THE HYPOTHESES UNDERLYING THE PRICING OF OPTIONS

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Abstract :

This paper gives a critical investigation on the hypotheses underlying the pricing of options. Even in recent articles on this subject many authors overlook the gaps in the original proof of the Black-Scholes option pricing formula, so the main goal of this note is to clarify firstly under which assumptions the formula remains true, and secondly repeat a correct proof of the mentioned formula. Probably this is justified by the fact that up to date even many text-books on this subject reproduce the wrong argumentation using erroneously the Black-Scholes hedge portfolio : obviously they missed the fact, that contrary to the claim of Black-Scholes, the change in value of the hedge portfolio in a short time interval is not riskless, an observation which was - as far as I know - firstly done by Y. Z. Bergman in his PH.D. Thesis, Berkeley 1982 and later on independently by W. Böge in Heidelberg in 1993.

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Option pricing formulae, hedging and arbitrage, continuous time models

These are the sections of the paper:

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(ii) The Merton self-financing-condition and the equivalence theorem of Bergman

An alternative derivation of the formula using the concept of consistency of the pricing process

Introduction

The influence of the Black-Scholes pricing formula on the traded options market is immense and since options can be used as instruments of portfolio insurance (similar to the classical stop loss contracts), the practical importance of the probabilistic aspects of options for "Actuaries of the Third Kind" is evident. Our study of the modern theory of contingent claim valuation was originally motivated by the desire to better understand the formula of Black-Scholes for the pricing of options. In studying the existing literature on this subject we find many inconsistencies and I have got the impression that not only "many actuaries find themselves applying a formula which they don´t understand properly, and which they have never seen demonstrated" (compare [11], page 417) but also most of the writers of text-books or of articles on this subject are just in the same unsatisfactory situation. So this paper is partly intended as a tutorial and partly as a short survey on Bergman´s results. In the last section it is shown, that under a suitable weakened form of the No-Arbitrage-Condition, the so-called consistency of pricing, the Black-Scholes formula can be developed by elementary methods completely avoiding the machinery of the Itô-calculus.

The Black-Scholes model

(i) The errors of the model

We restrict ourself to the most simple case of the European call option giving its owner the right to buy a unit of the underlying security (normally in this paper : an ordinary share) at a fixed price (the exercise price or striking price) at a fixed date (the expiration date). Using the originally notation of Black and Scholes [3] let w(x,t) denote value of the option at time t, so that at the expiration date t* the following equation holds

(1) $$w(x,t^*) = \mathrm{Max}(0,x(t^*)-c)$$

here x(t) is the value of the underlying security (i.e. in our case the share under consideration) at time t and c is the exercise price. The usual assumptions in the Black-Scholes model are the following :

- the short-term interest rate r is known and is constant through time,
- there are no dividends payable before the expiry date,
- there are no transaction costs in buying or selling the stock or the option,
- it is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate,
- there are no penalties to short selling : A seller who does not own a security will simply accept the price of the security from a buyer, and will agree to settle with the buyer on some future date by paying him an amount equal to the price of the security on that date,
- the stock-price follows as a stochastic process a geometric Brownian motion, that means the following equation holds :

(2) $$x(t) = x(0)\cdot e^{\mu\cdot t+\sigma\cdot B(t)}$$

where B(t) denotes the standard Brownian motion and μ and σ are constant. In the terminology of stochastic analysis the stochastic process x(t) follows the stochastic differential equation :

(3) $$dx = \sigma\cdot x\cdot dB + (\mu + \frac{1}{2}\sigma^2)\cdot x\cdot dt \qquad .$$

As a fair price for such an European call-option Black and Scholes derived the formula:

(4) $$w(x,t) = x \cdot N(d_1) - c \cdot e^{-r(t^*-t)} \cdot N(d_2)$$

where $$d_1 = \frac{\ln(x/c) + (r + \frac{1}{2} \cdot \sigma^2) \cdot (t^* - t)}{\sigma \cdot \sqrt{t^* - t}} \quad ,$$

$$d_2 = \frac{\ln(x/c) + (r - \frac{1}{2} \cdot \sigma^2) \cdot (t^* - t)}{\sigma \cdot \sqrt{t^* - t}}$$

and N(.) is the cumulative normal density function. The original derivation of the mentioned formula proceeds as follows : Let w_1 denote $\partial w / \partial x$, $w_{11} := \partial^2 w / \partial^2 x$ and $w_2 = \partial w / \partial t$. The Black-Scholes hedge portfolio consists of maintaining one share of stock long and $1/w_1$ options short, the value of this position is :

(5) $$x - \frac{w}{w_1} \quad = \quad x(t) - \frac{w(x,t)}{w_1\,(x,t)} \; .$$

They conclude that for an arbitrage free market the change in the value of the hedge portfolio must coincide with the riskless return given by the interest rate r, hence:

(6) $$(x - \frac{w}{w_1})\, r\, dt \quad = \quad d(x - \frac{w}{w_1}) \qquad .$$

Using Itô's lemma the right hand side of this stochastic differential equation was calculated to

$$(7) \qquad dx - \frac{dw}{w_1} \;=\ldots=\; -\left(\frac{1}{2} w_{11}\,\sigma^2 x^2 + w_2\right) dt\,\frac{1}{w_1}\,.$$

Together with equation (6) one obtains the partial differential equation

$$(8) \qquad w_2 \;= r\,w - r\,x\,w_1 - \frac{1}{2}\sigma^2 x^2\, w_{11}$$

with the boundary condition w(x,t*) = Max ((x-c),0), this gives the solution (4). The errors in this derivation of formula (4) are the following :
In equations (6) and (7) the proportion $1/_{w_1}$ of options short is considered as a constant - otherwise one should have calculated the differential $d(x - \frac{w}{w_1})$ in a different manner (see below) - , on the other hand w_1 is of course not constant as the partial derivative of the function w(x,t). The correct application of Itô´s lemma leads to

$$d(x - \frac{w}{w_1}) = dx - (\frac{w_1^{\,2} - w_{11}\,w}{w_1^{\,2}}\,dx + \frac{w_1\,w_2 - w_{12}\,w}{w_1^{\,2}}\,dt + \frac{1}{2}(\frac{2w_{11}^{\,2}ww_1 - w_1^{\,3}w_{11} - ww_1^{\,2}w_{111}}{w_1^{\,4}})\sigma^2 x^2 dt)$$

a term in which the stochastic term does not disappear. Fortunately this is not the only mistake in the derivation of formula (4). The conclusion,that from the absence of arbitrage opportunities it follows, that the rate of change in value of a riskless portfolio equals the riskless interest rate, is also not true, if one does not assume the extra condition of self-financing for the portfolio under consideration. Following Y. Bergman [2] this easily can be seen by the study of the following instructive example of a hedge portfolio :

Manage a portfolio according to the following rules : Take $\frac{e^{At}}{2x}$ shares of stocks and $\frac{e^{At}}{2w(x,t)}$ call options on the stock, this two proportions represents clearly a riskless portfolio and the total value of this hedge portfolio amounts to e^{At} . Since one can choose the constant A as large as one wants, the return certainly not coincides with the riskless external interest rate. The main point is that the above portfolio is not self-financing, i. e. it does require external net inflow of funds. But as Bergman observed [2], page 7 :

Proposition: The Black and Scholes hedge portfolio is not self-financing.

Curiously enough even in recent articles on this subject these two errors are reproduced by many authors when they use the Black-Scholes hedge portfolio.

One remark at the end of this section : Following Cox, Ross and Rubinstein many textbooks develop the option pricing formula using discrete binomial random walks and a limit argument where the time interval tends to zero cf. [4]. But this is too simplistic, since there is no reason to restrict our attention to a locally binary scenario. The situation becomes less pleasant as soon as one admits locally a third or even more possibilities for the value of the underlying security at the branching point: In that case a perfect hedge of a contingent claim is no longer possible, since three (or even more) linear equations generally can not be fullfilled by two unknowns. In more advanced terminology this can also be phrased as follows : A discrete random walk with more than two possibilities for the outcome at the branching points generally does not admit an equivalent martingal measure.

(ii) The Merton self financing condition and the equivalence theorem of Bergman

It´s largely owing to Robert C. Merton, that the self financing condition for portfolio strategies has been understood as one of the most important essentials for the derivation of option pricing formulas. Let us describe this condition in a more general context : A general portfolio strategy contains n "risky" assets, called stocks, and one asset, called the bond. The bond-price $P_0(t)$ evolves according to the differential equation

$$dP_0(t) \quad = \quad r\, P_0(t)\, dt \ ,$$

while the stock prices are given by the stochastic processes $P_i(t)$ for i=1,..,n, hence the total value of the portfolio at time t is given by

(9) $\quad V = \sum_{i=0}^{n} N_i(t)\, P_i(t)$, where $N_i(t)$ denotes the number of shares of asset i at time t.

Under suitable regularity conditions the change in value of the portfolio is calculated by Itô's lemma to :

(10) $$dV = \sum_{i=0}^{n} N_i(dP_i) + \sum_{i=0}^{n} (dN_i)(dP_i) + \sum_{i=0}^{n} (dN_i)P_i \ .$$

The first term of the right hand side of equation (10) can be considered as the contribution to change in value arising from changes in prices only, while the third term is a contribution arising from changes in the number of shares . By a *delicate*[1] limit argument considering the discrete analogon Merton identifies the middle term as belonging together with the third (cf. [9] page 124-127), so by definition the condition for a self-financing portfolio is :

(11) $$\sum_{i=0}^{n} (dN_i)(dP_i) + \sum_{i=0}^{n} (dN_i)P_i \ = 0 \ .$$

Coming back to the special case n=1 of a portfolio strategy consisting of two assets : the risky asset with stock price x(t) following a geometric Brownian motion (cf. (2)) and a riskless bond with price e^{rt} per unit with constant interest rate r, then the following beautiful observation is due to Bergman :

Equivalence theorem of Bergman: V(x,t) is the value of a self-financing portfolio strategy of stock and bonds if and only if V(x,t) satisfies Black-Scholes partial differntial equation (8), in which case the unique self-financing

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[1] : Crucial points concerning the limit process are for instance: Which convergence is considered, only the convergence of subsequences? Are the stochastic processes N.(t) previsible? What about right or left continuity of the involved stochastic processes?

strategy is maintaining $V_1 = \partial V / \partial x$ shares of stock and $e^{-rt}(V - (\partial V / \partial S)\, S)$ bonds.

For the proof firstly one calculates the change in the value of the portfolio according to Itô, secondly uses Merton´s self-financing condition and gets the desired partial differential equation (for the details compare [2]).
It is worth to mention, that the above mentioned equivalence between Merton´s condition and the Blach-Scholes partial differential equation uses not the absence of arbitrage opportunities and does not use the existence of options.
The calculation of option prices proceeds now as follows:
The idea is to construct a portfolio consisting of stock and bonds duplicating exactly the cash flow of say a call option. Since the option is self-financing so is the duplicating portfolio, hence its value V(x,t) satisfies the Black-Scholes partial differential equation. One gets (cf. [2] page 30):

Corollary: If arbitrage profits generating strategies in the stock, the riskless bond and the option are excluded, then the price w of the option is a function of the stock price and time alone : w = w(x,t) and it satisfies the partial differential equation (8) with the boundary condition w (x(t*),t*) = Max (0,x(t*)-c).

Bergman commented, that "the Black-Scholes derivation is an example of two wrongs which do make a (most important) right".

An alternative derivation of the formula using the concept of the consistency of the pricing process

With a slightly modified condition of "No-arbitrage" it is possible to derive the Black-Scholes option price formula by complete elementary methods avoiding Itô´s calculus. The assumption is the following:

Assumption C: The security is priced consistently (cf. [11]) in the following sense: the current price x(t) is the discounted expectation of the future price i.e.

(12) $$E\left[x(t^*)\middle|x(t)\right] = x(t)\, e^{r\,(t^*-t)} \qquad \text{for arbitrary } t < t^*.$$

So on the level of expectation values no arbitrage gains are possible.
In our special case of a geometric Brownian motion one has for t< t*

(13) $$E\left[x(t^*)\middle|x(t)\right] = E\left[x(t)e^{\mu\,(t^*-t)+\sigma(B(t^*)-B(t))}\middle|x(t)\right] = x(t)\, e^{\mu\,(t^*-t)}\, e^{\frac{1}{2}\sigma^2(t^*-t)},$$

so combining (12) and (13) one gets from assumption C the following relation between the external interest rate r, drift μ and volatility σ :

(14) $$r = \mu + \frac{1}{2}\sigma^2.$$

If one calculates the option price in a classical manner (cf. L. Bachelier [1]) as the conditional expected value

(15) $$w(x,t) = E\left[e^{-r(t^*-t)}(x(t^*)-c)_+ \middle|x(t)\right],$$

one gets the result (compare also [10]) :

(16) $$w(x,t) = e^{-r(t^*-t)}\left(x(t)\, e^{(\mu+\frac{\sigma^2}{2})(t^*-t)} N(\bar{d}_1) - cN(\bar{d}_2)\right)$$

where $\bar{d}_1 = (\ln(x/c) + (\mu+\sigma^2)(t^*-t)) / \sigma\sqrt{t^*-t}$ and

$$\bar{d}_2 = \frac{\ln(x/c) + \mu\,(t^*-t)}{\sigma\sqrt{t^*-t}}$$

(cf. [6] section 3 and [10]). Hence the formula (16) gives the same result as Black and Scholes if one assumes C, i.e. if the relation (14) holds. This is not surprising, since it is known that the Black-Scholes option price formula can be obtained as a conditional expected value after elimination of the drift term i.e. after changing to an equivalent martingal measure ([7], page 378).

The described elementary derivation of the formula has another aspect which should be finally remarked: The popularity of the Black-Scholes formula stems from the fact, that in (4) w(x,t) explicitely does not depend on the drift parameter μ , moreover in many publications this is seen as the main advantage compared with other option price formulas. The independence of the drift term of the underlying security from the external interest rate often is emphasized. But let us look to formula (4) in the limit case of arbitrary small volatilities $\sigma \to 0$:
If $\ln (x/c) + r (t^*-t) > 0$ one has on the one hand the limit

$$(17) \qquad w(x,t) \quad \to \qquad x\,N(\infty) - c\,e^{-r(t^*-t)} N(\infty) = x - c\,e^{-r(t^*-t)} ,$$

on the other hand the true value of the option at time t in case of volatility 0 is certainly the discounted value of the call option at time t*, i.e.

$$(18) \qquad e^{-r(t^*-t)}\,(x(t^*) - c)_+ = e^{-r(t^*-t)}\; Max((x(t^*) - c, 0) .$$

Because of $x(t) = e^{-\mu(t^*-t)}\, x(t^*)$ - according to equation (2) - equality i.e. consistence of formula (4) with (18) holds only in case $r = \mu$, i.e. if equation (14) holds in this case. It seems to us that this fact is not noticed in [4] when J. Cox and M. Rubinstein stated in loc. cit. page 216 : "All these implications of the formula are fully consistent with the general arbitrage relationships developed in Chapter 4. Of course, if they were not, our exact formula would be in error."

All this is not a surprising fact since the consistence condition is a martingal condition.

On the one hand it seems evident for me,that the external interest rate and the drift behaviour of shares of stocks are not independent, but on the other hand one certainly can not hope that the complex economic reality obeys such a simple relation as equation (14). In this connection compare also the empirical martingal tests of several option price models due to F. A. Longstaff in [8], where empirical evidence for a rejection of the Black-Scholes model is given. From this point of view one can agree with H. Föllmer when he stated in [5] : "In any case,there seems to be a need for a thorough look at the probabilistic structure of basic price fluctuations. This is a challenging program in itself. But it would also lesd to a reconsideration of hedging strategies. Furthermore, it would be a crucial step towards a more rigorous analysis of the impact of such strategies on the underlying price process."

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