

STOCHASTIC MODEL WITH POSSIBILITY OF RUIN AND DIVIDEND REPARTITION FOR INSURANCE AND BANK

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Abstract

This paper is related to the possible applications of a stochastic model of ALM for insurance (Ars and Janssen (1994)) to real life situations.

To begin with, we present a model that generalizes the one presented by Janssen (1993) and takes into account discontinuities (that is, catastrophes) for liabilities. A transformation of variables allows us to use the results of Dufresne (1989) and leads to important consequences for company management (ALM). We also consider the possibility for the company to avoid the bankruptcy by subscribing to a loan and evaluate the implications on its future financial wealth. Then, we investigate dividend repartition and important consequences for company management are presented like, for instance, the prevision of its financial position.

We illustrate our results by treating numerical examples with data coming from the balance sheet of a big belgian insurance company.

INTRODUCTION

In an earlier paper (Ars and Janssen(1994)), we developed some applications of a stochastic model of Asset Liability Management (ALM) for insurance to real life situations. This model was an extension of older ones studied by Cummins (1988 and 1990). The basic idea was to study the relations between the assets process (A) and the liabilities process (B) in order to point out some management principles. More exactly, we first pointed out the factors that make the ruin probability of the company increase, then determined the objectives to be achieved by the company and finally developed some tools needed to encounter these objectives.

The model presented in this paper extends the originally one presented by Janssen (1993) so as to take into account discontinuities (we mean *catastrophes*, that is, particularly important (set of) claims) for liabilities. We suppose that the assets and liabilities processes are governed by stochastic differential equations (of the Doléans-Dade type) with respect to processes with stationary independent increments. We assume the continuity of the assets process and that the trajectories of the liabilities process have a finite number of jumps on each finite interval. A change of variable allows us to use the results of Dufresne (1989) and so to determine the ultimate ruin probability.

The first goal (for ALM) of this paper is to look for objectives to be followed by the company in order to maximize the insurance company lifetime, which is a question of great interest for both the shareholders and policy-holders. The tools needed to achieve these objectives are based on the use of reinsurance options, reinsurance contracts, traditional financial products (such as options, swaps,...).

We then consider the possibility for the company to avoid the bankruptcy by subscribing to a loan and evaluate the implications on its future financial wealth. After, we investigate dividend repartition and important consequences for company management are presented such as, for instance, the modification of its financial position.

We illustrate our results by treating numerical examples with data coming from the balance sheet of a big belgian insurance company.

1. THE GENERALIZED JANSSEN'S MODEL AND LIFETIME OF THE COMPANY

1.1. PRESENTATION OF THE MODEL

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq \infty}, P)$ be a filtered complete probability space satisfying the *conditions habituelles* :

- 1) \mathcal{F}_0 contains all the P-null sets of \mathcal{F} ,
- 2) $\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s, \forall t, 0 \leq t \leq \infty$, that is, the filtration is right continuous.

Let us suppose that $\mathcal{F}_\infty = \bigvee_{0 \leq t < \infty} \mathcal{F}_t$.

Let $A = (A_t, t \geq 0)$ and $B = (B_t, t \geq 0)$ denote the assets and liabilities processes (defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq \infty}, P)$) like all processes considered in this paper). So A_t represents the amount of all that belongs to the company at time t and B_t represents the expected (total) amount of all the claims (evaluated at time t).

We make the following assumptions about the processes A and B :

- i) These processes are governed by the following integral equations :

$$A_t = A_0 + \int_0^t A_{s-} dX_s, \quad (1)$$

$$B_t = B_0 + \int_0^t B_{s-} dY_s, \quad (2)$$

where $Z = (X, Y) = (X_t, Y_t, t \geq 0)$ is a vector of semimartingales nul at $t = 0$. (We refer to Protter (1992) for the general theory of semimartingales and stochastic differential equations).

ii) The process Z is a bidimensional process with stationary independent increments with a finite number of jumps on each finite interval. We suppose further that X is a continuous process, i.e. its trajectories are a.s. continuous (this implies that A is also a continuous process). So discontinuities appear only for liabilities and have to be interpreted as the result of *catastrophes* (that is : particularly important claims), of which number may reasonably be supposed finite on each finite interval. Now, it is well-known that the processes X and Y , being stochastically continuous with stationary independent increments, admits the following representation (see Gihman & Skorohod (1969) or Paulsen (1993)):

$$X_t = \mu_A t + \sigma_A W_t^1, \quad (3)$$

$$Y_t = \mu_B t + \sigma_B W_t^2 + S_t, \quad (4)$$

where

μ_A and μ_B are real constants;

σ_A and σ_B are strictly positive real constants;

$W = (W_t^1, W_t^2, t \geq 0)$ is a bidimensional Brownian motion with variance-covariance matrix Q :

$$Q = \begin{pmatrix} 1 & \varphi \\ \varphi & 1 \end{pmatrix}, \quad |\varphi| \leq 1;$$

and $S = (S_t, t \geq 0)$ is a 1-dimensional compound Poisson process independent of W , that can be represented in the following way :

$$S_t = \sum_{i=1}^{N_t} Y_i$$

with $N = (N_t, t \geq 0)$ being a Poisson process of parameter λ and $(Y_i, i \geq 1)$ being i.i.d. random variables.

Such processes (with applications in A.L.M. and ruin theory) have been studied by Janssen (1993) and Ars & Janssen (1994). The model presented here is a generalization which take into account the possibility of *catastrophes*. It is for this reason that we call this model " Generalized Janssen's model". Similar processes were considered by Cummins (1988 et 1990) but with other applications.

1.2. THE ASSETS AND LIABILITIES PROCESSES AND THE MODIFIED RISK RESERVE PROCESS

The processes A and B are solutions of the stochastic integral (or differential) equations (1) and (2). These solutions are easily derived from the Doléans-Dade Stochastic Exponential formula (see Protter(1992, p77)). Using (3) and (4), we obtain the following expressions :

$$A_t = A_0 \exp\left[\left(\mu_A - \sigma_A^2/2\right)t + \sigma_A W_t^1\right] ; \quad (5)$$

$$B_t = B_0 \exp\left[\left(\mu_B - \sigma_B^2/2\right)t + \sigma_B W_t^2\right] \prod_{1 \leq i \leq N_t} (1 + Y_i); \quad (6)$$

We now considered the process of greatest interest for our concern : let $R = (R_t, t \geq 0)$ the process defined by :

$$R_t = \ln(A_t/B_t), \quad t \geq 0. \quad (7)$$

One can see the process R as a traditional surplus in the sense that the ruin occurs when A becomes less than B , thus when R becomes negatif. So, let us define the time of ruin T by :

$$T = \inf (t \geq 0 : R_t \leq 0).$$

By the use of Itô's Formula (see for instance Protter (1992, p. 74)), one can prove that the process R admits the following representation :

$$R_t = R_0 + \mu t + \sigma \bar{W}_t - \bar{S}_t, \quad t \geq 0, \quad (8)$$

where $\sigma^2 = \sigma_A^2 + \sigma_B^2 - 2 \rho \sigma_A \sigma_B$,

$$\mu = \mu_A - \mu_B - \frac{1}{2} (\sigma_A^2 - \sigma_B^2),$$

\bar{W} is a standard brownian motion;

\bar{S} is a compound Poisson process :

$$\bar{S}_t = \sum_{i=1}^{N_t} \bar{Y}_i, \quad \bar{Y}_i = \ln(1 + Y_i), \quad 1 \leq i \leq N_t.$$

Remark 1 : in the remainder of this paper, we will always suppose that $\mu > \lambda p_1 > 0$, where p_1 denotes the expectation of \bar{Y}_1 .

1.3. THE ULTIMATE RUIN PROBABILITY AND THE SEVERITY OF THE RUIN

The ruin model described by (8) is exactly the model studied by Dufresne in his thesis (1989). So by using Dufresne's results (see Dufresne (1989 p.146)), one can find the exact expression of the ruin probability when the distribution of \bar{Y} is a combination of exponential distributions, that is, its distribution function is of the form (with n being an integer different of 0) :

$$P[\bar{Y}_i \leq z] = 1 - \sum_{1 \leq i \leq n} a_i \exp(-\beta_i z), \quad (9)$$

where $\sum_{1 \leq i \leq n} a_i = 1$ and
 $a_i \geq 0, 1 \leq i \leq n.$

Let us note that in the last expression, the positivity of a_i ($1 \leq i \leq n$) is not absolutely necessary (see Dufresne (1989) for details) but if for (at least) one i ($1 \leq i \leq n$), a_i is negative then our conclusions for ALM will not be valid anymore (see Dufresne (1989, p.147) and below). The Dufresne's formula for the ultimate ruin probability is then:

$$\psi(R_0) = \sum_{1 \leq k \leq n+1} E_k \exp(-r_k R_0), \quad (10)$$

where :

$$E_k = \frac{1}{r_k} (\mu - \lambda p_1) \left(\frac{\sigma^2}{2} + \lambda \sum_{i=1}^n \frac{a_i}{(\beta_i - r_k)^2} \right)^{-1}, 1 \leq k \leq n+1; \quad (11)$$

$$p_1 = E[\bar{Y}_1];$$

r_1, \dots, r_{n+1} are the roots of :

$$\frac{\sigma^2}{2} r - \mu + \lambda \sum_{i=1}^n \frac{a_i}{\beta_i - r} = 0. \quad (12)$$

Other useful formulas are the expressions of the probabilities of ruin due to the diffusion ($\psi_d(R_0)$) and due to a catastrophe ($\psi_c(R_0)$):

$$\psi_d(R_0) = \sum_{1 \leq k \leq n+1} E'_k \exp(-r_k R_0), \quad (10')$$

where :

$$E'_k = \frac{\sigma^2}{2} \left(\frac{\sigma^2}{2} + \lambda \sum_{i=1}^n \frac{a_i}{(\beta_i - r_k)^2} \right)^{-1}, \quad 1 \leq k \leq n+1; \quad (11')$$

$$\text{and } \psi_c(R_0) = \sum_{1 \leq k \leq n+1} E''_k \exp(-r_k R_0), \quad (10'')$$

where :

$$E''_k = \frac{1}{r_k} \left(\mu - \lambda p_1 - \frac{\sigma^2}{2} r_k \right) \left(\frac{\sigma^2}{2} + \lambda \sum_{i=1}^n \frac{a_i}{(\beta_i - r_k)^2} \right)^{-1}, \quad 1 \leq k \leq n+1; \quad (11'')$$

We obviously have that :

$$\psi(R_0) = \psi_d(R_0) + \psi_c(R_0). \quad (13)$$

Dufresne's work gives us also the expression of the ruin severity. Otherwise stated, if, for y strictly positive, $\psi(R_0, y)$ denotes the probability that ruin occurs and that the value of the process R at the time of ruin is less than or equal to $(-y)$, we have (see Dufresne (1989, p.155)) :

$$\psi(R_0, y) = \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n+1} C_{jk} \exp(-r_k R_0 - \beta_j y) \quad (14)$$

where :

$$C_{jk} = \left(\frac{a_j}{\beta_j (\beta_j - r_k)} \right) \left(\frac{\sigma^2}{2\lambda} + \sum_{i=1}^n \frac{a_i}{(\beta_i - r_k)^2} \right)^{-1}, \quad 1 \leq k \leq n+1, 1 \leq j \leq n. \quad (15)$$

Example :

Let us suppose that \bar{Y} is exponentially distributed (this situation is called in what follows *exponential case*) with parameter $\beta = 5$ (this means that $n = 1$), that the parameter λ of the Poisson Process is equal to 0.2 and that the values of R_0 , μ and σ are respectively 0.1887, 0.0603 and 0.0186.

Then we obtain from relations (11) to (14), where $a=R_0$:

$$p_1 = E[\bar{Y}] = \frac{1}{\beta} = 0.2,$$

$$r_1 = 1.6620,$$

$$r_2 = 351.9337.$$

$$\psi_c(a) = 0.6602 e^{-1.662 a} - 0.6602 e^{-351.9337 a} = 0.4825,$$

$$\psi_d(a) = 0.0095 e^{-1.662 a} + 0.9905 e^{-351.9337 a} = 0.0069,$$

$$\psi(a) = \psi_s(a) + \psi_d(a) = 0.4825 + 0.0069 = 0.4894.$$

1.4. AN ASYMPTOTIC FORMULA

Unfortunately, if \bar{Y} is not distributed like a combination of exponential distributions, we do not dispose of an analytic solution. But, there is an

asymptotic formula generalizing (in some sense) Cramer's one that we are going now to present here. This formula is developed in Dufresne (1989, p.136).

Let us suppose that the distribution of \bar{Y} provides an adjustment coefficient r^* (see Gerber (1979)), that is, a real number r^* such that :

$$\{U_t, t \geq 0\} \equiv \{e^{-r^* R_t}, t \geq 0\}$$

is a martingale with respect to the filtration $(F_t)_{0 \leq t \leq \infty}$. One can prove that r^* is the strictly positive root of :

$$\frac{\sigma^2}{2} r^2 - \mu r - \lambda + \lambda M(r) = 0, \tag{16}$$

where M denotes the moment generating function of \bar{Y} .

Let us suppose that we make R_0 tend to infinity, we have :

$$\psi(R_0) \rightarrow \exp(-r^* R_0) \frac{1}{r^*} \frac{\mu - \lambda p_1}{\sigma^2/2 + \lambda \int_0^\infty y \exp(r^* y) (1 - P(y)) dy} \tag{17}$$

Let us remark that in the exponential case, the adjustment coefficient is the smallest root of (12) (as it is easily seen from (10)), and therefore, if this smallest root is, say, r_1 :

$$\psi(R_0) \rightarrow E_1 \exp(-r_1 R_0) \tag{18}$$

Example (continued):

Let us now come back to our example. The exact ruin probability $\psi(R_0)$ is:

$$\psi(R_0) = \psi_s(R_0) + \psi_d(R_0) = 0.4825 + 0.0069 = 0.4894.$$

The approximate ruin probability computed from (18) gives :

$$\begin{aligned} \psi(R_0) &\cong E_1 \exp(-r_1 R_0) = 0.6697 \exp(-1.662 \times 0.3136194) \\ &= 0.4894. \end{aligned}$$

So the approximation is very good.

1.5. APPLICATION TO ALM

Formula (17) gives us a interesting tool for the asset liability management of the company. Indeed we can see that the more r^* the less $\psi(R_0)$. So the goal of the company must be to increase r^* . Let us illustrate this by considering for instance the exponential case. The adjustment coefficient is then the smallest of the two roots of (12) and has the following expression :

$$r^* = \frac{\mu + \frac{\sigma^2}{2}\beta - \sqrt{\left(\mu - \frac{\sigma^2}{2}\beta\right)^2 + 2\lambda\sigma^2}}{\sigma^2} \quad (19)$$

One easily verifies (do not forget that $\lambda/\beta \leq \mu$ (see remark 1)) that $r^*(\mu, \sigma^2, \beta, \lambda)$ is an increasing function of μ , a decreasing function of λ and σ^2 .

If $(\mu - \sigma^2 \beta/2) \geq 0$, $r^*(\mu, \sigma^2, \beta, \lambda)$ is also a decreasing function of β and therefore an increasing function of p_1 . So the management goals for the company are :

- i) to increase μ ;
- ii) to decrease σ^2 , p_1 and λ ;
- iii) to be sure that $(\mu - \sigma^2 \beta/2) \geq 0$.

The connection with the initial parameters is made via the following formulas :

$$\sigma^2 = \sigma_A^2 + \sigma_B^2 - 2 \varphi \sigma_A \sigma_B,$$

$$\mu = \mu_A - \mu_B - \frac{1}{2} (\sigma_A^2 - \sigma_B^2).$$

If we are not in the exponential case, then obviously the first and second (very intuitive) objectives are always valid but others conditions similar to the third one may appear.

In an earlier paper (Ars & Janssen (1994)), we presented the management tools which allow to achieve the fixed objectives. Those are based on :

i) Reinsurance contracts : excess of loss contracts provide a diminution of p_1 but also a diminution of μ (or more exactly μ_A) ; stop-loss contracts imply a diminution of the ruin probability but the counterpart is a diminution of μ .

ii) Modification of assets and liabilities structures.

iii) Hedging (or speculation) via the purchase or the selling of traditional (put or call) options, swaps,...

iv) Hedging (or speculation) via the purchase or the selling of (put or call) reinsurance options.

The last (and most recent) possibility constitutes (from far) the most interesting innovation in insurance.

2. THE POSSIBILITY FOR THE COMPANY TO AVOID THE BANKRUPTCY BY SUBSCRIBING TO A LOAN

2.1. SUBSCRIPTION AND REPAYMENT OF THE LOAN

Now, if ruin occurs at time T and $R_T < 0$ then it is obvious that the ruin is caused by a catastrophe and the unlucky company (but not necessarily mismanaged) can perhaps avoid the ruin by subscribing to a loan allowing it to start again. In order to investigate this possibility we have to answer the three following questions :

i) How much has the company to borrow?

ii) How can the company pay off the loan?

iii) When can one expect to save the company? (Or what is the maximal severity of the ruin which allows to subscribe for a loan?)

We begin by answering the first question : the loan has to be sufficiently important in order to insure the company to start again from a good position. For this reason, we shall suppose in what follows that, immediately after the time of ruin T and the subscription for the loan, the reserve equals γ , $\gamma > 0$. So the amount of the loan, denoted by L in the remainder of this paper, is :

$$L = B_T - A_T + \gamma B_T = (1 + \gamma) B_T - A_T. \quad (20)$$

We are now going to answer the second question : let us suppose that the company pays off the loan by continuously paying the proportion θ of its assets : this means that the stochastic integral (or differential) equation followed by A , denoted after the ruin by \bar{A} , becomes :

$$\bar{A}_t = A_T + L + \int_T^t \bar{A}_s \left((\mu_A - \theta) ds + \sigma_A dW_s^1 \right), \quad t > T. \quad (21)$$

Now, the problem which naturally arises is the determination of θ . In order to solve it, we shall suppose for sake of simplification that the company will pay off on an infinite interval (in fact, the repayment only exists till the following time of ruin T') and that the implicated uncertainty is taken into account in the choice of the interest rate δ , that is, the factor θ is the solution of the following equation :

$$\mathbb{E} \left[\int_T^\infty e^{-\delta t} (\theta \bar{A}_t) dt \right] = L, \quad (22)$$

where \mathbb{E} denotes as usually expectation with respect to P .

Now, if $\delta > \mu_A - \theta$, it follows from classical results (see for instance Jazwinski (1970, p.68 and 70)) that :

$$\begin{aligned} \mathbb{E} \left[\int_T^\infty e^{-\delta t} (\theta \bar{A}_t) dt \right] &= \theta \int_T^\infty e^{-\delta t} \mathbb{E}[\bar{A}_t] dt \\ &= \theta \int_T^\infty e^{-\delta t} (A_T + L) e^{(\mu_A - \theta)t} dt \quad (23) \\ &= \frac{\theta (A_T + L)}{\delta - (\mu_A - \theta)}. \end{aligned}$$

One can compute the value of θ from (20), (22) and (23) :

$$\theta = \frac{(\delta - \mu_A) L}{A_T} = (\delta - \mu_A) \left((1 + \gamma) \exp(-R_T) - 1 \right). \quad (24)$$

Relation (24) gives us the answer to the third question : if we accept the intuitively logical principle that θ has to be less than μ_A then the smallest acceptable value of R_T is the unique solution (in x) of the following equation (if this solution is positive; if not, any ruin is fatal) :

$$\mu_A = (\delta - \mu_A) \left((1 + \gamma) \exp(-x) - 1 \right). \quad (25)$$

In the remainder of this paper, we shall denote this solution by y_{\min} . We obtain:

$$y_{\min} = \ln \left[\frac{(1 + \gamma) (\delta - \mu_A)}{\delta} \right] \vee 0. \quad (26)$$

2.2. THE DIMINUTION OF THE RUIN PROBABILITY

If the ruin occurs (say at time T) and the restarting takes place by the subscription to a loan, then, among the factor involved only μ and μ_A are modified. We then obtain a ultimate ruin probability which is function not only of R_0 but also of μ and μ_A . Therefore, we shall note in this section the ultimate (definitive) ruin probability by $\psi^*(R_0, \mu, \mu_A)$. Now, if y denotes the ruin severity (that is : $y = -R_T$) we can express (from (20)) the amount L of the loan in function of B_T :

$$L(y) = (e^y - e^{-y}) B_T. \quad (27)$$

The value of θ then follows from (24) :

$$\theta(y) = \frac{(\delta - \mu_A) L(y)}{A_T} = (\delta - \mu_A) (\exp(y + \gamma) - 1). \tag{28}$$

It results now from the formula of complete probability that:

$$\begin{aligned} \psi^*(R_0, \mu, \mu_A) &= \psi_d(R_0, \mu) \psi^*(\gamma, \mu - \theta(0), \mu_A - \theta(0)) + \\ &+ \int_0^{-y_{\min}} \psi^*(\gamma, \mu - \theta(y), \mu_A - \theta(y)) d_y [-\psi(R_0, \mu, y)] \tag{29} \\ &+ \psi(R_0, \mu, y_{\min}), \end{aligned}$$

where $\psi_d(R_0, \mu)$ and $\psi(R_0, \mu, y_{\min})$ are given by (10') and (14) (we insist on the dependence on μ because this factor may change).

Let Φ be the of the applications from $[0, \infty) \times [0, \mu] \times [\mu_A - \mu, \mu_A]$ into $[0, 1]$, not decreasing in each of its variables. Note that ψ^* belongs to Φ .

Now consider the operator U from Φ into Φ defined by :

$$\begin{aligned} U[f](u, v, w) &= \psi_d(u, v) f(\gamma, v - \theta(0), w - \theta(0)) + \\ &+ \int_0^{y_{\min}} f(\gamma, v - \theta(y), w - \theta(y)) d_y [-\psi(u, v, y)] \tag{30} \\ &+ \psi(u, v, y_{\min}), \end{aligned}$$

where f belongs to Φ and $(u, v, w) \in [0, \infty) \times [0, \mu] \times [\mu_A - \mu, \mu_A]$.

It is obvious that U satisfies the two following properties :

(i) U is increasing : let f, f' and f'' belong to Φ , then :

$$f \leq f' \leq f'' \Rightarrow U(f) \leq U(f') \leq U(f'');$$

$$(ii) \quad U(\psi^*) = \psi^*.$$

It immediately results that :

$$0 \leq U(0) \leq \psi^* \leq U(1) = \psi \leq 1, \text{ and for every integer greater than } 1$$

$$0 \leq U^n(0) \leq \psi^* \leq U^n(1) = U^{n-1}(\psi).$$

Particularly, we have :

$$\psi^* \leq U(\psi) \leq \psi. \quad (31)$$

So $U(\psi)$ represents a first step in the evaluation of the gain in lifetime. Further steps are more complicated to compute (because it needs numerical computation of an integral at each point).

Example (continued).

Our goal here is to show the gain (at the level of the probability of ruin) obtained by the described method (see (31)).

The exact ruin probability is 0.4894.

But the (numerical) computation of $U(\psi)$ (by (30)) gives $0.3854 < 0.4894$.

3. THE IMPACT OF DIVIDEND DISTRIBUTION

3.1. MODELLISATION OF THE DIVIDEND DISTRIBUTION

We suppose, in the second part of this paper, that both the processes X and Y are continuous. It implies that the (continuous) processes A and B are governed by the following system of stochastic integral equations :

$$A_t = A_0 + \int_0^t A_s (\mu_A ds + \sigma_A dW_s^1), \quad t \geq 0; \quad (32)$$

$$B_t = B_0 + \int_0^t B_s (\mu_B ds + \sigma_B dW_s^2), \quad t \geq 0. \quad (33)$$

We shall suppose that the impact of the dividend distribution on the lifetime of the company can be described by considering a reflecting barrier at the level $b > R_0$ (see for instance Cox & Miller (1965), p.223) for the process R . So, the trajectories of the process R are of the form presented in fig. 1 where $a = R_0$. Let us recall that there is also an absorbing barrier at 0 (see Ars and Janssen (1994)).

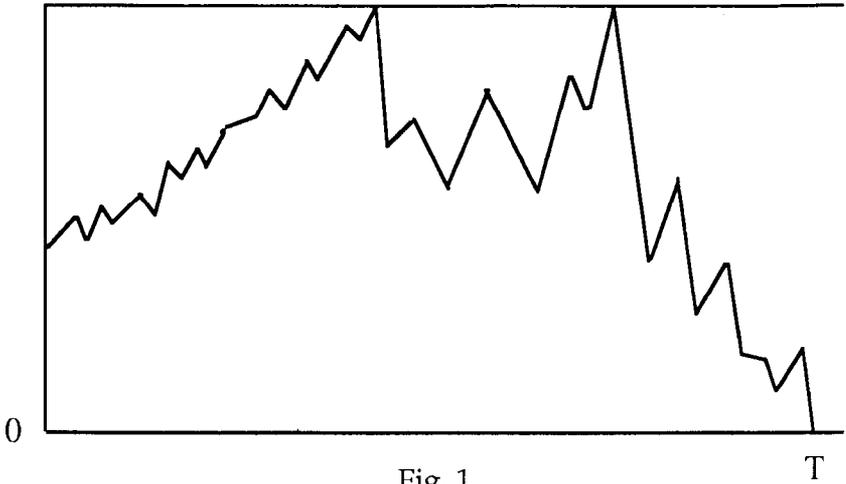


Fig. 1

3.2. LIFETIME OF THE COMPANY

We are here interested in the determination of the company ultimate ruin probability but also, if this probability equals 1, in the computation of the density of the time of ruin which are important elements in the choice of the level of the reflecting barrier. Let T denotes the time of ruin and $\gamma(\alpha ; a)$ denotes its Laplace Transform :

$$\gamma(\alpha ; a) = \mathbb{E}^a(e^{-\alpha T}) \quad , \quad (34)$$

where we note a for R_0 (for the remainder of this paper) in order to simplify the notations.

Now, it is well-known that γ is solution of the following differential equation (see for instance Cox & Miller (1965, p.) or Itô (1961, p.41 with also the use of a corollary of Dynkin's formula) :

$$\frac{1}{2} \sigma^2 \gamma'' + \mu \gamma' = \alpha \gamma, \quad (35)$$

where ' denotes differentiation with respect to a , subject to the boundary conditions :

$$(i) \gamma(\alpha; 0) = 1, \quad (36)$$

$$(ii) \gamma(\alpha; b) = 0. \quad (37)$$

By solving this (easy) differential equation by classical methods, one obtains :

$$\gamma(\alpha; x) = \frac{e^{\theta_1 x} (\theta_1 - \theta_2 e^{(\theta_2 - \theta_1)(b-x)})}{\theta_1 e^{-(\theta_2 - \theta_1)x} - \theta_2 e^{(\theta_2 - \theta_1)(b-x)}}, \quad (38)$$

$$\text{where } \theta_1 \equiv \theta_1(\alpha) = \frac{-\mu + \sqrt{\mu^2 + 2\alpha\sigma^2}}{\sigma^2}, \quad (39)$$

$$\theta_2 \equiv \theta_2(\alpha) = \frac{-\mu - \sqrt{\mu^2 + 2\alpha\sigma^2}}{\sigma^2}.$$

By making α tend to 0 in (38), we obtain the ultimate probability of ruin :

$$P[T < \infty] = 1. \quad (40)$$

So the ruin becomes certain.

By differentiation with respect to α , we get the moments of T :

$$E[T^k] = (-1)^k \left. \frac{d^k \gamma(\alpha; a)}{d\alpha^k} \right|_{\alpha=0} \quad (41)$$

This gives us the expressions of the two first moments for the only interesting practical case $\mu \geq 0$ (see Ars & Janssen (1994)) where b' denotes the difference between b and a , that is $b' = b - a$:

$$ET = \left(e^{\frac{2\mu a}{\sigma^2}} - 1 \right) \frac{\sigma^2}{2\mu^2} e^{\frac{2\mu b'}{\sigma^2}} - \frac{a}{\mu}, \quad (42)$$

$$ET^2 = \frac{D^2 N'' - D N D'' - 2 D' D N' + 2 D'^2 N}{D^3}, \quad (43)$$

where

$$\begin{aligned} N &= \frac{2\mu}{\sigma^2} e^{-\frac{2\mu b'}{\sigma^2}}, \\ N' &= \frac{e^{-\frac{2\mu b'}{\sigma^2}}}{\mu} \left[\frac{2\mu a}{\sigma^2} + e^{\frac{2\mu b'}{\sigma^2}} + 1 - \frac{4\mu b'}{\sigma^2} \right], \\ N'' &= \frac{e^{-\frac{2\mu b'}{\sigma^2}}}{\mu^2} \left[\left(a - \frac{\sigma^2}{\mu} \right) \left(e^{\frac{2\mu b'}{\sigma^2}} - (2b' - a) \left(\frac{2\mu}{\sigma^2} \right) - 1 \right) + \right. \\ &\quad \left. \left(a e^{\frac{2\mu b'}{\sigma^2}} + a - \frac{4\mu a b'}{\sigma^2} - 4b + \frac{8\mu b'^2}{\sigma^2} \right) \right]; \end{aligned}$$

$$D = \frac{2\mu}{\sigma^2} e^{-\frac{2\mu b'}{\sigma^2}},$$

$$D' = \frac{e^{-\frac{2\mu b'}{\sigma^2}}}{\mu} \left[1 + e^{\frac{2\mu b}{\sigma^2}} - \frac{4\mu b'}{\sigma^2} \right],$$

$$D'' = -\frac{\sigma^2 e^{-\frac{2\mu b'}{\sigma^2}}}{\mu^3} \left[1 + e^{\frac{2\mu b}{\sigma^2}} - \frac{4\mu b'}{\sigma^2} \right] + \frac{e^{-\frac{2\mu b'}{\sigma^2}}}{\mu^2} \left[4a e^{\frac{2\mu b}{\sigma^2}} + \frac{8\mu b'^2}{\sigma^2} - 4b' \right].$$

Note that if we make b tend to infinity, we will of course find back the results of Ars and Janssen(1994).

We now know that the random variable T admits a density (denoted in what follows by g). This can be calculated by the computation of the Inverse Laplace Transform of $\gamma(\alpha; a)$ (see formula(38)). This gives (see for the method Williams (1973, p. 67)) :

$$\text{Let } m = \frac{\sigma^2}{\mu b}; \text{ if } m \geq 1, \text{ then}$$

$$g(t) = \sum_{j=0}^{\infty} \alpha_j e^{(-\lambda_j t)},$$

with :

$$\alpha_j = \frac{e^{-\frac{\mu a}{\sigma^2}} (1 + m^2 \omega_j^2) (\mu m \omega_j) \sin\left(\frac{\mu m \omega_j a}{\sigma^2}\right)}{b(1 + m^2 \omega_j^2) - \frac{\sigma^2}{\mu}} ; \quad (44)$$

$$\lambda_j = \frac{\mu^2}{2\sigma^2} + \frac{\sigma^2}{2b^2} \omega_j^2 = \frac{\mu^2}{2\sigma^2} (1 + m^2 \omega_j^2) \quad ; \quad (45)$$

ω_j is the solution between $j\pi$ and $(j+1)\pi$ of the equation :

$$\operatorname{tg}(x) = m x; \quad (46)$$

if $m < 1$, then :

$$g(t) = \alpha_0 e^{(-\lambda_0 t)} + \sum_{j=1}^{\infty} \alpha_j e^{(-\lambda_j t)}, \quad (47)$$

with :

α_j and λ_j having for $j \geq 1$ the same meaning than before,

$\lambda_0 = \frac{\mu^2}{2\sigma^2} (1 - m^2 v^2)$ where v is the real solution, which exists

because $m < 1$, of the following equation :

$$\operatorname{Th} x = m x; \quad (48)$$

$$\alpha_0 = \frac{N(-\lambda_0)}{D'(-\lambda_0)} \quad ; \quad (49)$$

$$D' = \frac{e^{-(\theta_2 - \theta_1)a} (1 + 2a\theta_1) - e^{(\theta_2 - \theta_1)b'} (1 + 2b'\theta_2)}{\mu m v}, \quad (50)$$

$$\text{with } \theta_1 = \frac{-\mu + \mu m v}{\sigma^2} \text{ and } \theta_2 = \frac{-\mu - \mu m v}{\sigma^2}.$$

Example :

The company parameters are (see Ars & Janssen (1994)) :

$$a = 0.1887, \quad \mu = 0.0103, \quad \sigma = 0.0186.$$

Let us fix the level of the reflecting barrier (b) at 0.1987. The computation of m gives 0.169 and we obtain the following expression of the density ($t \geq 0$) :

$$g(t) = 4.460939 \cdot 10^{-6} e^{-4.4606579 \cdot 10^{-6} t} - 4.882037 \cdot 10^{-5} e^{-0.2133256 t} \\ + 1.216118 \cdot 10^{-4} e^{-0.378168 t} - 1.756491 \cdot 10^{-4} e^{-0.634685 t} \\ + \dots$$

We can also express g as the sum of densities of exponential random variables :

$$g(t) = 1.000063113 p_0(t) - 2.288538 \cdot 10^{-4} p_1(t) \\ + 3.215814 \cdot 10^{-4} p_2(t) - 2.7675 \cdot 10^{-4} p_3(t) + \dots$$

where p_i denotes for $i \geq 0$ the density of an exponential random variable of parameter λ_i .

We easily obtain the expected lifetime of the company :

$$ET = 224196 \text{ (ans).}$$

The probability of ruin before t' is :

$$P[T \leq t'] = \int_0^{t'} g(t) dt = \sum_{j=0}^{\infty} \beta_j (1 - e^{-\lambda_j t'}),$$

and the probability of survival to time t' :

$$P[T \geq t'] = \int_{t'}^{\infty} g(t) dt = \sum_{j=0}^{\infty} \beta_j e^{-\lambda_j t'}$$

So the probability of survival to 20 years is 0.999974 and to 100 years is 0.99962.

3.3. THE TIME BEFORE THE FIRST DIVIDEND REPARTITION

Another thing of great interest for the company is the time that the shareholders have to wait before the first dividend repartition. This time, denoted in what follows by S , depends obviously on the choice of the parameter b .

One easily finds the Laplace Transform $\delta(\alpha ; a)$ of S : it is the solution of the following differential equation :

$$\frac{1}{2} \sigma^2 \delta'' + \mu \delta' = \alpha \delta, \quad (51)$$

subject to the boundary conditions (see Cox & Miller (1965)) :

$$\delta(\alpha ; 0) = 0, \quad (52)$$

$$\delta(\alpha ; b) = 1. \quad (53)$$

The analytic form of δ is then :

$$\delta(\alpha ; a) = \frac{e^{-\theta_1 b'} (1 - e^{-(\theta_2 - \theta_1) b'})}{e^{(\theta_2 - \theta_1) b'} - e^{-(\theta_2 - \theta_1) a}}, \quad (54)$$

$$\begin{aligned} \text{where } \theta_1 = \theta_1(\alpha) &= \frac{-\mu - \sqrt{\mu^2 + 2\alpha\sigma^2}}{\sigma^2}, \\ \theta_2 = \theta_2(\alpha) &= \frac{-\mu + \sqrt{\mu^2 + 2\alpha\sigma^2}}{\sigma^2}. \end{aligned} \tag{55}$$

By taking $\alpha=0$, we obtain the probability that a dividend repartition precedes the ruin :

$$P[S < T] = \frac{e^{\frac{2\mu b'}{\sigma^2}} - e^{\frac{2\mu(b'-a)}{\sigma^2}}}{e^{\frac{2\mu b'}{\sigma^2}} - e^{\frac{-2\mu a}{\sigma^2}}} = \frac{e^{\frac{2\mu a}{\sigma^2}} - 1}{e^{\frac{2\mu a}{\sigma^2}} - e^{\frac{-2\mu b'}{\sigma^2}}}. \tag{56}$$

The computation of the Inverse Laplace Transform of (54), denoted by f , gives : (see Cox & Miller (1965, p 222)) :

$$f(t) = \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n t}, \tag{57}$$

where

$$\begin{aligned} \alpha_n &= (-1)^{n+1} \left(\frac{\sigma^2 n \pi}{b^2} \right) e^{\frac{\mu b'}{\sigma^2}} \sin\left(\frac{n \pi a}{b}\right), \\ \lambda_n &= \frac{\mu^2}{2\sigma^2} \left[1 + \left(\frac{\sigma^2 n \pi}{\mu b} \right)^2 \right]. \end{aligned} \tag{58}$$

So, we can compute for every $s \geq 0$, the following probability :

$$P[S \leq s | S < T] = \left(e^{2\mu a/\sigma^2} - 1 \right) \left(e^{2\mu a/\sigma^2} - e^{-2\mu b'/\sigma^2} \right)^{-1} \int_0^s f(t) dt. \tag{59}$$

So, the managers of the company have two parameters which are useful in the choice of the level of the reflecting barrier : the density function (and therefore mean, variance, ...) and the waiting time for the first dividend distribution.

Another thing of great importance for the shareholders is the expected discounted value of the distributed dividends. Unfortunately, we can't compute it with this model without bringing about a modification. We postpone it until a future paper.

CONCLUSION

This paper is related to the possible applications of a stochastic model of Asset Liability Management (ALM) for insurance (Ars and Janssen (1994)) to real life situations.

We study a model generalizing the one presented by Janssen (1993) and taking into account the occurrence of catastrophes for liabilities. The assumptions made about the (assets and liabilities) processes combining with a very useful change of variable allow us to use the results of Dufresne (1989) and so to determine the ultimate ruin probability. This leads to important consequences for company management (ALM) such as the determination of the objectives to be followed by the company and we recall the (possible) tools needed to achieve them.

We then consider the possibility for the company to avoid the bankruptcy by subscribing to a loan and evaluate the implications on its future financial wealth. We show by an example that this implies a diminution for the ultimate ruin probability.

After, we investigate dividend repartition by considering a reflecting barrier. We show that this implies the ruin with probability one and that the density of the random variable can be expressed as the sum of exponential densities. We also compute the expected time before the first dividend repartition with the particular dividend strategy considered.

Some interesting problems are left open :

- how to integrate (in our model) discontinuities (also) for the assets process ?

- how to generalize our results to semimartingales (X, Y) with independent but not necessarily stationary increments (with always a finite number of jumps on a finite interval) ?

- is it possible to generalize the considered dividend strategy ?

- how to construct optimal strategies based on reinsurance options in order to encounter the (management) objectives ?

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