above (see [10]), the long-term return is shown to converge almost everywhere to a constant:

$$\frac{1}{t}\int_0^t r_s ds = \frac{1}{t}\int_0^t \frac{\sigma^2}{4} X_s ds \xrightarrow{a.e.} \gamma^*.$$

We conclude that the long-term return in these two-factors models of short interest rate satisfies the strong convergence property as defined by Dybvig, Ingersoll and Ross. The average accumulating factor, where the average is taken over a period t, is found to converge almost everywhere to a constant as the period t tends to infinity, and this constant is independent of the current state of the economy:

$$\left(e^{\int_0^t r_u du}\right)^{1/t} \xrightarrow{a.e.} e^{\gamma^*}.$$

Let us now choose a stochastic reversion level process from another family: we express the reversion level dynamics by a Longstaff double square root process [17]:

$$d\gamma_t = \tilde{\kappa}(\gamma^* - \sqrt{\gamma_t})dt + \tilde{\sigma}\sqrt{\gamma_t}d\tilde{B}_t.$$

In this case $\frac{1}{t}\int_0^t \gamma_s ds \xrightarrow{a.e.} \gamma^{*2}$ and consequently

$$\frac{1}{t}\int_0^t r_s ds \xrightarrow{a.e.} \gamma^{*2}.$$

The same conclusion holds: we find the SCP and

$$\left(e^{\int_0^t r_u du}\right)^{1/t} \xrightarrow{a.e.} e^{\gamma^{*2}},$$

where $e^{\gamma^{*2}}$ is a constant independent of the current state of the economy.

As a last example in this section, we treat the two-factor model proposed by Cox, Ingersoll and Ross [3]. They assumed a stochastic reversion level process depending on Y, the state variable which describes the change in the production opportunities, namely

$$\begin{aligned} dr_t &= \kappa(\gamma_t - r_t)dt + \sigma\sqrt{r_t}dB_t \\ d\gamma_t &= \tilde{\kappa}(Y_t - \gamma_t)dt \\ dY_t &= -\xi(\frac{-\zeta}{\xi} - Y_t)dt + \tilde{\sigma}\sqrt{Y_t}dB_t', \end{aligned}$$

with $\kappa, \sigma, \xi, \zeta$ and $\tilde{\sigma}$ constants. Remark that this model has a serious drawback as the explanatory variable Y is not directly observable. We here only theoretically show that this model also has the SCP for the long-term return: Since $\frac{1}{t}\int_0^t Y_s ds \xrightarrow{a.e.} \frac{-\zeta}{\xi}$, we have that $\frac{1}{t}\int_0^t \gamma_s ds \xrightarrow{a.e.} \frac{\zeta}{\xi}$ and by the same reasoning as above, we obtain

$$\frac{1}{t}\int_0^t r_s ds \xrightarrow{a.e.} \frac{\zeta}{\xi}.$$

We conclude that two-factor models do not inevitably lack the SCP. It is not surprising that these examples satisfy the SCP since in each model, the reversion level process itself is elastically pulled to a constant independent of the economic state. We recall that the convergence theorem from [10] has no such strong hypothesis, on the contrary, the assumptions are very general. For example, the reversion level process does not have to be continuous. The convergence theorem only assumes a positive, predictable reversion $(\delta_u)_{u\geq 0}$ such that $\frac{1}{s}\int_0^s \delta_u du \xrightarrow{a.e.} \overline{\delta}$, where $\overline{\delta}$ may be a random variable. Models in which this $\overline{\delta}$ really is a random variable, would imply the WCP for the long-term return but not the SCP.

3 Approximation of the long-term return and of bond prices.

In this section, we give a slightly generalised version of the central limit theorem from [11]. We study the convergence in law since it is always useful to know how the long-term return is distributed in the limit so that approximations can be deduced. We are particularly interested in an approximation of $\int_0^t r_u du$ since this term appears in bond prices, accumulation and discounting factors,...An approximation is proposed, which offers very good estimations for bond prices of all maturities. Simulation results are presented.

In order to obtain convergence in law, we have to make some more assumptions about our family:

Theorem

Suppose that a probability space $(\mathrm{A}, (\mathcal{F}_t)_{t\geq 0}, I\!P)$ is given and that a *stochastic process* $X : \mathbf{A} \times I\!R^+ \to I\!R^+$ *is defined by the stochastic differential equation*

$$dX_s = (2\beta X_s + \delta_s)ds + v\sqrt{X_s}dB_s \quad \forall s \in I\!R^+$$

with $(B_s)_{s\geq 0}$ *a Brownian motion with respect to* $(\mathcal{F}_t)_{t\geq 0}$, v *a constant and* $\beta < 0$.
Let us make the following assumptions about the adapted and measurable process $\delta : \mathbf{A} \times I\!R^+ \to I\!R^+$:

- $\frac{1}{s}\int_0^s \delta_u du \xrightarrow{a.e.} \overline{\delta}$ *where* $\overline{\delta}$ *is a real number,* $\overline{\delta} > 0$
- *There is a constant* k *such that* $\sup_{t\geq 1} \frac{1}{t}\int_0^t I\!E\left[\delta_u^2\right] du \leq k$
- *For all* $a \in I\!R^+$ $\lim_{t\to\infty} \frac{1}{t}\int_{t-a}^t I\!E[\delta_u^2]du = 0$.

Under these conditions, the following convergence in distribution holds:

$$\left(\sqrt{\frac{-8\beta^3}{v^2\overline{\delta}n}}\int_0^{nt}\left(X_u + \frac{\delta_u}{2\beta}\right)du\right)_{t\geq 0} \xrightarrow{\mathcal{L}} (B_t)_{t\geq 0}$$

where $(B_t)_{t\geq 0}$ *denotes a Brownian motion and where* '$\xrightarrow{\mathcal{L}}$' *denotes convergence in law.*

Since this theorem is an immediate consequence of the theorem in [11], the proof is omitted. We refer the interested reader to [11] and we immediately turn to the applications.

Inspired by this theorem, we estimate

$$\int_0^t X_u du \qquad \text{by} \qquad \int_0^t \frac{-\delta_u}{2\beta}du + \sqrt{\frac{-v^2\overline{\delta}}{8\beta^3}}B_t$$

for t large enough. In [11], we used the hypothesis $\frac{1}{t}\int_0^t \delta_u du \xrightarrow{a.e.} \overline{\delta}$, to approximate $\int_0^t X_u du$ by the sum of the long-term constant $\frac{-\overline{\delta}}{2\beta}$, to which the long-term return a.e. converges, multiplied by t and a scaled Brownian motion:

$$\int_0^t X_u du \qquad \text{by} \qquad \frac{-\overline{\delta}}{2\beta}t + \sqrt{\frac{-v^2\overline{\delta}}{8\beta^3}}B_t. \tag{1}$$

A drawback of this estimator is that the moments of $\int_0^t X_u du$ do not equal those of the estimator, although they are the same asymptotically. If the period observed is large enough, this is satisfactory. If the objective is to approximate the distribution of the long-term return of an investment made at time 0, it seems to be appropriate to approximate $\int_0^t X_u du$ by a scaled Brownian motion with drift since the Central Limit Theorems are applicable on long-term.

For practical reasons, we want to know how large the period should be to have reasonable approximations. Therefore, we did some simulations in case of the Cox-Ingersoll-Ross square root process $(r_u)_{u\geq 0}$. We recall from the previous section that this process can be transformed to a stochastic process $(X_u)_{u\geq 0}$ of our family observed, namely by the transformations $r_u = \frac{\sigma^2}{4} X_u$, $v = 2$, $\overline{\delta} = \frac{4\kappa\gamma}{\sigma^2}$ and $\beta = \frac{-\kappa}{2}$. Substituting these relations reduces the approximation to

$$\int_0^t r_u du \quad \sim \quad \gamma t + \sqrt{\frac{\sigma^2 \gamma}{\kappa^2}} B_t.$$

Using the parameters estimated within the empirical work of Chan, Karolyi, Longstaff and Sanders [15], namely $\kappa = 0.23394, \gamma = 0.0808$ and $\sigma = 0.0854$, we have simulated the long-term return $\frac{1}{t}\int_0^t r_u du$ and the approximation $\gamma + \sqrt{\frac{\sigma^2\gamma}{t\kappa^2}} B_1$ for t equal to 10, 20, 30 and 40 years. We applied a χ^2-test to check if the distributions of the long-term return and of the approximation are equal. Therefore, we calculated the probabilities of being in the intervals $[\ln(1+(i-1)/100), \ln(1+i/100)]$ for $1 \leq i \leq 20$. For periods smaller than 30 years, the null-hypothesis that the distributions are equal, is rejected at every reasonable significance level. For a period of 40 years, the χ^2-statistic equals 24.342 and the null-hypothesis is accepted at the significance level $\alpha = 0.05$. We conclude that if one wants to approximate the distribution of the long-term return, then one should observe a period larger than 40 years.

However, one of our objectives is to look at the approximation

$$\int_0^t X_u du \qquad \text{by} \qquad \frac{-\overline{\delta}}{2\beta} t + \sqrt{\frac{-v^2\overline{\delta}}{8\beta^3}} B_t$$

to find estimations of bond prices for all maturities. Therefore, the moments of $\int_0^t X_u du$ and of the estimator should be equal for all t. A second drawback of the approximation immediately appears in the bond price, namely

$$\mathbb{E}_{X_0}\left[e^{-\int_0^t X_u du}\right] \sim \exp\left(\frac{\overline{\delta}}{2\beta}t - \frac{v^2\overline{\delta}}{16\beta^3}t\right).$$

It is not realistic that the estimating bond price is independent of the current short interest rate X_0. Remark that we work with the preference-free bond prices with no market price of risk, since we want to compare different approximations theoretically. In the sequel, we omit without notice the adjective "preference-free".

In case of the Cox-Ingersoll-Ross square root process, the approximating bond price equals:

$$\mathbb{E}_{r_0}\left[e^{-\int_0^t r_u du}\right] \sim \exp\left(\gamma t(\frac{\sigma^2}{2\kappa^2} - 1)\right).$$

This estimating bond price is a decreasing function of the speed of adjustment parameter κ, where in the case of the Cox-Ingersoll-Ross model, two cases are distinguished: for $r_0 < \gamma$, the bond price is a decreasing function of the parameter κ, and for $r_0 > \gamma$, it is an increasing function of κ. In [11], we showed by simulation of these approximating bond prices with the parameters of Chan, et al. [15], that there is an underestimation of bond prices if $r_0 < \gamma$ and an overestimation if $r_0 > \gamma$.

Therefore, we searched for an improved approximation. It seems logical to propose the approximation

$$\int_0^t X_u du \quad \sim \quad \int_0^t \mathbb{E}[X_u]du + \sqrt{\frac{-v^2\overline{\delta}}{8\beta^3}}B_t.$$

Then, the expectation value is equal for all t and the variance still is asymptotically equal.

Since $(X_u)_{u\geq 0}$ is defined by the stochastic differential equation

$$dX_s = (2\beta X_s + \delta_s)ds + v\sqrt{X_s}dB_s,$$

the expectation value of X_s equals:

$$\mathbb{E}[X_s] = e^{2\beta s}X_0 + e^{2\beta s}\int_0^s e^{-2\beta u}\mathbb{E}[\delta_u]du,$$

which can only be calculated if $\mathbb{E}[\delta_u]$ is known. But then, it is immediately clear that the current state X_0 is introduced in the approximation. If the expected value of the reversion level δ_u is unknown, we use the hypothesis that $\frac{1}{t}\int_0^t \delta_u du \xrightarrow{a.e.} \overline{\delta}$ to propose the approximation

$$\mathbb{E}[\delta_u] \sim \overline{\delta}$$

and consequently

$$\int_0^t X_u du \sim \int_0^t \left(e^{2\beta s}X_0 + e^{2\beta s}\int_0^s \overline{\delta}e^{-2\beta u}du\right) + \sqrt{\frac{-v^2\overline{\delta}}{8\beta^3}}B_t.$$

An easy calculation shows that:

$$\int_0^t X_u du \sim \frac{-\overline{\delta}}{2\beta}t + \frac{e^{2\beta t}-1}{2\beta}\left(X_0 + \frac{\overline{\delta}}{2\beta}\right) + \sqrt{\frac{-v^2\overline{\delta}}{8\beta^3}}B_t.$$

In this case, X_0 is introduced in the approximation and the moments still are asymptotically equal.

As an example of the approximation, let us look again at the Cox-Ingersoll-Ross two-factor model:

$$\begin{aligned} dr_t &= \kappa(\gamma_t - r_t)dt + \sigma\sqrt{r_t}dB_t, \\ d\gamma_t &= \tilde{\kappa}(\gamma^* - \gamma_t)dt + \tilde{\sigma}\sqrt{\gamma_t}d\tilde{B}_t. \end{aligned}$$

The approximation becomes

$$\begin{aligned} \int_0^t r_u du &\sim \int_0^t \mathbb{E}[r_u]du + \sqrt{\frac{\sigma^2\gamma^*}{\kappa^2}}B_t \\ &\sim \gamma^* t + \frac{1-e^{-\kappa t}}{\kappa}\left(r_0 - \gamma^* - \frac{\gamma_0 - \gamma^*}{\kappa - \tilde{\kappa}}\kappa\right) \\ &\quad + \frac{1-e^{-\tilde{\kappa}t}}{\tilde{\kappa}}\left(\frac{\gamma_0-\gamma^*}{\kappa-\tilde{\kappa}}\right)\kappa + \sqrt{\frac{\sigma^2\gamma^*}{\kappa^2}}B_t. \end{aligned}$$

The bond price is estimated by:

$$\mathbb{E}_{r_0}\left[e^{-\int_0^t r_u du}\right] \sim \exp\left(\gamma^* t(\frac{\sigma^2}{2\kappa^2} - 1)\right).$$

$$\exp\left(-\frac{1-e^{-\kappa t}}{\kappa}\left(r_0 - \gamma^* - \frac{\gamma_0-\gamma^*}{\kappa-\tilde{\kappa}}\kappa\right) - \frac{1-e^{-\tilde{\kappa}t}}{\tilde{\kappa}}\left(\frac{\gamma_0-\gamma^*}{\kappa-\tilde{\kappa}}\right)\kappa\right).$$

Let us evaluate the approximation in case of the Cox-Ingersoll-Ross single-factor model:

$$dr_t = \kappa(\gamma - r_t)dt + \sigma\sqrt{r_t}dB_t.$$

An anonymous referee remarked (see [11]) that in this case, the moments of the first proposal (1) are equal for all t, as soon as the current short interest rate r_0 is distributed according to the steady state distribution of the square root process, namely the gamma-function with parameters $\alpha = \frac{2\kappa\gamma}{\sigma^2}$ and $\beta = \frac{2\kappa}{\sigma^2}$:

$$\mathbb{E}_{r_0}\left[\int_0^t r_u du\right] = \gamma t = \mathbb{E}_0\left[\gamma t + \sqrt{\frac{\sigma^2}{2\kappa^2}}B_t\right].$$

In reality, r_0 is not distributed this way, so an improvement is also necessary here to obtain good estimations of bond prices:

$$\int_0^t r_u du \sim \int_0^t \mathbb{E}[r_u]du + \sqrt{\frac{\sigma^2\gamma}{\kappa^2}}B_t.$$

Substituting the mean of the short interest rate, gives the expression

$$\int_0^t r_u du \sim \gamma t + \frac{1-e^{-\kappa t}}{\kappa}(r_0 - \gamma) + \sqrt{\frac{\sigma^2\gamma}{\kappa^2}}B_t$$

and the estimating bond price is found to be

$$\mathbb{E}_{r_0}\left[e^{-\int_0^t r_u du}\right] \sim \exp\left(\gamma t(\frac{\sigma^2}{2\kappa^2} - 1) - \frac{1-e^{-\kappa t}}{\kappa}(r_0 - \gamma)\right).$$

In case of the previous approximation (1), we found for $r_0 < \gamma$ an underestimation of the bond prices. The approximation in this paper is larger since for $r_0 < \gamma$, a positive term is added to the exponent, namely $-\frac{1-e^{-\kappa t}}{\kappa}(r_0 - \gamma)$. In the same way, the underestimation in case of $r_0 > \gamma$ is solved.

For the Cox-Ingersoll-Ross square root process, an explicit formula for the bond price is given by Pitman-Yor [8] and Cox, Ingersoll and

Table 1: Bond prices: The exact values and the approximations.

t	$r_0 = 0.04$			$r_0 = 0.1$		
	Exact	Approx	Approx-Exact	Exact	Approx	Approx-Exact
1	.9565	.9617	.0051	.9068	.9116	.0048
6	.7061	.7254	.0192	.5843	.5978	.0134
7	.6587	.6788	.0200	.5386	.5521	.0134
8	.6135	.6339	.0204	.4970	.5102	.0131
9	.5708	.5912	.0204	.4591	.4719	.0127
10	.5305	.5507	.0201	.4244	.4367	.0122
20	.2503	.2630	.0127	.1968	.2040	.0071
30	.1171	.1239	.0068	.0919	.0959	.0039
40	.0547	.0582	.0035	.0430	.0451	.0021

Ross [3]. We recall the bond price from Pitman-Yor [8]:

$$\begin{aligned} &\mathbb{E}_{r_0}\left[\exp\left(-\int_0^t r_u du\right)\right] \\ &= \frac{\exp\left\{-\frac{r_0}{\sigma^2} w \frac{1+\kappa/w \coth(wt/2)}{\coth(wt/2)+\kappa/w}\right\} e^{\kappa x/\sigma^2} e^{\kappa^2\gamma^* t/\sigma^2}}{\left(\cosh(wt/2)+\kappa/w \sinh(wt/2)\right)^{\frac{2\kappa\gamma^*}{\sigma^2}}} \end{aligned}$$

with $w = \sqrt{\kappa^2 + 2\sigma^2}$.

Using various values for the parameters, we have calculated this exact bond price and the improved approximation, for a large range of maturities. The deviations are always very small. The largest deviations appear when the bond price has a value about 0.5. The reason therefore is that the bond price is a decreasing convex function of maturity and that the endpoints are fixed, namely for $t = 0$, the bond price equals 1, and for $t = \infty$, the bond price converges to 0. Consequently, the largest deviations are to be expected around one half.

In table 1, the exact bond prices and the estimating bond prices are calculated with the parameters estimated by Chan et al. [15]. The results are given for $r_0 = 0.04$ and for $r_0 = 0.1$. It is clear that the underestimation and overestimation are solved. In both cases, there only is a small overestimation. The maturities between 6 and 10 are given since then, the bond price is approximately 0.5 and the largest deviations appear. We do not present prices for bonds with maturities larger than 40 years: these are of course very good since the equality of the distribution of the long-term return and of its approximation is not rejected by the χ^2-test mentioned above.

We conclude that the approximation is very good in case of the Cox-Ingersoll-Ross model and that the approximation proposed in this section forms a good solution for models in which explicit formula's for bond prices are unknown.

4 Applications in life assurance.

In this section, we show the usefulness of the approximations of section 3 in insurance, and in particularly in life assurance. This section is based on chapter 4 and 5 of the Ph. D. thesis of Parker, namely "An application of stochastic interest rates models in life assurance."[12].

Following the notation of [12], we denote by K the integer-valued discrete random variable which represents the number of completed years to be lived by a life assured, whose age is exactly x year at the issue of the contract. We let $\mathcal{Z}$ be the present value of the benefit payable under a given assurance contract. As the precise definition of $\mathcal{Z}$ depends on the specific assurance under consideration, we have a look at some examples: the n-year temporary assurance, the whole-life assurance and the endowment assurance.

Under the n-year temporary assurance, the benefit of 1 is payable at the end of the year of death of a life assured, if the death occurs within n years from the date of issue. Thus $\mathcal{Z}$ is defined to be:

$$\mathcal{Z} = \begin{cases} \exp\left(-\int_0^{K+1} X_u du\right) & K = 0, 1, \ldots, n-1 \\ 0 & K = n, n+1, \ldots \end{cases}$$

where $(X_u)_{u \geq 0}$ denotes as before the short interest rate, defined by the stochastic differential equation

$$dX_t = (2\beta X_t + \delta_t)dt + v\sqrt{X_t} d\tilde{B}_t.$$

The m-th non-centered moment of $\mathcal{Z}$ is given by:

$$I\!E\left[\mathcal{Z}^m\right]=\sum_{k=0}^{n-1} I\!E\left[\exp\left(-m\int_0^{k+1} X_u du\right)\right]\ {}_{k|}q_x,$$

where ${}_{k|}q_x$ denotes the probability that the life assured dies between his $(x+k)$-th and his $(x+k+1)$-th birthday.

Remark that for an whole-life assurance, the benefit certainly will be paid once, namely at the end of the year of death. Consequently,

$$\mathcal{Z}=\exp\left(-\int_0^{K+1} X_u du\right) \qquad K=0,1,\cdots,\omega-x-1,$$

where ω is the least age so that $l_x=0$. The m-th non-centered moment is given by:

$$I\!E\left[\mathcal{Z}^m\right]=\sum_{k=0}^{\omega-x-1} I\!E\left[\exp\left(-m\int_0^{k+1} X_u du\right)\right]\ {}_{k|}q_x.$$

Under the endowment assurance contract, the benefit is payable at the end of the year of death if death occurs within n years of the issue date or, if the insured person survives n years, the benefit is payable at time n. Consequently, the present value $\mathcal{Z}$ of an endowment assurance is defined as:

$$\mathcal{Z}=\begin{cases}\exp\left(-\int_0^{K+1} X_u du\right) & K=0,1,\ldots,n-1\\ \exp\left(-\int_0^{n} X_u du\right) & K=n,n+1,\ldots\end{cases}$$

The m-th non-centered moment of the present value is given by:

$$I\!E\left[\mathcal{Z}^m\right]=\sum_{k=0}^{n-1} I\!E\left[\exp\left(-m\int_0^{k+1} X_u du\right)\right]\ {}_{k|}q_x+I\!E\left[\exp\left(-m\int_0^{n} X_u du\right)\right]\ {}_{n}p_x.$$

Approximations of the net single premium of each contract are easily calculated. Indeed, approximations of the expected value of $\mathcal{Z}$ are obtained by taking $m=1$ and by substituting the estimating bond price, proposed in the previous section.

We have evaluated this approximation in case of the Cox-Ingersoll-Ross single factor model, with the parameters estimated within Chan et

Table 2: Net single premiums: The exact values and the approximations.

n	life assurance			endowment assurance		
	Exact	Approx	Approx-Exact	Exact	Approx	Approx-Exact
1	.00154	.00155	.000008	.9313	.9363	.0049
10	.01453	.01484	.000314	.4785	.4944	.0158
20	.02896	.02985	.000887	.2354	.2453	.0098
40	.06222	.06479	.002572	.0894	.0935	.0041
60	.07635	.07979	.003439	.0767	.0801	.0034
80	.07664	.08010	.003459	.0766	.0801	.0034

al. [15] and with $r_0 = 0.07$. We used the mortality table HD(1968-72), which is commonly used in Belgium and which is based on Makeham's formula $l_x = k\, s^x\, g^{c^x}$ with for the ages between 0 and 69: $k = 1,000,268$, $s = 0.999147835528$, $g = 0.999731696667$ and $c = 1.115094352734$; and otherwise $k = 1,292,726$, $g = 0.995564574228$, $c = 1.077130677635$ and the same value of s.

In table 2, the exact values and the approximations are given for the net single premiums of n-year temporary life assurances and endowment contracts. Remark that for n larger than 60 year, both assurances become whole-life assurances since the life assured is aged $x = 30$ at the date of issue. We conclude that the approximations of the single net premiums are very good.

The variance and the skewness of $\mathcal{Z}$ also are easy to find since the variance is defined as

$$\text{var}\,[\mathcal{Z}] = \mathbb{E}\left[\mathcal{Z}^2\right] - \mathbb{E}[\mathcal{Z}]^2,$$

and the skewness is defined as

$$\begin{aligned}\text{sk}[\mathcal{Z}] &= \frac{\mathbb{E}\left[(\mathcal{Z} - \mathbb{E}[\mathcal{Z}])^3\right]}{\text{var}[\mathcal{Z}]^{3/2}} \\ &= \frac{\mathbb{E}\left[\mathcal{Z}^3\right] - 3\mathbb{E}\left[\mathcal{Z}^2\right]\,\mathbb{E}[\mathcal{Z}] + 2\mathbb{E}[\mathcal{Z}]^3}{\text{var}[\mathcal{Z}]^{3/2}}.\end{aligned}$$

Each of these terms can be calculated by substituting m =1, 2 or 3 in $\mathbb{E}\left[\mathcal{Z}^m\right]$ and by using the approximation of the m-th non-centered

Table 3: The variances: The exact values and the approximations.

n	life assurance			endowment assurance		
	Exact	Approx	Exact-Approx	Exact	Approx	Exact-Approx
1	.00153	.00147	.00006	.00949	.05587	.04638
10	.01182	.01071	.00111	.02844	.07711	.04867
20	.01763	.01587	.00175	.01849	.02994	.01148
40	.02021	.01796	.00225	.01567	.01833	.00266
60	.01902	.01658	.00243	.01653	.01897	.00244
80	.01898	.01654	.00244	.01654	.01898	.00244

moment of the discounting factor, namely

$$\mathbb{E}\left[\exp\left(-m\int_0^t X_u du\right)\right] \sim \exp\left(-m\int_0^t \mathbb{E}[X_u]du - \frac{m^2\bar{\delta}v}{16\beta^3}t\right).$$

In table 3 and 4, the variance and the skewness of $\mathcal{Z}$ is calculated, for $\mathcal{Z}$ being an n-year temporary life-assurance, an endowment assurance and an whole-life assurance (if n is very large). Again, we used the formula of Makeham and the Cox-Ingersoll-Ross model with the same parameters as above. The approximations of the variance seem to be excellent in both cases. Also the approximation of the skewness in case of the life assurance is very good. In case of the endowment, the approximation is not perfect but in our opinion, it is satisfactory.

With regard to the problems of setting contingency reserves and assessing the solvency of life assurance companies, it is interesting to study portfolio's of assurance policies. In his Ph. D. thesis [12], Parker considers a portfolio of c identical policies, one policy being issued to each of c independent lives of the same age. Denoting $\mathcal{Z}_i$ the present value of the benefit that is payable with respect to the i-th life assured and denoting ${}_c\mathcal{Z}$ the total present value of all the benefits to be paid within the portfolio, he obtains the relation:

$${}_c\mathcal{Z} = \sum_{i=1}^{c} \mathcal{Z}_i.$$

Table 4: The skewness: The exact values and the approximations.

n	life assurance			endowment assurance		
	Exact	Approx	Exact-Approx	Exact	Approx	Exact-Approx
1	25.245	24.866	0.379	-3.685	0.313	-3.998
10	7.540	7.421	0.119	-0.957	1.048	-2.005
20	5.019	5.002	0.017	0.443	2.046	-1.602
40	3.655	3.719	-0.064	3.668	3.961	-0.293
60	3.724	3.891	-0.166	3.733	3.904	-0.171
80	3.731	3.902	-0.170	3.731	3.902	-0.170

Denoting K_i the curtate-future-lifetime of the i-th assured, $t(k)$ the time of payment function and δ_u the force of interest, Parker studies the random variable ${}_c\mathcal{Z}$ under the following three assumptions:

1. $\{K_i\}_{i=1}^{c}$ are i.i.d.

2. $\{\mathcal{Z}_i\}_{i=1}^{c}$ are identical distributed

3. $\left\{\mathcal{Z}_i \mid \int_0^{t(k)} \delta_u du\right\}_{i=1}^{c}$ are i.i.d.

In the case that the force of interest rate follows an Ornstein-Uhlenbeck process, Parker calculates the expected value, the variance and the skewness of ${}_c\mathcal{Z}$ and discusses them (see chapter 5 of [12]).

By our approximation method, we can obtain the expected value, the variance and the skewness of the total present value of all benefits to be paid within the portfolio, namely ${}_c\mathcal{Z}$, in very general models of the short interest rate. It would be interesting to evaluate the approximations in case of the Cox-Ingersoll-Ross model.

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References

[1] P.H. Dybvig, J.E. Ingersoll and S.A. Ross, Do Interest Rates Converge?, *working paper*, (1986).

[2] P.H. Dybvig, J.E. Ingersoll and S.A. Ross, Long Forward Rates can Never Fall, *forthcoming.*

[3] J.C. Cox, J.E. Ingersoll and S.A. Ross, A theory of the term structure of interest rates, *Econometrica* 58 (1985), 385-407.

[4] M.J. Brennan and E.S. Schwartz, An equilibrium Model of Bond Pricing and a Test of Market Efficiency, *Journal of Financial and Quantitative Analysis* 17, no 3 (1982), 301-329.

[5] S.M. Schaefer and E.S. Schwartz, A two-factor model of the term structure: An approximate analytical solution, *Journal of Financial and Quantitative Analysis* 19 (1984), 413-424.

[6] F.A. Longstaff and E.S. Schwartz, Interest rate volatility and the term structure: a two-factor general equilibrium model, *The Journal of Finance* 47, nr 4 (1992), 1259-1282.

[7] G. Deelstra and F. Delbaen, Existence of Solutions of Stochastic Differential Equations related to the Bessel process, *forthcoming.*

[8] J. Pitman and M. Yor, A decomposition of Bessel Bridges, *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 59 (1982), 425-457.

[9] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion.* Springer-Verlag Berlin, Heidelberg, New York, 1991.

[10] G. Deelstra and F. Delbaen, Long-term returns in stochastic interest rate models, *forthcoming.*

[11] G. Deelstra and F. Delbaen, Long-term returns in stochastic interest rate models:Convergence in law, *forthcoming.*

[12] G. Parker, An application of Stochastic Interest Rates Models in Life Assurance, Ph.D. thesis, Heriot-Watt University, 1992.

[13] N.L. Bowers, H.U. Gerber, J.C. Hickman, D.A. Jones and C.J. Nesbitt, *Actuarial Mathematics*, Society of Actuaries, Schaumburg, 1986.

[14] G. Courtadon, The Pricing of Options on Default-Free Bonds, *Financial and Quantitative Analysis*, 17 (1982), 75-100.

[15] K.C. Chan, G.A. Karolyi, F.A. Longstaff and A.B. Sanders, An Empirical Comparison of Alternative Models of the Short-Term Interest Rate, *Journal of Finance*, 47 (1992), 1209-1227.

[16] M. Uhrig and U. Walter, Term Structure of Interest Rates, Consistent Risk Attitudes, and Preference-Free Option Prices, *Working paper*, 1993.

[17] F.A. Longstaff, A Nonlinear General Equilibrium Model of the Term Structure of Interest Rates, *Journal of Financial Economics*, 23 (1989), 195-224.

[18] G. Deelstra, Long-term returns in stochastic interest rate models, Ph.D. thesis, *in preparation.*