Dynamic Fund Protection

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Presentation based on two papers:

Hans U. Gerber and Gerard Pafumi, “Pricing Dynamic Investment Fund Protection,” *North American Actuarial Journal*, Vol 4 (2), 2000

Hans U. Gerber and Elias S.W. Shiu, “Pricing Perpetual Fund Protection with Withdrawal Protection,” *North American Actuarial Journal*, Vol 7 (2), 2003

Presentation based on two papers by Hans U. Gerber:

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- Stock Index value at time t:

$$I(t) = I(0)e^{Y(t)}$$

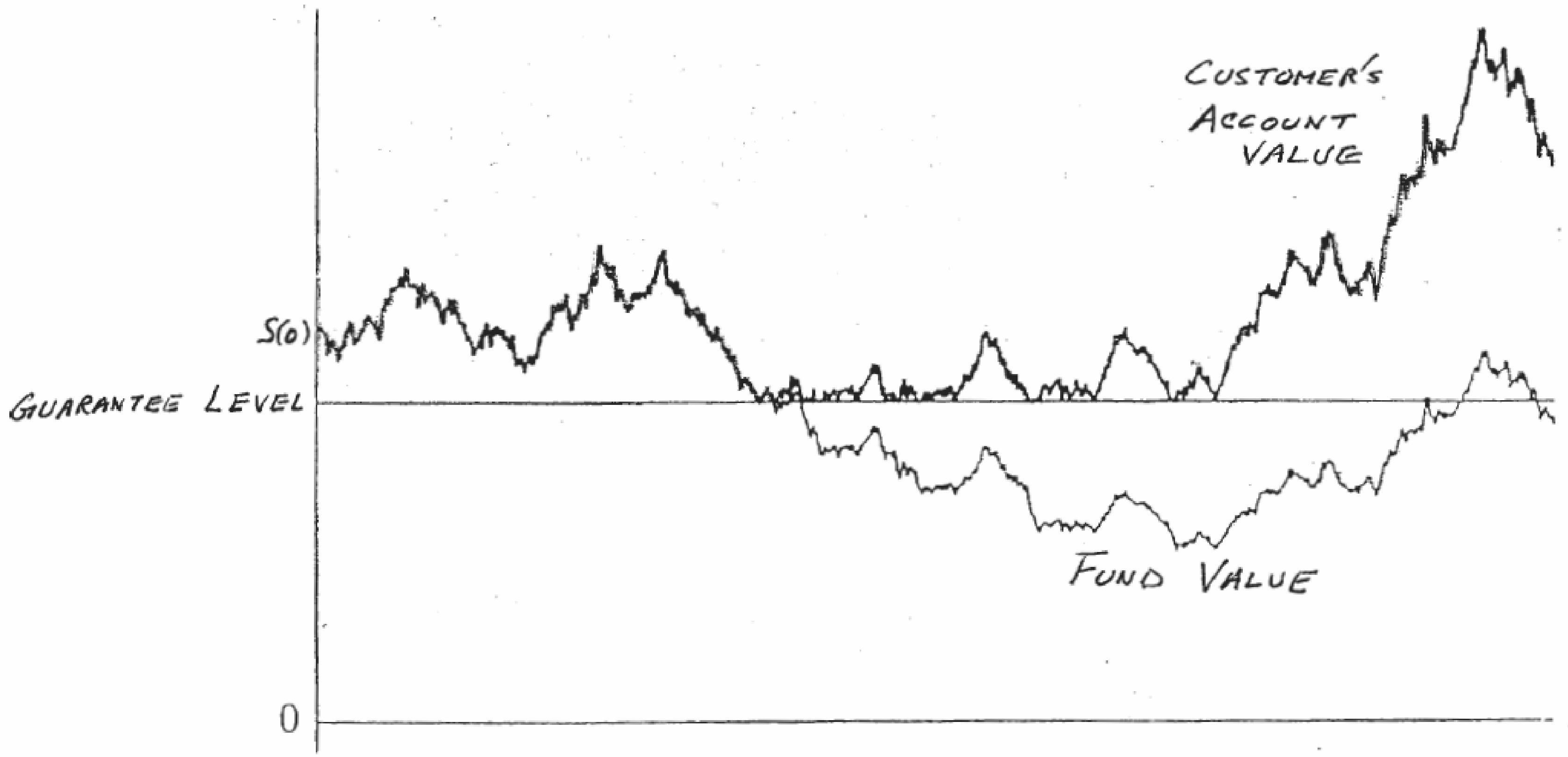
e.g., S&P 500

Assume {Y(t)} is a *Brownian motion* (*Wiener process*).

- Value of one unit of fund at time t:

$$S(t) = S(0)e^{\alpha Y(t)}$$

where α, called the *participation rate*, is usually < 1.


CUSTOMER'S ACCOUNT VALUE
S(0)
GUARANTEE LEVEL
FUND VALUE
0
t

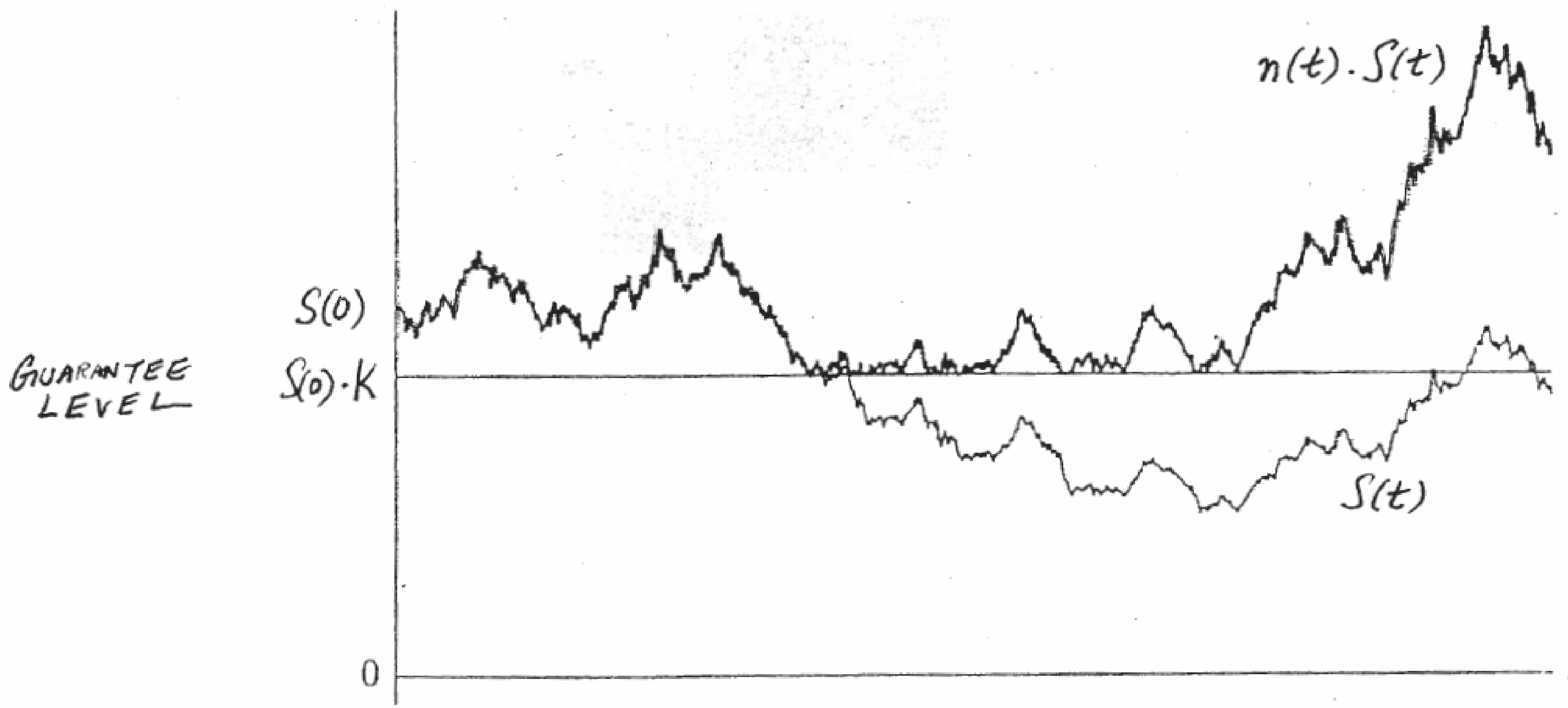
$n(t)$ = number of fund units in the customer's account at time t

$n(0) = 1$

$n(t)S(t) \geq S(0)K(t)$ guarantee boundary

e.g., $K(t) = 0.9\,(1.03)^t$ for satisfying U.S. nonforfeiture laws

A Typical Sample Path of the Fund Values



- What is n(t)?

$n(0) = 1$

$$n(t) \geq \frac{S(0)K(t)}{S(t)}$$

n(t) being *non-decreasing* means:

For all $t \geq 0$, $\displaystyle n(t) \geq \max_{0 \leq \tau \leq t} n(\tau)$

Hence,

$$n(t) \geq \max\left\{1, \max_{0 \leq \tau \leq t} \frac{S(0)K(\tau)}{S(\tau)}\right\}$$

Therefore, the number of fund units in the customer's account at time t is

$$n(t) = \max\left\{1, \max_{0 \le \tau \le t} \frac{S(0)K(\tau)}{S(\tau)}\right\}$$

The customer's account value at time t is $n(t)S(t)$.

By the *Fundamental Theorem of Asset Pricing*, the time-0 “value” of this Dynamic Fund Protection contract is

$$E^*[e^{-rT} n(T) S(T)]$$

where * signifies that the expectation is taken with respect to the *risk-neutral* probability measure, r is the risk-free force of interest, and T is the maturity date of the contract.

This “value” should be compared with S(0), the premium for one unit of the fund at time 0.

Because

$$S(T) = S(0)e^{\alpha Y(T)}$$

we have

$$E^{*}[e^{-rT} n(T) S(T)]$$

$$= S(0)e^{-rT} E^{*}[n(T)e^{\alpha Y(T)}]$$

What is $E^{*}[n(T)e^{\alpha Y(T)}]$?

$$E^*[n(T)e^{\alpha Y(T)}]$$

$$= \frac{E^*[n(T)e^{\alpha Y(T)}]}{E^*[e^{\alpha Y(T)}]} E^*[e^{\alpha Y(T)}]$$

$$= E^{**}[n(T)]\ E^*[e^{\alpha Y(T)}]$$

where ** signifies a changed probability measure (an Esscher transform).

Now, $E^*[e^{\alpha Y(T)}]$ is the *moment-generating function* of the random variable Y(T) (with respect to the risk-neutral probability measure) at the value α.

It is assumed that $\{Y(t)\}$ is a Brownian motion with *diffusion coefficient* σ. Under the risk-neutral probability measure, Y(T) is a *normal* random variable, and

$$E^*[e^{\alpha Y(T)}]$$

$$= \exp(\alpha E^*[Y(T)] + \frac{1}{2}\alpha^2 \mathrm{Var}^*[Y(T)]),$$

where

$$E^*[Y(T)] = \left(r - \frac{\sigma^2}{2} - \zeta\right)T,$$

and

$$\mathrm{Var}^*[Y(T)] = \mathrm{Var}[Y(T)] = \sigma^2 T.$$

Here, r is the risk-free force of interest, and ζ is the (constant) dividend-yield rate.

So it remains to determine $E^{**}[n(T)]$.

To determine E**[n(T)], we consider the case

$$K(t) = \beta e^{gt} \qquad [\text{e.g., } K(t) = 0.9(1.03)^t]$$

Then,

$$n(t) = \max\left\{1, \max_{0\le\tau\le T} \frac{S(0)K(\tau)}{S(\tau)}\right\}$$

$$= \max\left\{1, \max_{0\le\tau\le T} \frac{\beta e^{g\tau}}{e^{\alpha Y(\tau)}}\right\}$$

$$= \max\left\{1, \beta \exp\left[\max_{0\le\tau\le T} \big(g\tau - \alpha Y(\tau)\big)\right]\right\}$$

Define $X(\tau) = g\tau - \alpha Y(\tau)$.

Let

$$M(t) = \max_{0 \leq \tau \leq T} X(\tau)$$

be the *running maximum* of the process $\{X(\tau)\}$. Then,

$$n(t) = \max\{1, \beta e^{M(t)}\}.$$

Thus, to determine $E^{**}[n(T)]$, we need to know the probability distribution of running maximum $M(T)$ under the ** probability measure.

For a Wiener process $\{X(\tau)\}$ with *drift* μ and *diffusion coefficient* σ, it is known that

$$\Pr[M(t) \le m] = \Phi\left(\frac{m - \mu t}{\sigma\sqrt{t}}\right) - e^{2m\mu/\sigma^2}\Phi\left(\frac{-m - \mu t}{\sigma\sqrt{t}}\right)$$

where $\Phi(\cdot)$ is the c.d.f. of the Normal (0, 1) random variable. For $X(\tau) = g\tau - \alpha Y(\tau)$, what are the drift and diffusion coefficient of $\{X(\tau)\}$ under the ** probability measure?

Recall:

$$\frac{E^{*}[n(T)e^{\alpha Y(T)}]}{E^{*}[e^{\alpha Y(T)}]} = E^{**}[n(T)]$$

Under the ** probability measure, the Brownian motion $\{Y(\tau)\}$ has drift

$$E^{*}[Y(1)] + \alpha\sigma^{2}$$

$$= (r - \frac{\sigma^{2}}{2} - \zeta) + \alpha\sigma^{2}$$

and (unchanged) diffusion coefficient σ.

Now, $X(\tau) = g\tau - \alpha Y(\tau)$. Thus, under the ** probability measure, the process $\{X(\tau)\}$ has drift

$$g - \alpha[(r - \frac{\sigma^2}{2} - \zeta) + \alpha\sigma^2]$$

and diffusion coefficient $\alpha\sigma$.

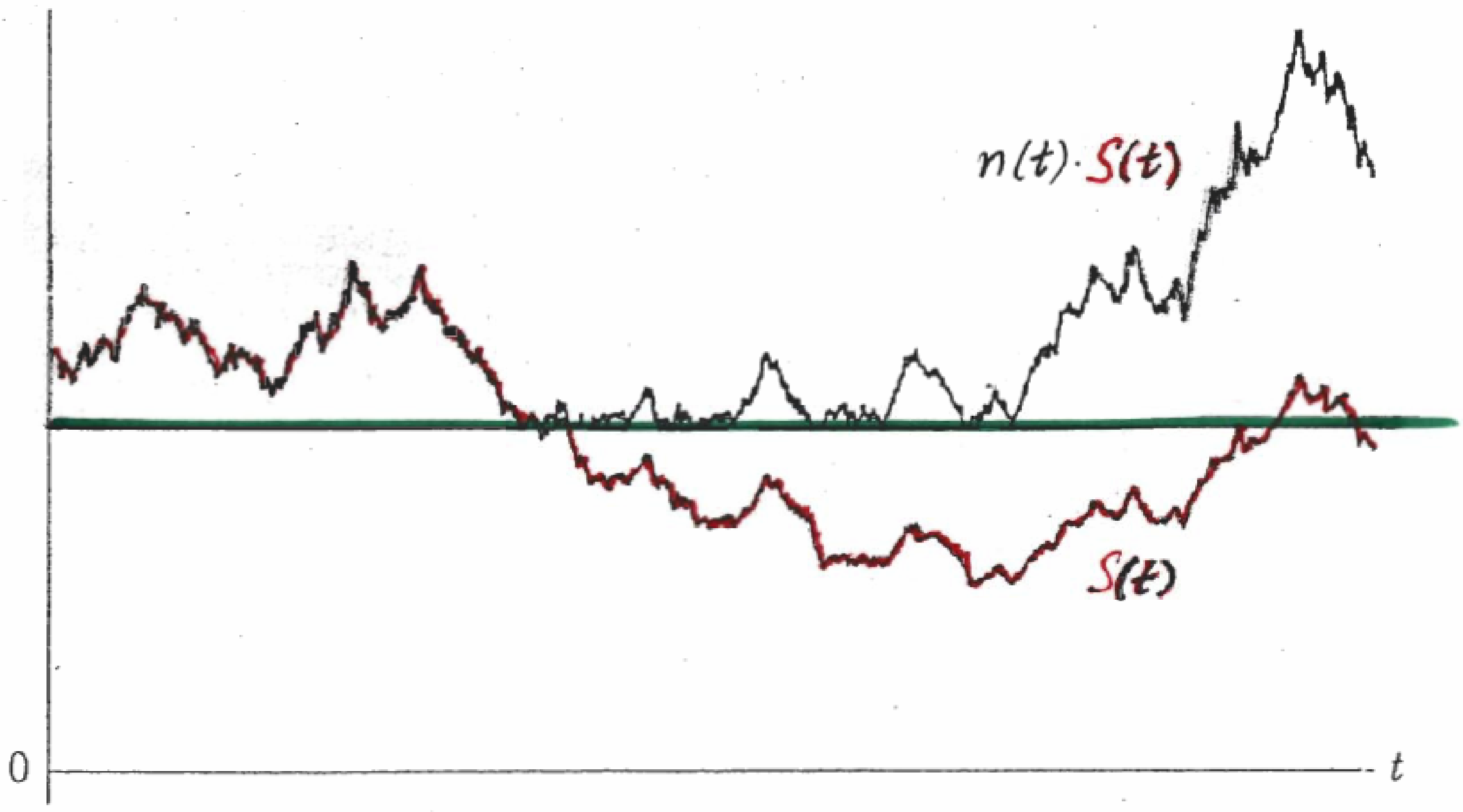
Hence, E**[n(T)] can be evaluated, and one can then write down a closed-form formula for $E^*[e^{-rT}n(T)S(T)]$.

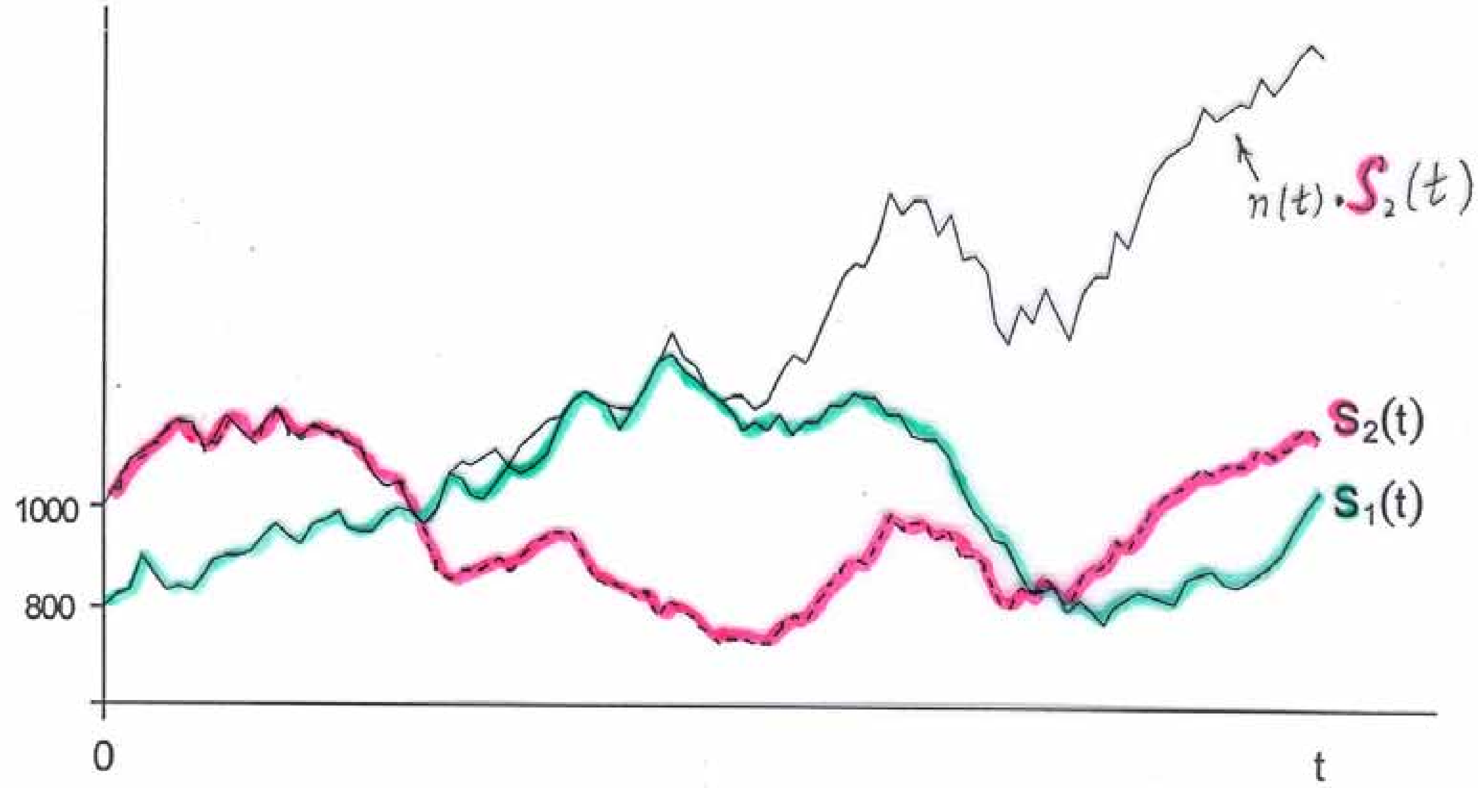
$$E^*\left[e^{-rT} \max\left\{1, \max_{0\le\tau\le T} \frac{S(0)K(\tau)}{S(\tau)}\right\} S(T) \right]$$

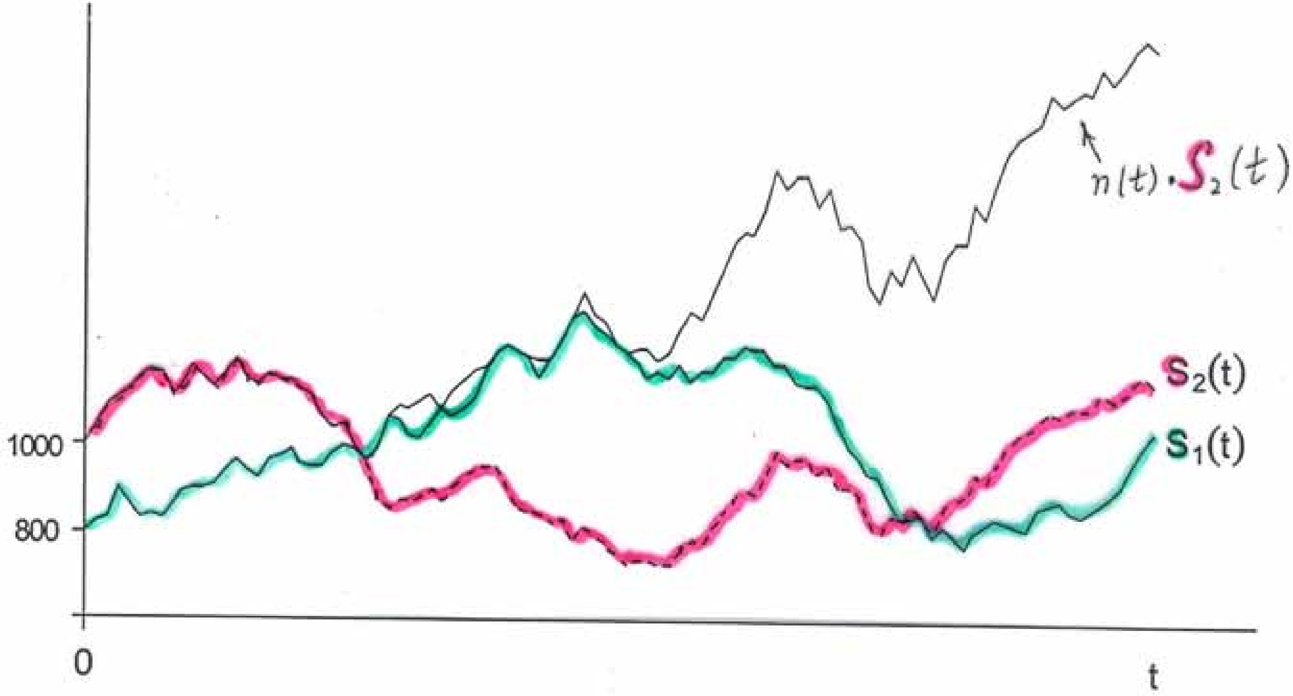
Generalize to stochastic guarantee boundary:

$$E^*\left[e^{-rT} \max\left\{1, \max_{0\le\tau\le T} \frac{S_1(\tau)}{S_2(\tau)}\right\} S_2(\tau) \right]$$

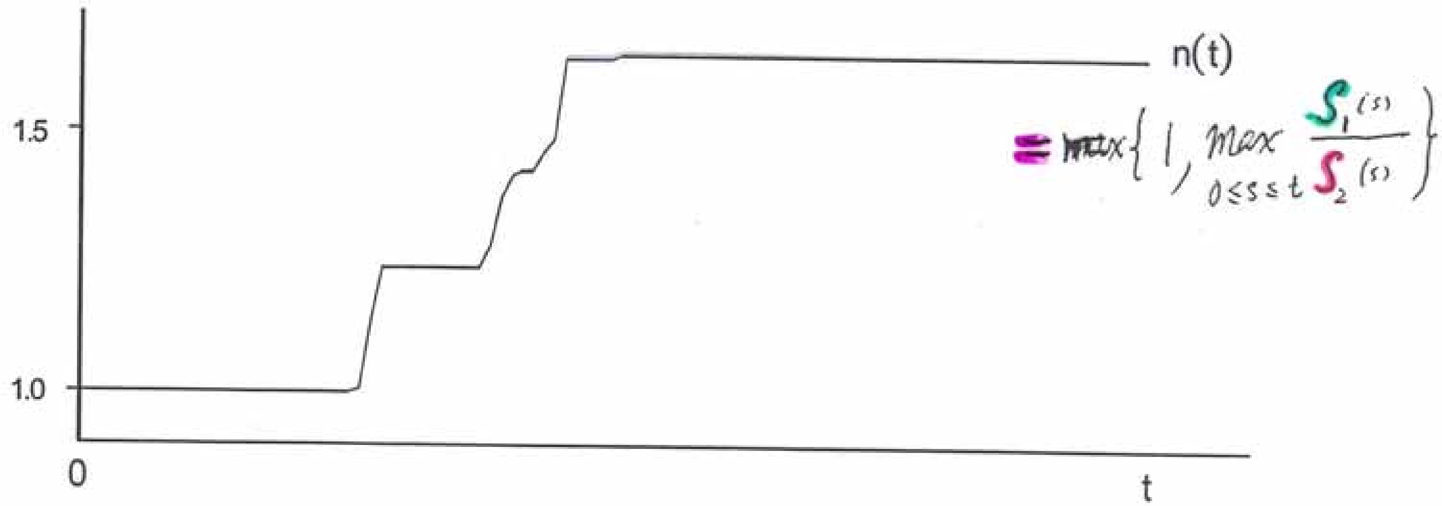
A Typical Sample Path of the Fund Values




n(t) · S2(t)
S2(t)
S1(t)
1000
800
0
t



n(t) · S₂(t)
S₂(t)
S₁(t)
1000
800
0
t



n(t)
= max { 1, max_{0 ≤ s ≤ t} S₁(s) / S₂(s) }
1.5
1.0
0
t

Earlier, T was a fixed maturity date, i.e., we were pricing “European” options.

How about “American” options?

$$\sup_{\substack{\text{All stopping} \\ \text{times } T}} E^*\left[e^{-rT} \max\left\{1, \max_{0 \le t \le T} \frac{S_1(t)}{S_2(t)}\right\} S_2(T) \right]$$

Note: The payoff is *path-dependent*.

Two special cases:

$S_2(t) = S(t)$, $S_1(t) = S(0)(0.9)(1.03)^t$

$S_1(t) = S(t)$, $S_2(t) = \text{constant}$

L. Shepp & A. N. Shiryaev, "The Russian Option: Reduced Regret," *Annals of Probability*, Vol 3, 1993.

The Russian option is an American *lookback* or *high watermark* option without maturity date. Its time-0 price is:

$$\sup_{T} E^{*}\left[e^{-rT} \max\left\{ k, \max_{0 \le t \le T} S(t) \right\} \right]$$

The constant k can be viewed as the historical maximum of the stock prices (the maximum before time 0).

- Let $S_j(t) = S_j(0)\, e^{X_j(t)}$, $j = 1, 2$, and we assume that $\{X_1(t), X_2(t)\}$ is a bivariate Brownian motion.
- Assume that each stock (or stock index) pays dividends continuously at a rate proportional to its price. That is, for $j=1, 2$, there is a constant $\zeta_j > 0$, such that stock j pays dividends of amount $\zeta_j S_j(t)dt$ between time t and time t+dt.

- Then, under the risk-neutral probability measure, $\left\{e^{-(r-\zeta_j)t} S_j(t)\right\}$, j = 1, 2, are *martingales*.

- Again, r is the risk-free force of interest.

- Then, under the risk-neutral probability measure, $\left\{ e^{-(r-\zeta_j)t} S_j(t) \right\}$, $j = 1, 2$, are *martingales*.

Address of the Society of Actuaries is:

475 North Martingale Road

Schaumburg, Illinois, U.S.A.

- Address of the Institute of Actuaries of Australia is:

  4 Martin Place, Sydney

But ….

- Address of the Institute of Actuaries of Australia is:

4 Martin Place, Sydney

But its current president is

Andrew Gale

- How about the Swiss Association of Actuaries?

- How about the Swiss Association of Actuaries?

Hans U. Gerber, “Martingales in Risk Theory,” *Bulletin of the Swiss Association of Actuaries* (1973), 205-216.

Under the risk-neutral probability measure,

$$\left\{e^{-(r-\zeta_j)t} S_j(t)\right\}, \; j = 1,2,$$

are martingales. Also, there are two martingales of the form

$$\left\{e^{-rt} [S_1(t)]^{\theta} [S_2(t)]^{1-\theta}\right\}.$$

The martingale condition is

$$e^{-rt} E^*[e^{\theta X_1(t)+(1-\theta)X_2(t)}] = 1.$$

This leads to the quadratic equation

$$-r + E^*[\theta X_1(1) + (1-\theta)X_2(1)] + \frac{1}{2}\mathrm{Var}[\theta X_1(1) + (1-\theta)X_2(1)] = 0.$$

Its solutions are $\theta_1 < 0$ and $\theta_2 > 1$. Thus, the two processes,

$$\left\{e^{-rt}[S_1(t)]^{\theta_j}[S_2(t)]^{1-\theta_j}\right\}, \quad j = 1,\ 2,$$

are martingales under the risk-neutral probability measure.

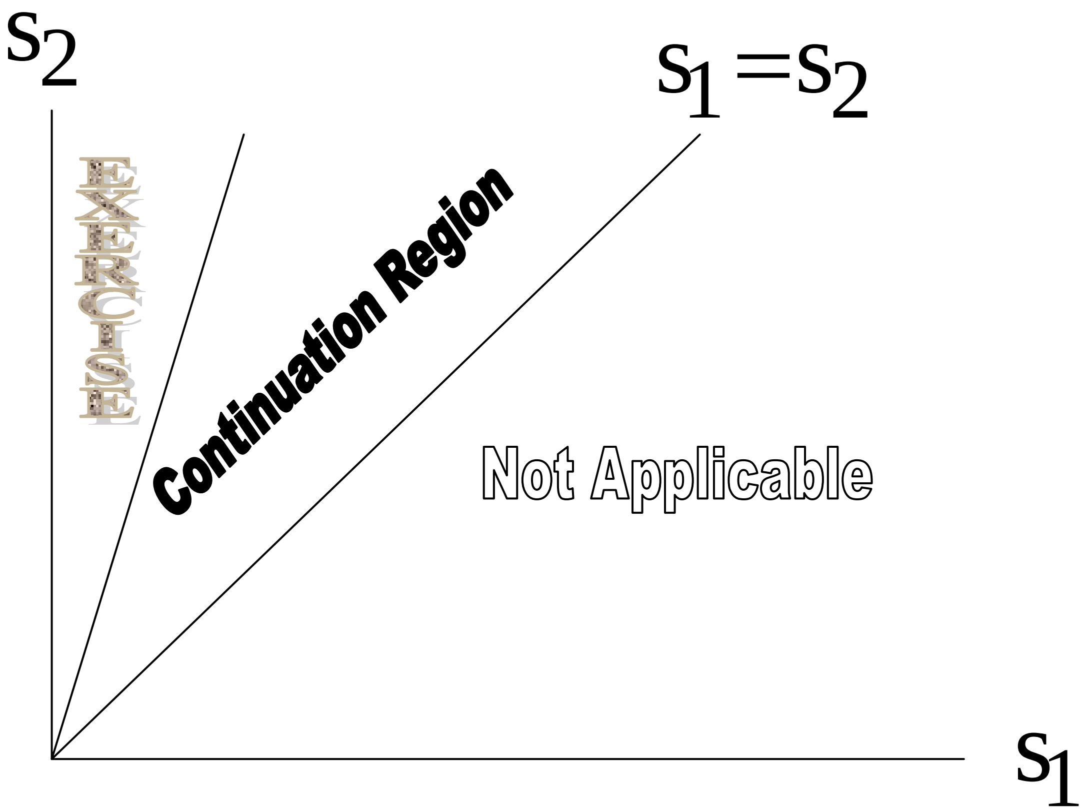
Let $$n(T) = \max\left\{1, \max_{0 \le \tau \le T} \frac{S_1(\tau)}{S_2(\tau)}\right\}$$

For $s_1 > 0$, $s_2 > 0$, define

$$V(s_1, s_2)$$
$$= \sup_T E^*[e^{-rT} n(T) S_2(T) \mid S_1(0) = s_1, S_2(0) = s_2]$$

The supremum is taken over all stopping times T. There is no fixed expiry date.

This is the price of the perpetual *dynamic protection option*.



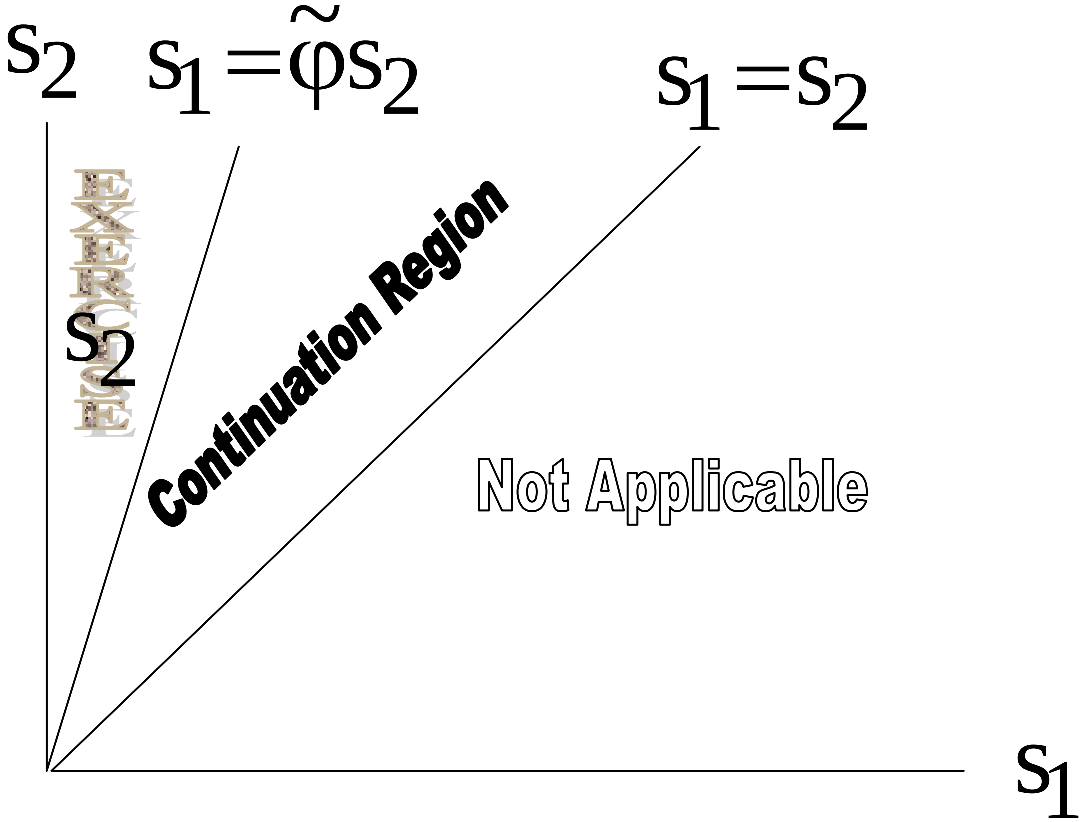
S_2
$S_1 = S_2$
EXERCISE
Continuation Region
Not Applicable
S_1

$$h(s) = (\theta_2 - 1)s^{\theta_1} + (1 - \theta_1)s^{\theta_2}, \quad s > 0$$

$$\tilde{\varphi} = \left(\frac{-\theta_1(\theta_2 - 1)}{(1 - \theta_1)\theta_2} \right)^{\frac{1}{\theta_2 - \theta_1}}$$

$$0 < \tilde{\varphi} < 1$$

$$V(s_1, s_2) = \begin{cases} \dfrac{h(s_1 / s_2)}{h(\tilde{\varphi})} s_2 & \text{if } \tilde{\varphi} < \dfrac{s_1}{s_2} \le 1 \\ s_2 & \text{if } 0 < \dfrac{s_1}{s_2} \le \tilde{\varphi} \end{cases}$$


s_2
$s_1 = \tilde{\varphi} s_2$
$s_1 = s_2$
s_2
Continuation Region
Not Applicable
s_1

Instead of $$n(T) = \max\left\{1, \max_{0 \le t \le T} \frac{S_1(t)}{S_2(t)}\right\},$$

we now consider $$n(T) = \max\left\{1, \frac{S_1(T)}{S_2(T)}\right\}.$$

Then, $n(T)S_2(T) = \max\{S_1(T), S_2(T)\}$,

which is the payoff of the *maximum option* (also called *alternative option* or *greater-of option*). This is a simpler option since the payoff is not path-dependent.

The price of the American maximum option without a fixed expiry date is:

$$W(s_1, s_2)$$

$$= \sup_{T} E^*[e^{-rT} \max\{S_1(T), S_2(T)\} \mid S_1(0)=s_1, S_2(0)=s_2],$$

$$s_1 > 0 \text{ and } s_2 > 0.$$

This option has been evaluated in the paper Gerber and Shiu, “Martingale Approach to Pricing Perpetual American Options on Two Stocks,” *Mathematical Finance*,” Vol 6 (1996).

$$\sup_{T} E^{*}[e^{-rT} n(T) S_2(T) \mid S_1(0) = s_1, S_2(0) = s_2]$$

For $V(s_1, s_2)$, $\quad n(T) = \max\left\{1, \max_{0 \le t \le T} \frac{S_1(t)}{S_2(t)}\right\}.$

For $W(s_1, s_2)$, $\quad n(T) = \max\left\{1, \frac{S_1(T)}{S_2(T)}\right\}.$

Obviously, $\quad V(s_1, s_2) \ge W(s_1, s_2).$

$$\sup_{T} E^{*}[e^{-rT} n(T) S_2(T) | S_1(0) = s_1, S_2(0) = s_2]$$

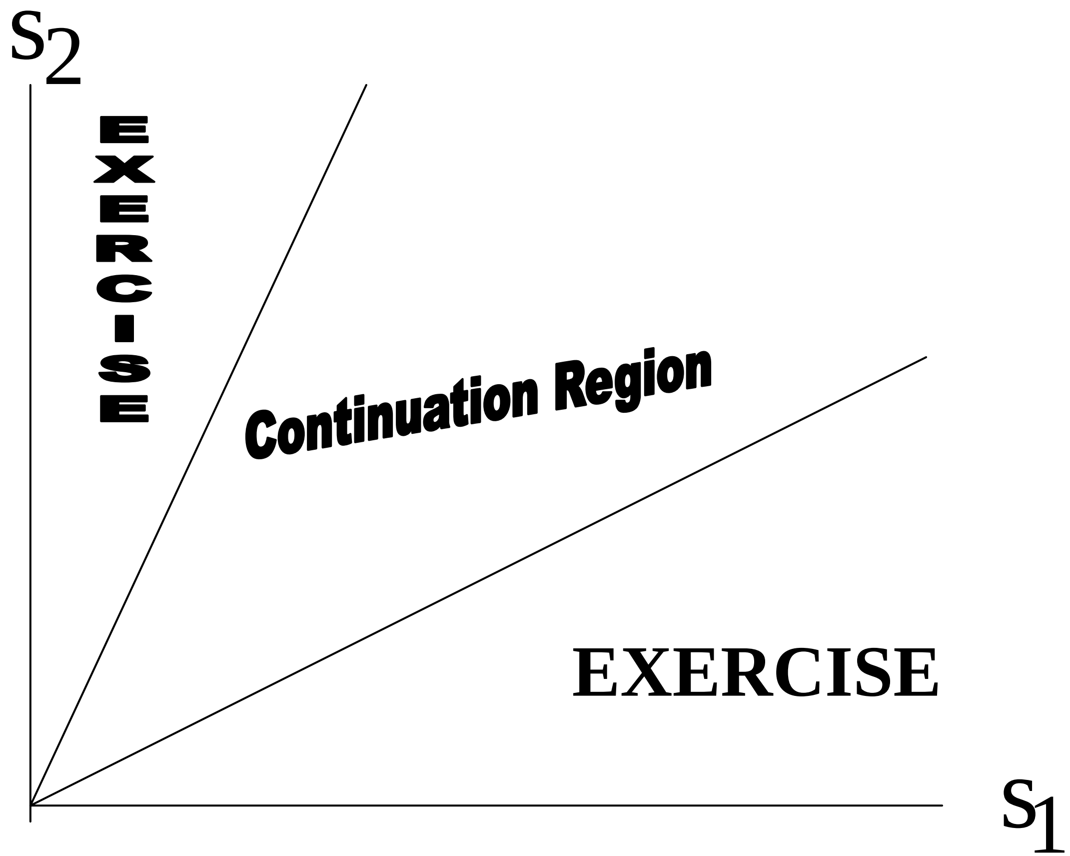
For $V(s_1, s_2)$, $\quad n(T) = \max\left\{1, \max_{0 \le t \le T} \frac{S_1(t)}{S_2(t)}\right\}$.

For $W(s_1, s_2)$, $\quad n(T) = \max\left\{1, \frac{S_1(T)}{S_2(T)}\right\}$.

Obviously, $\quad V(s_1, s_2) \ge W(s_1, s_2)$.

Surprisingly, there is a constant $\tilde{c} > 1$, such that

$$V(s_1, s_2) = W(\tilde{c} s_1, s_2).$$



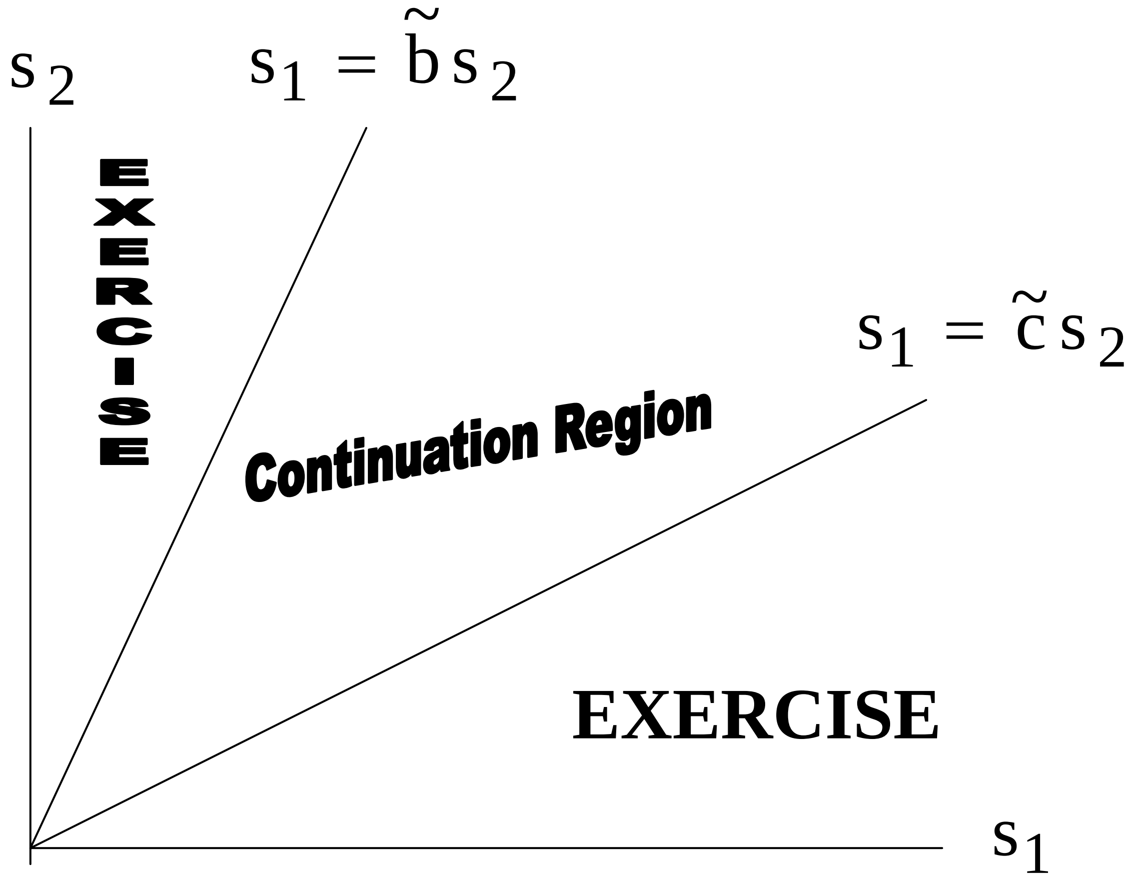
S2
EXERCISE
Continuation Region
EXERCISE
S1

$$\tilde{b} = \left(\frac{-\theta_1}{1-\theta_1}\right)^{(1-\theta_1)/(\theta_2-\theta_1)} \left(\frac{\theta_2}{\theta_2-1}\right)^{(\theta_2-1)/(\theta_2-\theta_1)}$$

$$\tilde{c} = \left(\frac{-\theta_1}{1-\theta_1}\right)^{-\theta_1/(\theta_2-\theta_1)} \left(\frac{\theta_2}{\theta_2-1}\right)^{\theta_2/(\theta_2-\theta_1)}$$

$$k(x) = \frac{\theta_2 (x/\tilde{b})^{\theta_1} - \theta_1 (x/\tilde{b})^{\theta_2}}{\theta_2 - \theta_1} \quad x > 0$$

$$W(s_1, s_2) = \begin{cases} s_2 & \text{if } \dfrac{s_1}{s_2} \leq \tilde{b} \\ s_2\, k\left(\dfrac{s_1}{s_2}\right) & \text{if } \tilde{b} < \dfrac{s_1}{s_2} < \tilde{c} \\ s_1 & \text{if } \dfrac{s_1}{s_2} \geq \tilde{c} \end{cases}$$



s_2
$s_1 = \tilde{b} s_2$
EXERCISE
Continuation Region
$s_1 = \tilde{c} s_2$
EXERCISE
s_1

It can be readily checked that

$$\tilde{\varphi} = \frac{\tilde{b}}{\tilde{c}}.$$

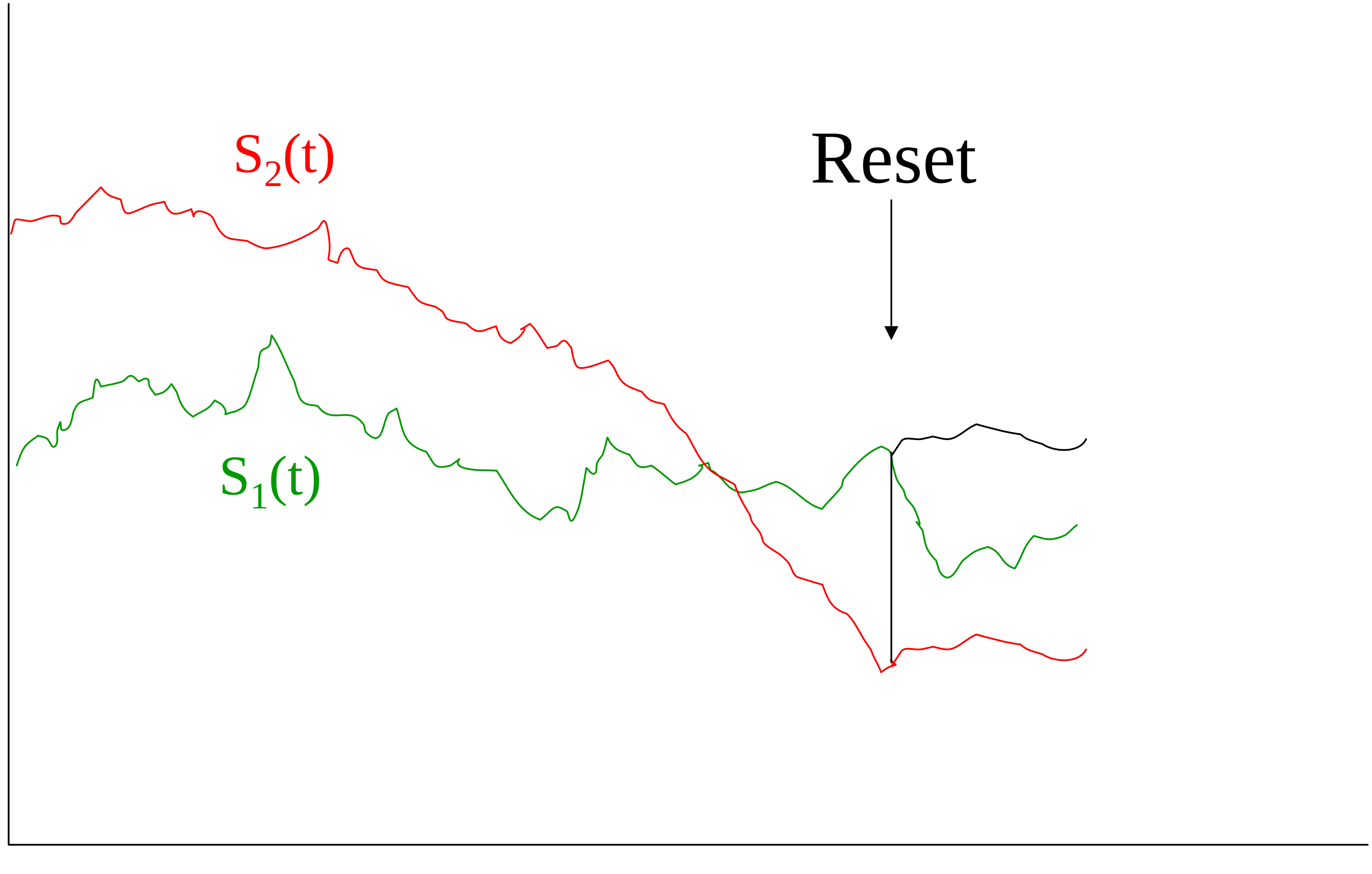
From this, we realized that

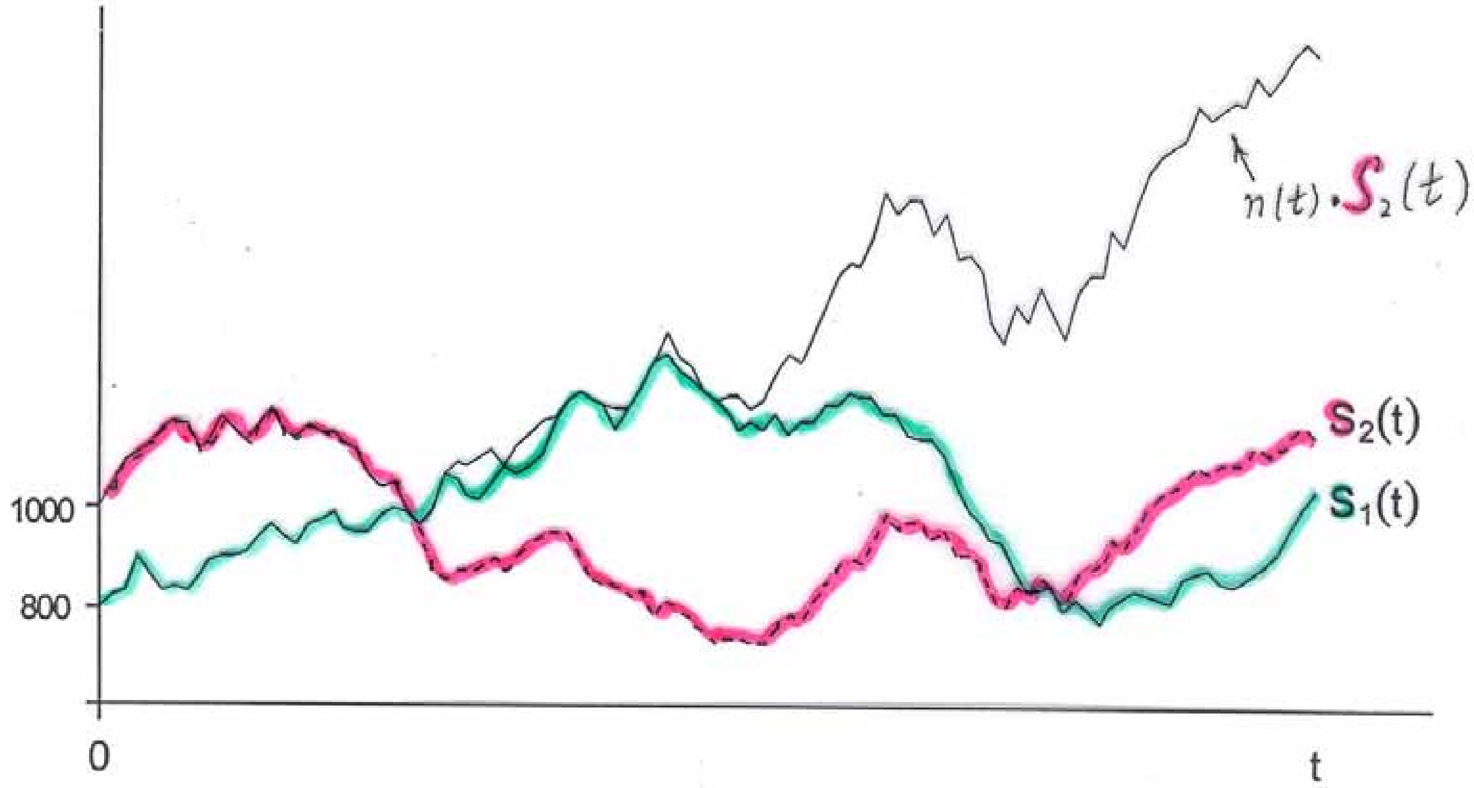
$$V(s_1, s_2) = W(\tilde{c}s_1, s_2)$$

But why is this formula true?

Y K Kwok and C C Chu wrote a discussion on “Pricing Perpetual Fund Protection with Withdrawal Protection,” *North American Actuarial Journal*, Vol 7 (2), 2003. They introduced the concept of a perpetual option with “up to n resets”. When the number of possible resets n becomes ∞, we have

$$V(s_1, s_2) = W(\tilde{c}\, s_1, s_2)$$


$S_2(t)$
Reset
$S_1(t)$



n(t) · S₂(t)
S₂(t)
S₁(t)
1000
800
0
t

For n = 1, 2, 3, ... , and let $V_n(s_1, s_2)$ denote the price of the option with *up to* n resets, where $s_1 = S_1(0) > 0$ and $s_2 = S_2(0) > 0$. The option has no fixed expiry date. Thus,

$$V_{n+1}(s_1, s_2) = \sup_T E^*[e^{-rT} \max\{V_n(S_1(T), S_1(T)), S_2(T)\} \mid S_1(0) = s_1, S_2(0) = s_2].$$

Because $V_n(s_1, s_2)$ is a homogeneous function of degree 1,

$$V_n(S_1(T), S_1(T)) = V_n(1, 1)S_1(T)$$

Define

$$\kappa_n = V_n(1, 1).$$

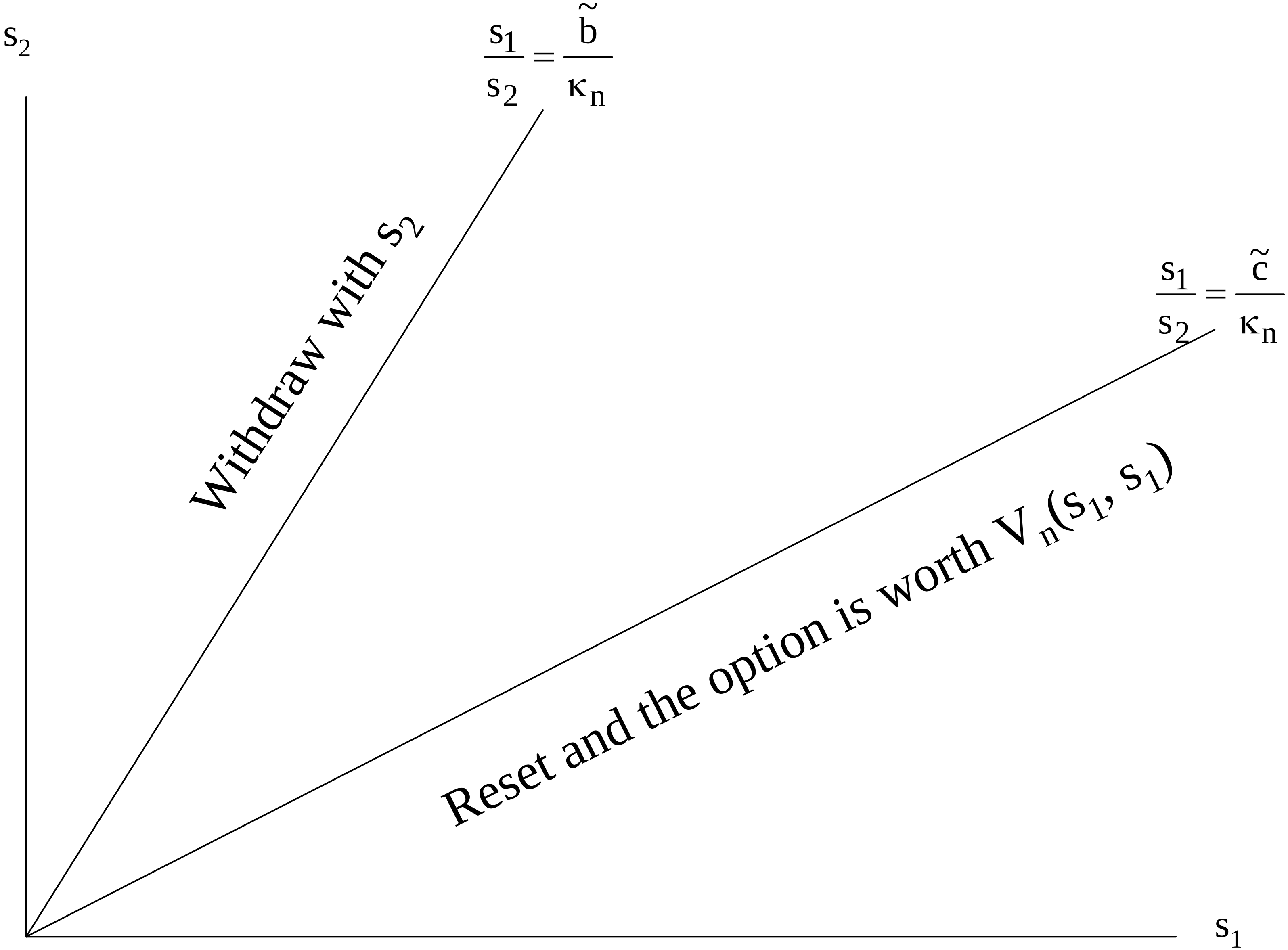
Then,

$$V_{n+1}(s_1, s_2)$$

$$= \sup_T E^*[e^{-rT} \max\{\kappa_n S_1(T), S_2(T)\} \mid S_1(0) = s_1, S_2(0) = s_2]$$

$$= \sup_T E^*[e^{-rT} \max\{S_1(T), S_2(T)\} \mid S_1(0) = \kappa_n s_1, S_2(0) = s_2]$$

$$= W(\kappa_n s_1, s_2).$$



$$V_{n+1}(s_1, s_2) = W(\kappa_n s_1, s_2)$$

$$V_{n+1}(s_1, s_2) = W(\kappa_n s_1, s_2)$$

Put $s_1 = s_2 = 1$. Then

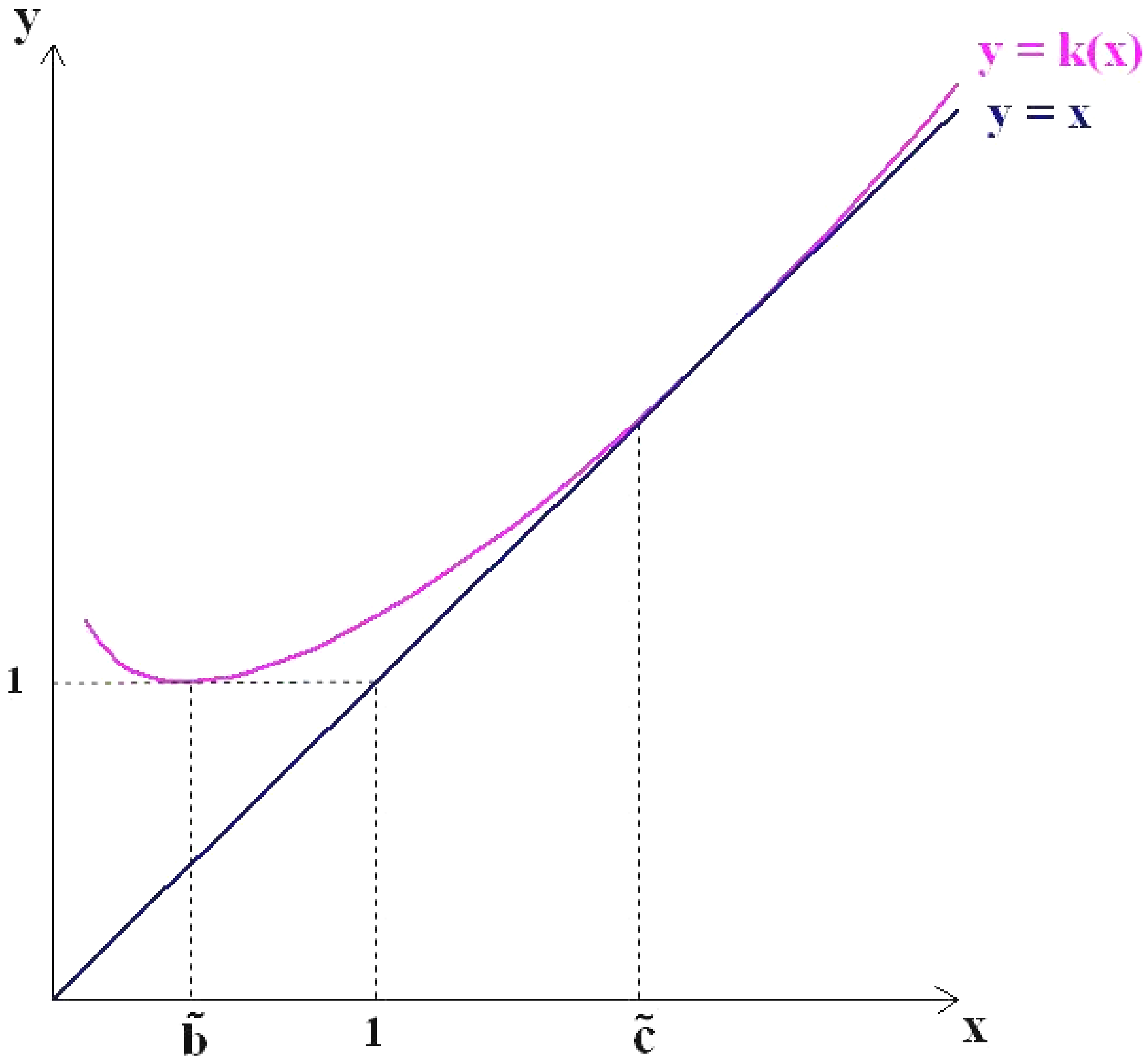
$$V_{n+1}(1, 1) = W(\kappa_n, 1),$$

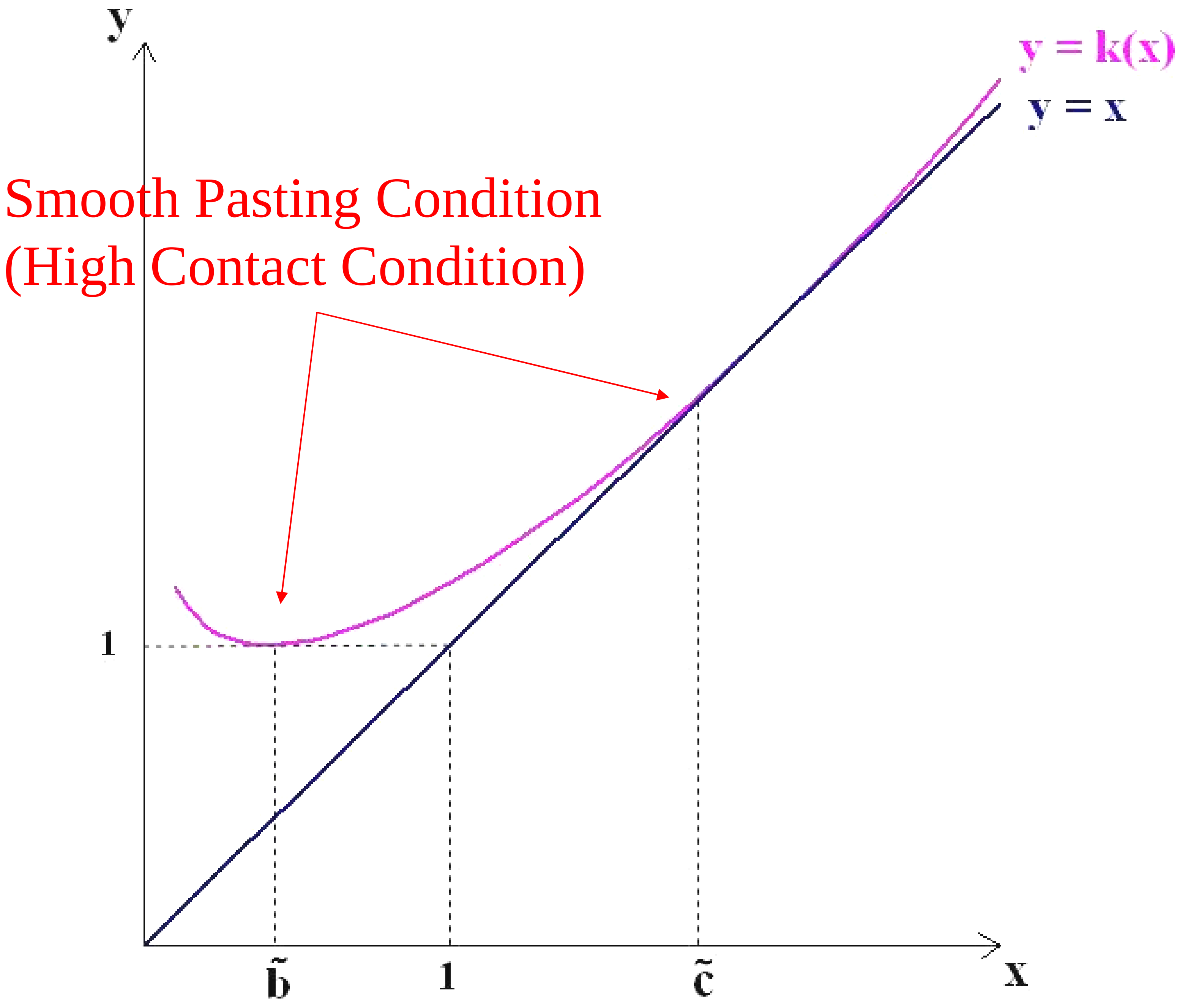
or

$$\kappa_{n+1} = W(\kappa_n, 1)$$
$$= k(\kappa_n),$$

where

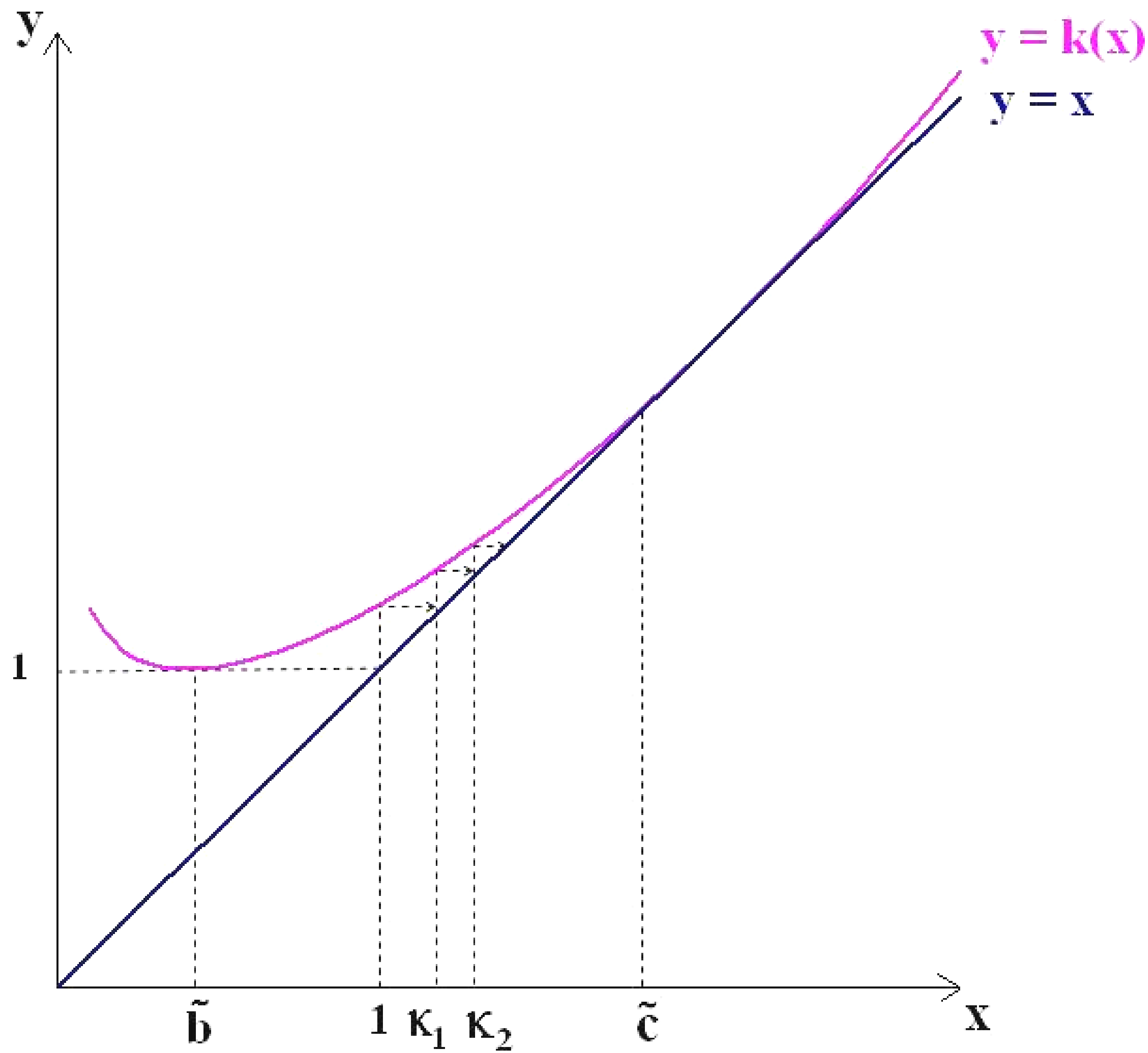
$$k(x) = \frac{\theta_2 (x/\tilde{b})^{\theta_1} - \theta_1 (x/\tilde{b})^{\theta_2}}{\theta_2 - \theta_1} \qquad x > 0.$$


y
y = k(x)
y = x
1
b̃
1
c̃
x

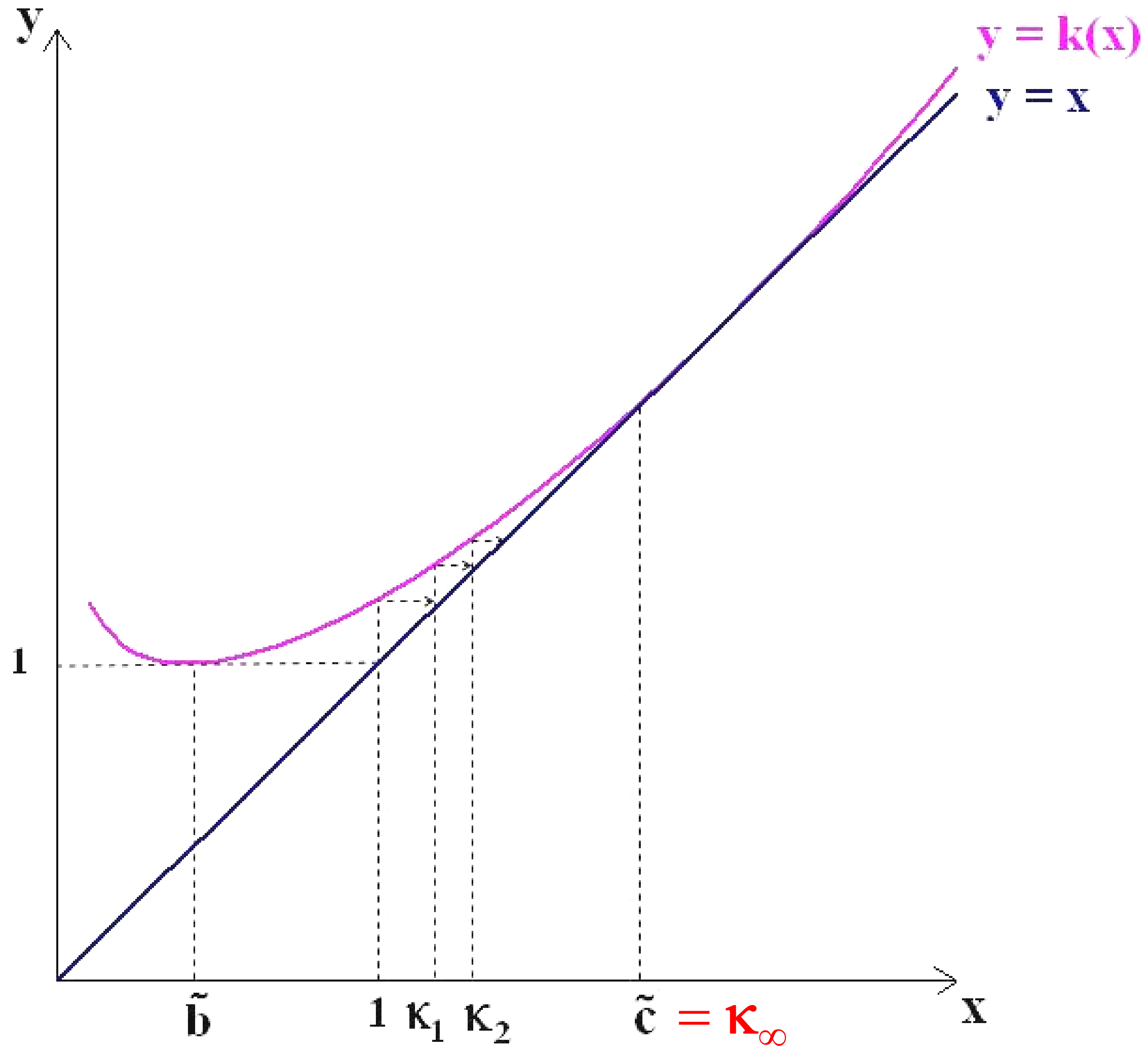


Smooth Pasting Condition
(High Contact Condition)
y
y = k(x)
y = x
1
x
$\tilde{b}$
1
$\tilde{c}$

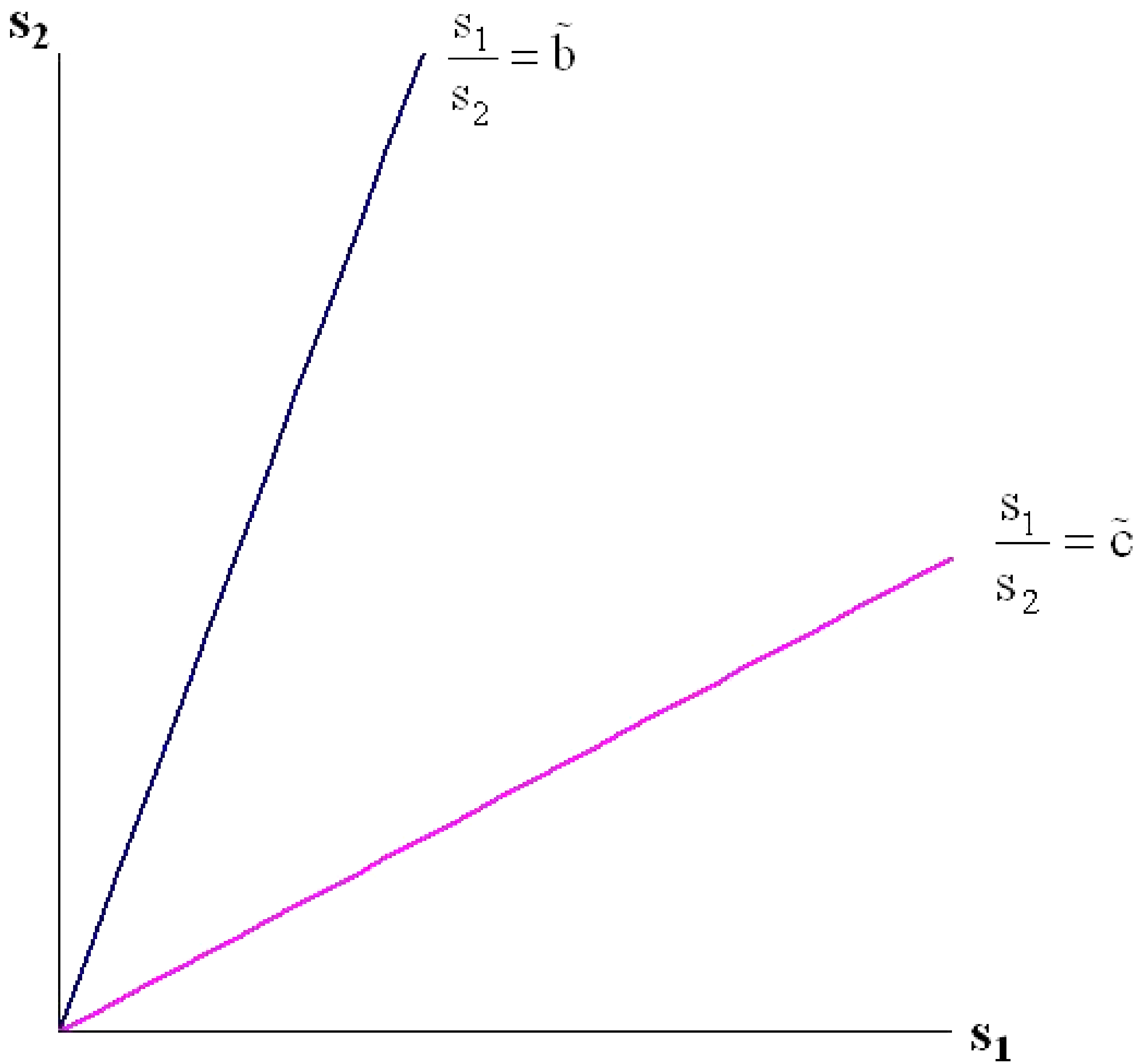
$$
\begin{aligned}
\kappa_1 &= W(1, 1) = k(1) \\
\kappa_2 &= k(\kappa_1) \\
\kappa_3 &= k(\kappa_2) \\
&\ \vdots \\
\kappa_{n+1} &= k(\kappa_n), \\
&\ \vdots
\end{aligned}
$$



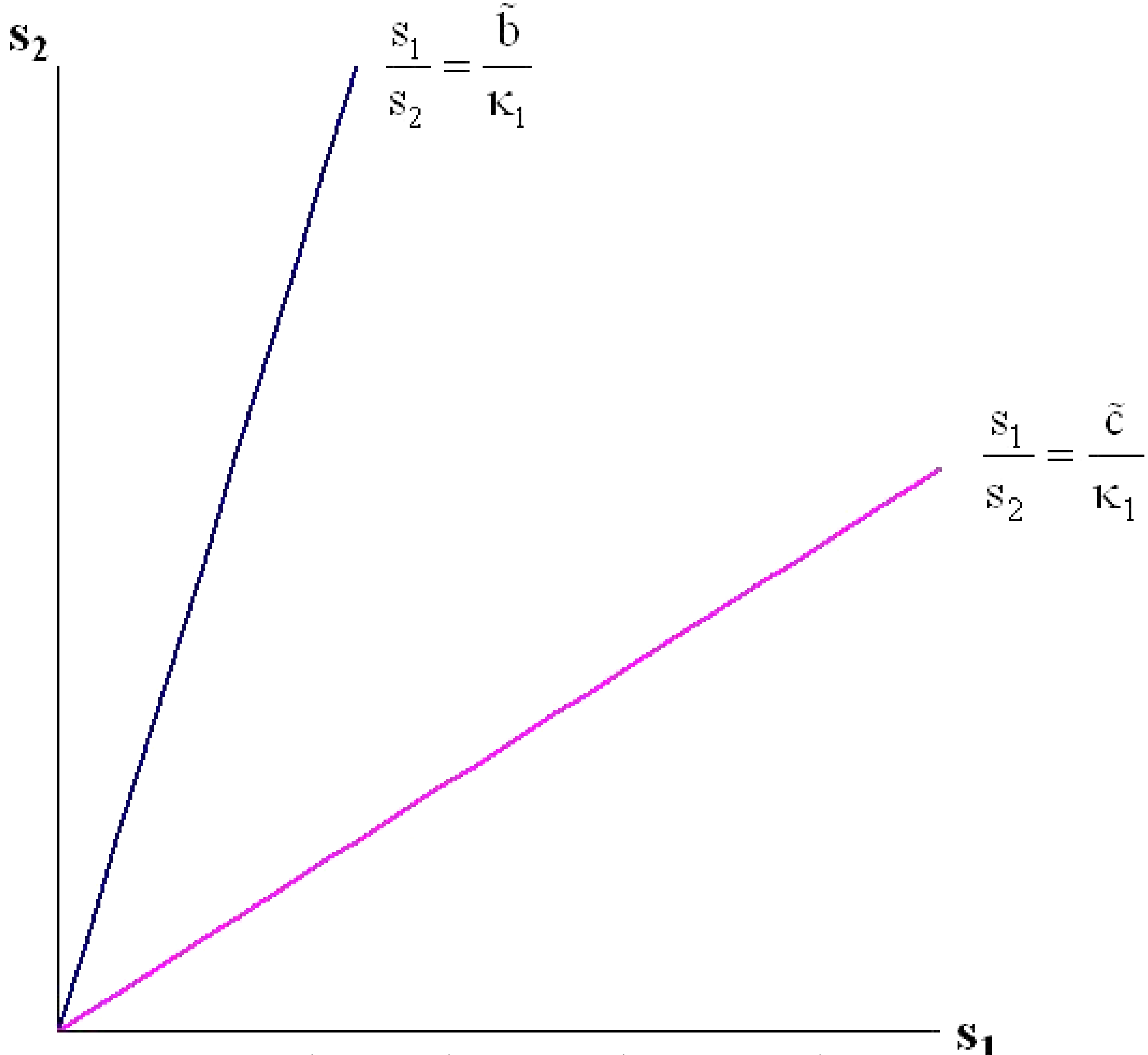
y
y = k(x)
y = x
1
x
b̃
1
κ1
κ2
c̃



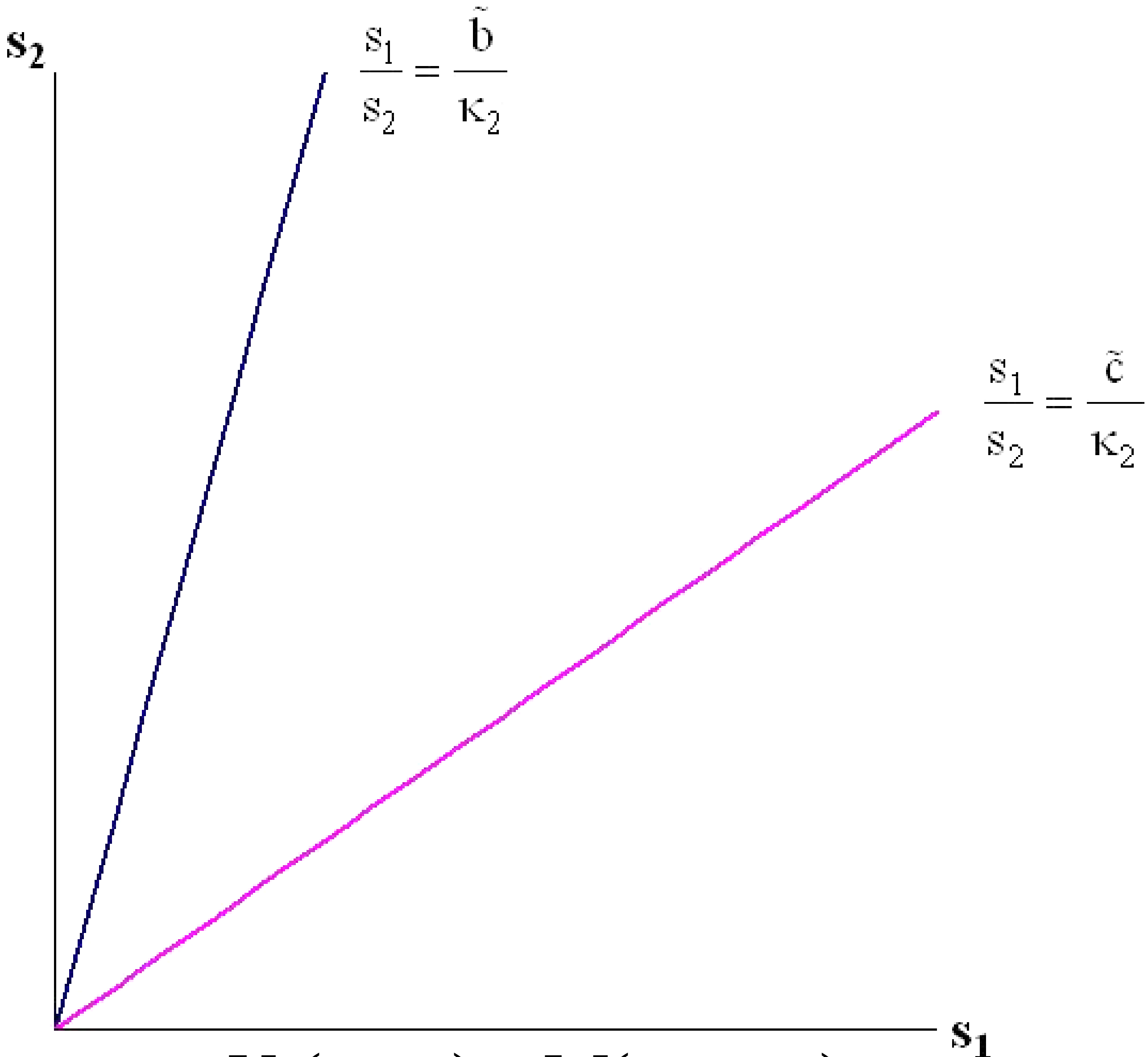
y
y = k(x)
y = x
1
x
$\tilde{b}$
1
κ_1
κ_2
$\tilde{c} = \kappa_\infty$



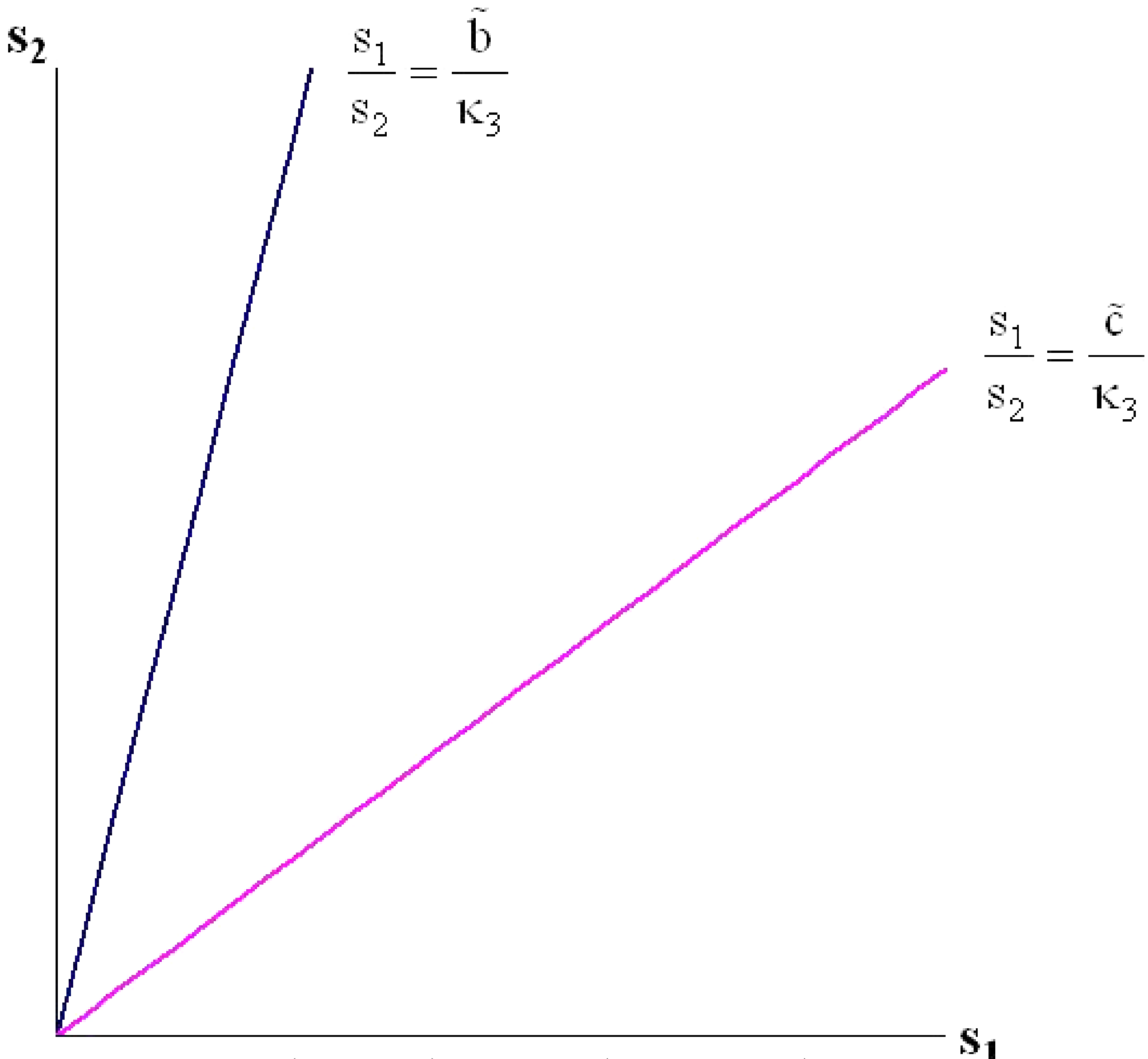
$V_1(s_1, s_2) = W(s_1, s_2)$



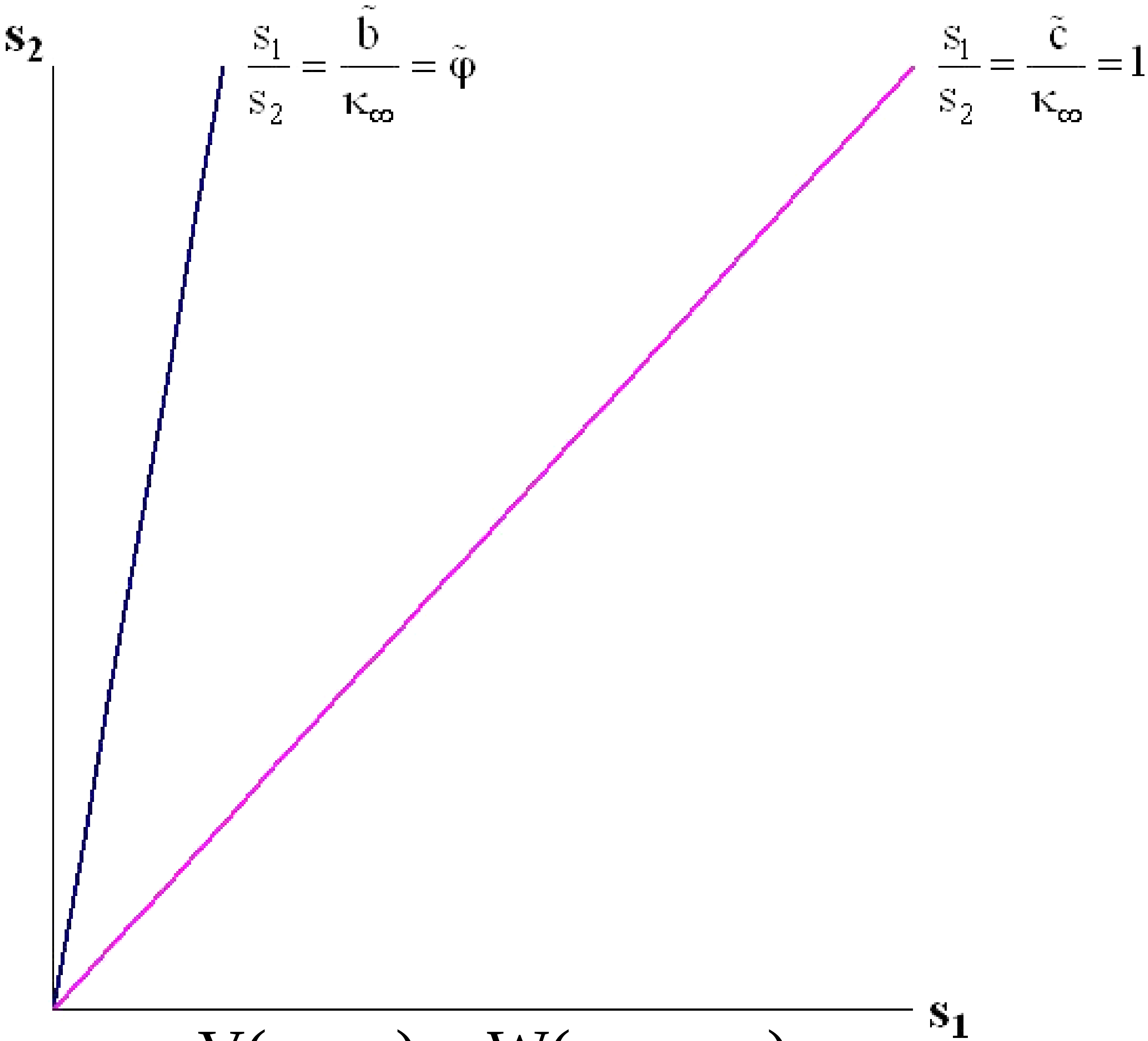
$$V_2(s_1, s_2) = W(\kappa_1 s_1, s_2)$$



$$V_3(s_1, s_2) = W(\kappa_2 s_1, s_2)$$



$$V_4(s_1, s_2) = W(\kappa_3 s_1, s_2)$$



$$V(s_1, s_2) = W(\kappa_\infty s_1, s_2)$$

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Thank you for your patience

Time for lunch