PRICING PERPETUAL FUND PROTECTION WITH WITHDRAWAL OPTION

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ABSTRACT

Consider an American option that provides the amount

$$F(t) = S_2(t) \max\left\{1, \max_{0 \leq \tau \leq t} \frac{S_1(\tau)}{S_2(\tau)}\right\},$$

if it is exercised at time t, $t \geq 0$. For simplicity of language, we interpret $S_1(t)$ and $S_2(t)$ as the prices of two stocks. The option payoff is guaranteed not to fall below the price of stock 1 and is indexed by the price of stock 2 in the sense that, if $F(t) > S_1(t)$, the instantaneous growth rate of $F(t)$ is that of $S_2(t)$. We call this option the dynamic fund protection option. For the two stock prices, the bivariate Black-Scholes model with constant dividend-yield rates is assumed. In the case of a perpetual option, closed-form expressions for the optimal exercise strategy and the price of the option are given. Furthermore, this price is compared with the price of the perpetual maximum option, and it is shown that the optimal exercise of the maximum option occurs before that of the dynamic fund protection option.

   Two general concepts in the theory of option pricing are illustrated: the smooth pasting condition and the construction of the replicating portfolio. The general result can be applied to two special cases. One is where the guaranteed level $S_1(t)$ is a deterministic exponential or constant function. The other is where $S_2(t)$ is an exponential or constant function; in this case, known results concerning the pricing of Russian options are retrieved. Finally, we consider a generalization of the perpetual lookback put option that has payoff $[F(t) - \kappa S_1(t)]$, if it is exercised at time t. This option can be priced with the same technique.

1. INTRODUCTION

Equity-indexed annuities (EIAs) can be viewed as mutual funds wrapped around with various guarantees. An overview of EIAs can be found in the Society of Actuaries study note by Mitchell and Slater (1996), a Task Force Report of the American Academy of Actuaries (AAA 1997), and the book by Streiff and DiBiase (1999). In the context of the classical Black-Scholes (1973) model, Tiong (2000a, b) and Lee (2002) have discussed the pricing of some such guarantees.

   In a recent paper in this journal, Gerber and Pafumi (2000) have proposed the concept of a dynamic guarantee or protection, which is applicable to EIA products. The *primary* (or *naked*) fund is replaced by a *protected* (or *upgraded*) fund. The guarantee is that the value of the protected fund does not fall below a guaranteed level or floor at all times before the maturity date of the contract. Assuming a geometric Brownian motion for the primary fund, Gerber and Pafumi (2000) derived a closed-form formula for pricing this dynamic guarantee if early withdrawal from the fund (before maturity) is not permitted. In a previous paper in this journal, Imai and Boyle (2001, Section 3) suppose that an early

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withdrawal from the fund is possible and then show that it is optimal not to exercise this early withdrawal option and to cash in the fund accumulation only at maturity. However, their conclusion is based on a crucial assumption. It is assumed that all dividends are reinvested in the primary fund, so that the discounted value of a unit of the primary fund is a martingale with respect to the risk-neutral probability measure (which is the probability measure used for pricing purposes). Without this assumption, it may very well be optimal to exercise the withdrawal option early, that is, to cash in the fund accumulation before maturity. In Section 6 we shall give an explicit illustration of this in the case of a perpetual option.

This paper studies the pricing of dynamic fund protection without maturity date. The investor chooses the date to cash in the fund accumulation. Furthermore, the guaranteed level is not necessarily constant or exponential but can be stochastic, such as that given by a stock price or stock index.

For simplicity of language, let us explain the general problem considered in this paper in terms of stocks. For $t \geq 0$, let $S_1(t)$ and $S_2(t)$ be the time-t prices of two stocks. Assuming $S_1(0) \leq S_2(0)$, we consider a security that gives its owner the amount

$$F(t) = S_2(t) \max\left\{1, \max_{0 \leq \tau \leq t} \frac{S_1(\tau)}{S_2(\tau)}\right\} \tag{1.1}$$

at a time t of his choice. Hence the security is an American option with a *path-dependent* payoff, and we call it the *dynamic fund protection option.* In the context of dynamic fund protection, $S_2(t)$ is the time-t value of one unit of the primary fund, $S_1(t)$ is the guaranteed level at time t, and the quantity (1.1) is the time-t value of the protected fund. What is the price of this option, and what is the optimal early exercise strategy? Under the assumption that the option is *perpetual,* that is, it is an American option without an expiration date, we present explicit answers in Section 2 and their proof in Section 3. Furthermore, detailed explanation and motivation for the payoff (1.1) are given in Section 2.

Russian options were introduced by Shepp and Shiryaev (1993); they are discussed in Section 10.11 of Panjer et al. (1998). A Russian option can be viewed as a special case of the perpetual dynamic fund protection option. Hence the main results concerning Russian options can be easily retrieved from the results in Section 2; this is done in Section 7. Shepp and Shiryaev's (1993) result created a surprise because it gives a closed-form formula for pricing an American option with a path-dependent payoff.

Section 8 considers a generalization of the perpetual *lookback put option* whose payoff is $[F(t) - \kappa S_1(t)]$, if it is exercised at time t. Here κ is a constant between 0 and 1. This option can be priced with the same technique, generalizing the result in Section 10.12 of Panjer et al. (1998).

Note that the dynamic fund protection option should be distinguished from the *maximum option* (also called an *alternative option* or *greater-of option*), whose payoff is

$$\max\{S_1(t), S_2(t)\} = S_2(t) \max\left\{1, \frac{S_1(t)}{S_2(t)}\right\}, \tag{1.2}$$

if it is exercised at time t. A comparison of (1.2) with (1.1) reveals that the maximum option is a less expensive security. In Section 4 an explicit comparison of the prices of the *perpetual* dynamic fund protection option and the *perpetual* maximum option is provided. This is highlighted by the theorem at the end of Section 4. Furthermore, it is shown that the optimal exercise of the maximum option occurs before that of the dynamic fund protection option.

2. THE GENERAL PROBLEM AND ITS SOLUTION

For $t \geq 0$, let $S_1(t)$ and $S_2(t)$ be the time-t prices of two stocks. We are interested in an option whose payoff is indexed by the price of stock 2 and is guaranteed not to fall below the price of stock 1. Let the option payoff be denoted by $F(t)$, if the option is exercised at time t. We start with

$$F(0) = S_2(0), \tag{2.1}$$

and assume that

$$S_2(0) \geq S_1(0). \tag{2.2}$$

Whenever $F(t) > S_1(t)$, the instantaneous rate of growth of $F(t)$ is that of $S_2(t)$,

$$\frac{dF(t)}{F(t)} = \frac{dS_2(t)}{S_2(t)}. \tag{2.3}$$

And $\{S_1(t)\}$ is a barrier for $\{F(t)\}$ so that $F(t) \geq S_1(t)$ at all times. This leads to formula (1.1).

For a more precise derivation of (1.1), consider a contract that would provide a sufficient number of units of stock 2 so that the total value of these units is at least the value of one unit of stock 1 at any time. Let $n(\tau)$ denote the aggregate number of units of stock 2 at time τ, $\tau \geq 0$. The following three conditions must be satisfied:

1. $n(0) = 1$
2. $n(\tau)$ is a nondecreasing function of τ
3. $n(\tau)S_2(\tau) \geq S_1(\tau)$ for all τ.

Condition 1 merely states that we start with one unit of stock 2, which is condition (2.1). Condition 2 means that additional units can be credited, but they can never be taken away afterwards. Condition 3 is the guarantee. From conditions 2 and 3, it follows that

$$n(t) \geq n(\tau) \geq \frac{S_1(\tau)}{S_2(\tau)}, \quad \text{for } 0 \leq \tau \leq t,$$

and, hence,

$$n(t) \geq \max_{0 \leq \tau \leq t} \frac{S_1(\tau)}{S_2(\tau)}.$$

Because of condition 1 we have

$$n(t) \geq \max\left\{1, \max_{0 \leq \tau \leq t} \frac{S_1(\tau)}{S_2(\tau)}\right\}. \tag{2.4}$$

Evidently there is an infinite number of functions $n(t)$ that satisfy these three conditions. To obtain the guarantee with the least cost, we choose the smallest such function, that is, the one with equality in (2.4):

$$n(t) = \max\left\{1, \max_{0 \leq \tau \leq t} \frac{S_1(\tau)}{S_2(\tau)}\right\}. \tag{2.5}$$

Then the value of the aggregate units at time t, $n(t)S_2(t)$, is the right-hand side of formula (1.1). Formulas (2.5) and (1.1) are illustrated in the second and third panels of Figure 1.

A main goal in this paper is to price the perpetual option with payoff $F(t)$ given by formula (1.1). We follow the classical Black-Scholes–type model. We assume a constant risk-free force of interest $r > 0$. For $j = 1, 2$, let

$$S_j(t) = S_j(0)e^{X_j(t)}, \quad t \geq 0. \tag{2.6}$$

It is assumed that $\{X_1(t), X_2(t)\}$ is a bivariate Wiener process in the Q-measure (the probability measure that is used for pricing derivatives), with correlation ρ $(-1 < \rho < 1)$, instantaneous variances σ_1^2 and σ_2^2, and drift parameters

Figure 1
Definitions of $n(t)$ and $F(t)$
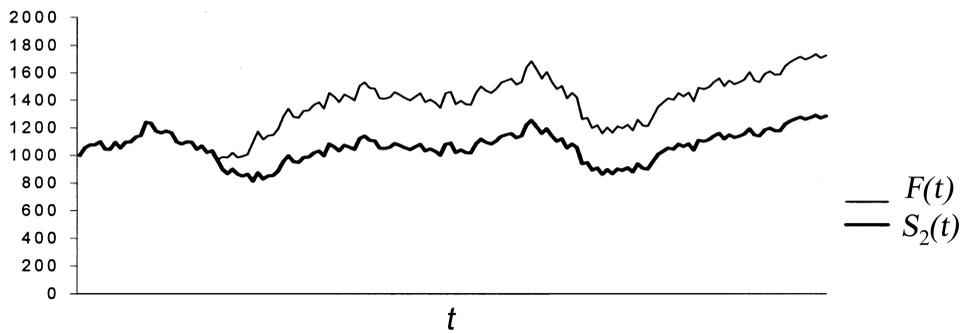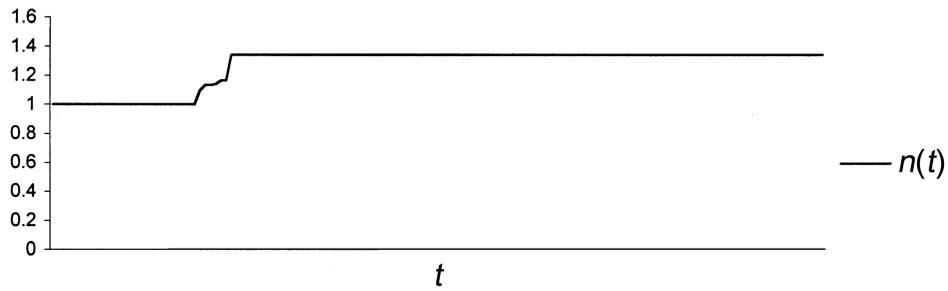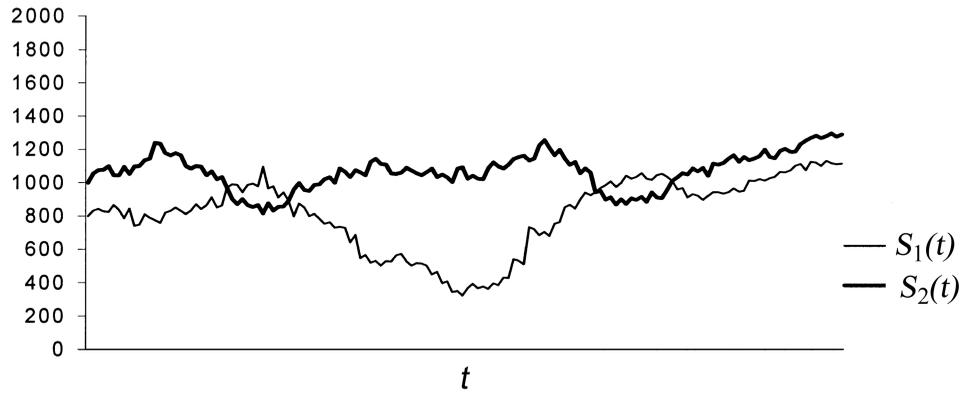
$$\mu_j = r - \frac{\sigma_j^2}{2} - \zeta_j, \quad j = 1, 2. \tag{2.7}$$

We assume that ζ_1 and ζ_2 are positive. The constant ζ_j can be interpreted as the dividend-yield rate of stock j; that is, one may consider that dividends of amount $\zeta_j S_j(t)dt$ are paid between time t and time $t + dt$. However, this interpretation of the constants ζ_1 and ζ_2 is not needed in the analysis below, except for Section 5 where replicating portfolios are discussed.

The option is perpetual. Thus its time-0 price is

$$V(s_1, s_2) = \sup_T \mathrm{E}[e^{-rT}F(T)], \tag{2.8}$$

where T is any exercise time, $F(T)$ is given by (1.1), and $s_j = S_j(0), j = 1, 2$. The *stopping time T* for which the maximum is attained is called the *optimal exercise strategy*. As explained in the Appendix, the optimal exercise strategy is one of the form

$$T_\varphi = \min\{t|S_1(t) = \varphi F(t)\} \tag{2.9}$$

with $0 < \varphi < 1$. That is, the option is exercised as soon as the ratio $S_1(t)/F(t)$ falls to the level φ.

Let

$$V(s_1, s_2; \varphi) = \mathrm{E}[e^{-rT_\varphi}F(T_\varphi)], \quad \varphi s_2 \le s_1 \le s_2 \tag{2.10}$$

denote the value of such a strategy, with $s_j = S_j(0), j = 1, 2$. In Section 3 we shall show that

$$V(s_1, s_2; \varphi) = \frac{h(s_1/s_2)}{h(\varphi)} s_2, \tag{2.11}$$

where

$$h(z) = (\theta_2 - 1)z^{\theta_1} + (1 - \theta_1)z^{\theta_2}, \quad z > 0. \tag{2.12}$$

Here θ_1 and θ_2 are the solutions of the quadratic equation

$$0 = -r + \mathrm{E}[\theta X_1(1) + (1 - \theta)X_2(1)] + \frac{1}{2}\mathrm{Var}[\theta X_1(1) + (1 - \theta)X_2(1)]$$

$$= -r + \mu_1\theta + \mu_2(1 - \theta) + \frac{\sigma_1^2}{2}\theta^2 + \frac{\sigma_2^2}{2}(1 - \theta)^2 + \rho\sigma_1\sigma_2\theta(1 - \theta), \tag{2.13}$$

with μ_1 and μ_2 given by (2.7). Note that, because of (2.7), the expression on the right-hand side of (2.13) equals $-\zeta_2$ for $\theta = 0$ and equals $-\zeta_1$ for $\theta = 1$. Hence, one solution of (2.13) is negative, and the other is greater than one, say, $\theta_1 < 0, \theta_2 > 1$. As a check, we observe that

$$V(\varphi s_2, s_2; \varphi) = \frac{h(\varphi)}{h(\varphi)} s_2 = s_2. \tag{2.14}$$

Note that $h(z) \to \infty$ for $z \to \infty$ and for $z \to 0$. Furthermore,

$$h''(z) = (\theta_2 - 1)(1 - \theta_1)(-\theta_1 z^{\theta_1 - 2} + \theta_2 z^{\theta_2 - 2}) > 0.$$

Hence, the graph of the function $h(z), z > 0$, is U-shaped, and the function $h(z)$ has a unique minimum. Let $\breve{\varphi}$ denote the optimal value of φ, that is, the value that maximizes the right-hand side of (2.11) as a function of φ, or, equivalently, that minimizes its denominator $h(\varphi)$. This leads to

$$h'(\breve{\varphi}) = 0, \tag{2.15}$$

or

$$\breve{\varphi} = \left(\frac{-\theta_1(\theta_2 - 1)}{\theta_2(1 - \theta_1)}\right)^{1/(\theta_2 - \theta_1)}. \tag{2.16}$$

The number $\breve{\varphi}$ is between 0 and 1 because both $-\theta_1/(1-\theta_1)$ and $(\theta_2-1)/\theta_2$ are between 0 and 1. The optimal exercise strategy is to exercise the option the first time when $S_1(t) = \breve{\varphi}F(t)$ if $s_1 > \breve{\varphi}s_2$, and to exercise it immediately if $s_1 \leq \breve{\varphi}s_2$. Hence, the *price* of the option is

$$V(s_1, s_2) = \begin{cases} \dfrac{h(s_1/s_2)}{h(\breve{\varphi})} \, s_2 & \text{if } \breve{\varphi} < \dfrac{s_1}{s_2} \leq 1 \\[2ex] s_2 & \text{if } 0 < \dfrac{s_1}{s_2} \leq \breve{\varphi} \end{cases}, \tag{2.17}$$

with $h(\cdot)$ given by (2.12). In Section 4 we shall indicate an alternative expression for the price of the option.

The payoff of the option is path dependent, but with a simple structure: If the option has not been exercised by time t, $t \geq 0$, then $\breve{\varphi} < S_1(t)/F(t) \leq 1$, and the only relevant information for the future consists of the values of $S_1(t)$ and $F(t)$. The time-t option price follows immediately from (2.17):

$$\frac{h(S_1(t)/F(t))}{h(\breve{\varphi})} \, F(t). \tag{2.18}$$

Furthermore, the amount

$$\left[\frac{h(S_1(t)/F(t))}{h(\breve{\varphi})} - 1 \right] F(t) \tag{2.19}$$

is the time-t value for having the possibility to cash in the funds at a future date of the investor's choice.

Remarks

We have assumed that $\zeta_1 > 0$ and $\zeta_2 > 0$. Consider what happens when one of these conditions is violated:

(a) If $\zeta_1 > 0$ and $\zeta_2 = 0$, it follows that $\theta_1 = 0$ and $\theta_2 > 1$. Hence, from (2.11) and (2.12),

$$V(s_1, s_2; \varphi) = \frac{\theta_2 - 1 + (s_1/s_2)^{\theta_2}}{\theta_2 - 1 + \varphi^{\theta_2}} \, s_2. \tag{2.20}$$

The supremum is obtained for $\varphi = 0$. It is

$$V(s_1, s_2; 0) = s_2 + \frac{s_2}{\theta_2 - 1} \left(\frac{s_1}{s_2} \right)^{\theta_2} = s_2 + \frac{s_1}{R} \left(\frac{s_1}{s_2} \right)^{R}, \tag{2.21}$$

with

$$R = \theta_2 - 1 = \frac{2\zeta_1}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}. \tag{2.22}$$

In this case there is no optimal withdrawal strategy. This phenomenon can be explained as follows. Because stock 2 does not pay any dividends, there is no incentive for early withdrawal (cf. Imai and Boyle 2001, Section 3). On the other hand, in order to take advantage of the guarantee, the option is exercised as late as possible, that is, . . . never. Ingersoll (1987, p. 373) calls this type of situation the "problem of infinities." Expression (2.21) should be interpreted as the sum of the price for one share of stock 2 and the price for perpetual dynamic fund protection. In fact, formula (2.21) generalizes formula (2.13) of Gerber and Pafumi (2000). Note that the denominator in (2.22) is $\mathrm{Var}[X_1(1) - X_2(1)]$.

(b) If $\zeta_1 = 0$ and $\zeta_2 \geq 0$, it follows that $\theta_1 \leq 0$ and $\theta_2 = 1$. Hence, from (2.11) and (2.12),

$$V(s_1, s_2; \varphi) = \frac{s_1}{\varphi}. \tag{2.23}$$

In this case the supremum is obviously infinite. Formula (2.23) can be directly obtained from formula (2.10), which, by (2.9), is

$$V(s_1, s_2; \varphi) = \frac{E[e^{-rT_\varphi}S_1(T_\varphi)]}{\varphi}. \tag{2.24}$$

With $\zeta_1 = 0$, $\{e^{-rt}S_1(t)\}$ is a martingale. Hence the expectation in (2.24) is s_1 by the *optional sampling theorem;* this proves (2.23).

3. DERIVATION OF FORMULA (2.11)

The function $V(s_1, s_2; \varphi)$ can be evaluated by means of martingales as in Gerber and Shiu (1994, 1996a, 1997). First note that, because $\{S_1(t)\}$ and $\{S_2(t)\}$ are geometric Brownian motions, the function $V(s_1, s_2; \varphi)$, defined by (2.10), is homogeneous of degree 1, that is, for each $z > 0$,

$$V(zs_1, zs_2; \varphi) = zV(s_1, s_2; \varphi), \quad \varphi s_2 \leq s_1 \leq s_2. \tag{3.1}$$

Consider

$$T = \min\{t | S_1(t) = \varphi S_2(t) \text{ or } S_1(t) = S_2(t)\}, \tag{3.2}$$

the first time when the ratio of prices, $S_1(t)/S_2(t)$, attains the value φ or 1 (see Figure 2).

By conditioning on when the ratio first attains the value φ or 1, we can rewrite the expectation (2.10) as

$$V(s_1, s_2; \varphi) = E[e^{-rT}S_2(T)I(S_1(T) = \varphi S_2(T))]$$

$$+ E[e^{-rT}V(S_2(T), S_2(T); \varphi)I(S_1(T) = S_2(T))]$$

$$= E[e^{-rT}S_2(T)I(S_1(T) = \varphi S_2(T))]$$

$$+ E[e^{-rT}S_2(T)I(S_1(T) = S_2(T))]V(1, 1; \varphi) \tag{3.3}$$

by (3.1) with $z = S_2(T)$. Here $I(\cdot)$ denotes the indicator function of an event. This suggests the following definitions:

$$A(s_1, s_2; \varphi) = E[e^{-rT}S_2(T)I(S_1(T) = \varphi S_2(T))] \tag{3.4}$$

and

$$B(s_1, s_2; \varphi) = E[e^{-rT}S_2(T)I(S_1(T) = S_2(T))]. \tag{3.5}$$

Then formula (3.3) becomes

$$V(s_1, s_2; \varphi) = A(s_1, s_2; \varphi) + B(s_1, s_2; \varphi)V(1, 1; \varphi). \tag{3.6}$$
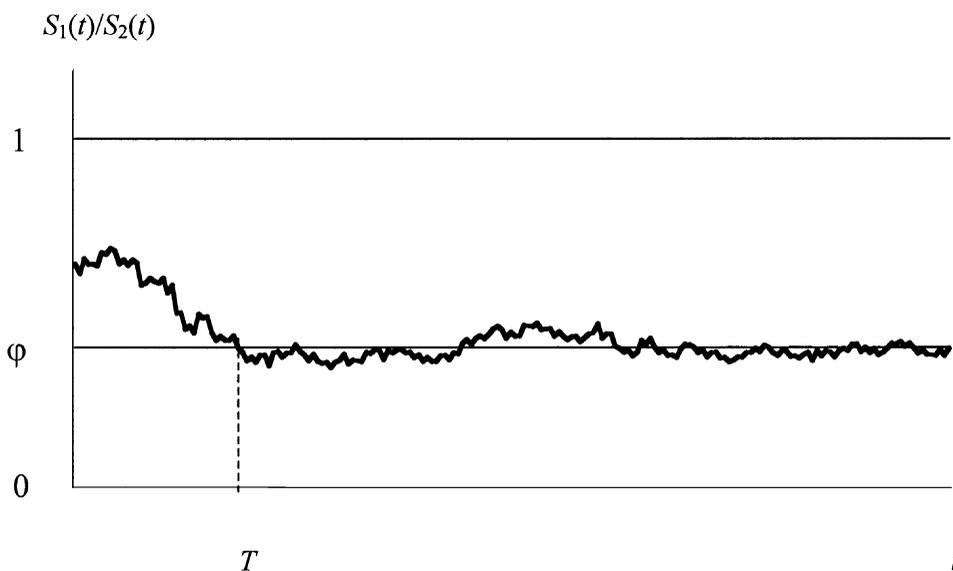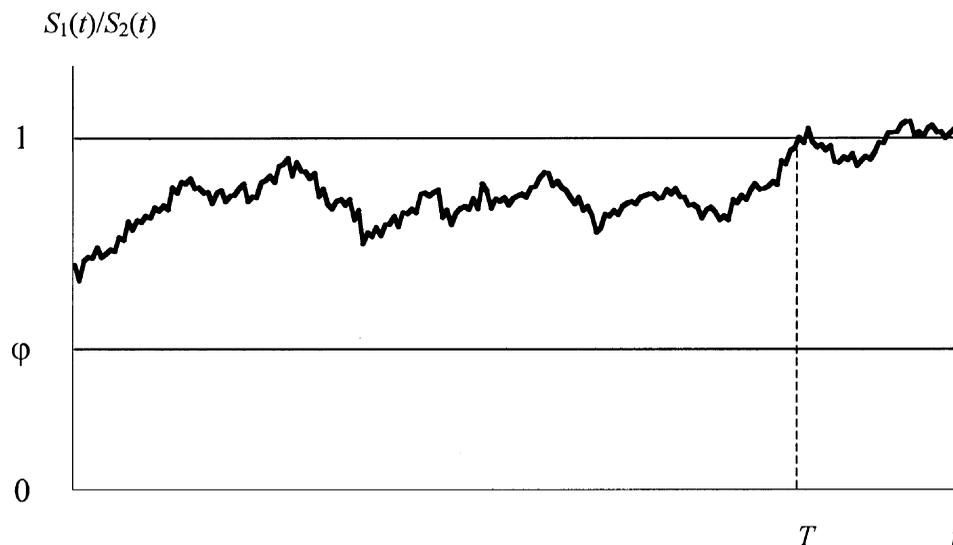
To get closed-form expressions for $A(s_1, s_2; \varphi)$ and $B(s_1, s_2; \varphi)$, we seek θ so that the stochastic process

$$\{e^{-rt}S_2(t)[S_1(t)/S_2(t)]^\theta\} \tag{3.7}$$

becomes a martingale. Equivalently, we seek θ so that

$$\{e^{-rt+\theta X_1(t)+(1-\theta)X_2(t)}\} \tag{3.8}$$

Figure 2
Definition of *T* in Formula (3.2)

is a martingale, which is the case if

$$e^{-r} \, \mathrm{E}[e^{\theta X_1(1)+(1-\theta)X_2(1)}] = 1. \tag{3.9}$$

This, in turn, leads to the quadratic equation (2.13), which has two solutions, $\theta_1 < 0$ and $\theta_2 > 1$. With $\theta = \theta_j$, the stochastic process (3.7) is a martingale; if we stop it at time T and apply the optional sampling theorem, we obtain

$$s_1^{\theta_j} s_2^{1-\theta_j} = \mathrm{E}[e^{-rT}S_2(T)[S_1(T)/S_2(T)]^{\theta_j}] = A(s_1, s_2; \varphi)\varphi^{\theta_j} + B(s_1, s_2; \varphi), \quad j = 1, 2. \tag{3.10}$$

These are two linear equations for A and B. Their solution is

$$A(s_1, s_2; \varphi) = \frac{s_1^{\theta_1} s_2^{1-\theta_1} - s_1^{\theta_2} s_2^{1-\theta_2}}{\varphi^{\theta_1} - \varphi^{\theta_2}}, \tag{3.11}$$

$$B(s_1, s_2; \varphi) = \frac{s_1^{\theta_2} s_2^{1-\theta_2} \varphi^{\theta_1} - s_1^{\theta_1} s_2^{1-\theta_1} \varphi^{\theta_2}}{\varphi^{\theta_1} - \varphi^{\theta_2}}. \tag{3.12}$$

Substituting (3.11) and (3.12) in the right-hand side of (3.6) yields

$$V(s_1, s_2; \varphi) = \frac{(s_1^{\theta_1} s_2^{1-\theta_1} - s_1^{\theta_2} s_2^{1-\theta_2}) + (s_1^{\theta_2} s_2^{1-\theta_2} \varphi^{\theta_1} - s_1^{\theta_1} s_2^{1-\theta_1} \varphi^{\theta_2})V(1, 1; \varphi)}{\varphi^{\theta_1} - \varphi^{\theta_2}}. \tag{3.13}$$

To determine $V(1, 1; \varphi)$, we use the condition

$$\left. \frac{\partial V(s_1, s_2; \varphi)}{\partial s_2} \right|_{s_1 = s_2} = 0. \tag{3.14}$$

(An intuitive explanation of this condition is that, when s_2 is "close" to s_1, the guarantee will be used instantaneously so the value of V is unaffected by marginal changes in s_2. For a rigorous derivation of a similar condition, see Goldman, Sosin, and Gatto 1979.) Differentiating (3.13) with respect to s_2 and applying (3.14), we obtain the equation

$$0 = [(1 - \theta_1) - (1 - \theta_2)] + [(1 - \theta_2)\varphi^{\theta_1} - (1 - \theta_1)\varphi^{\theta_2}]V(1, 1; \varphi) = \theta_2 - \theta_1 - h(\varphi)V(1, 1; \varphi),$$

where $h(\varphi)$ is defined by formula (2.12). Hence,

$$V(1, 1; \varphi) = \frac{\theta_2 - \theta_1}{h(\varphi)}. \tag{3.15}$$

Finally, we substitute (3.15) in (3.13) to obtain (2.11) after some simplifications.

4. COMPARISON WITH THE PERPETUAL MAXIMUM OPTION

By comparing (1.1) with (1.2), we have seen that the maximum option is cheaper than the dynamic fund protection option. If the options are perpetual, an analytical comparison of the prices is possible.

Gerber and Shiu (1996a, 1997) have shown that the price of the perpetual maximum option is

$$W(s_1, s_2) = \begin{cases} s_2 & \text{if } \dfrac{s_1}{s_2} \le \breve{b} \\[2mm] s_2\, g\left(\dfrac{s_1}{\breve{b} s_2}\right) & \text{if } \breve{b} < \dfrac{s_1}{s_2} < \breve{c}, \\[2mm] s_1 & \text{if } \dfrac{s_1}{s_2} \ge \breve{c} \end{cases} \tag{4.1}$$

where

$$g(x) = \frac{\theta_2 x^{\theta_1} - \theta_1 x^{\theta_2}}{\theta_2 - \theta_1}, \quad x > 0, \tag{4.2}$$

and the endpoints of the optimal continuation (nonexercise) interval are

$$\breve{b} = \left(\frac{-\theta_1}{1-\theta_1}\right)^{(1-\theta_1)/(\theta_2-\theta_1)} \left(\frac{\theta_2}{\theta_2-1}\right)^{(\theta_2-1)/(\theta_2-\theta_1)}, \tag{4.3}$$

$$\check{c} = \left(\frac{-\theta_1}{1-\theta_1}\right)^{-\theta_1/(\theta_2-\theta_1)} \left(\frac{\theta_2}{\theta_2-1}\right)^{\theta_2/(\theta_2-\theta_1)}. \tag{4.4}$$

Note that

$$\frac{\breve{b}}{\check{c}} = \check{\varphi} \tag{4.5}$$

by (2.16), and that

$$0 < \breve{b} < 1 < \check{c}. \tag{4.6}$$

Thus,

$$0 < \check{\varphi} < \breve{b} < 1. \tag{4.7}$$

Formula (4.1) cannot be compared immediately with (2.17). Therefore, we now derive an alternative expression for $V(s_1, s_2)$. The first-order condition (2.15) is the same as

$$(1 - \theta_1)\theta_2 \check{\varphi}^{\theta_2} = -(\theta_2 - 1)\theta_1 \check{\varphi}^{\theta_1}, \tag{4.8}$$

applying which to (2.12) yields the formulas

$$h(\check{\varphi}) = \frac{\theta_2 - \theta_1}{\theta_2} (\theta_2 - 1)\check{\varphi}^{\theta_1} \tag{4.9}$$

and

$$h(\check{\varphi}) = -\frac{\theta_2 - \theta_1}{\theta_1} (1 - \theta_1)\check{\varphi}^{\theta_2}. \tag{4.10}$$

It follows from (4.9) that

$$\frac{(\theta_2 - 1)z^{\theta_1}}{h(\check{\varphi})} = \frac{\theta_2}{\theta_2 - \theta_1} \left(\frac{z}{\check{\varphi}}\right)^{\theta_1}.$$

Similarly, it follows from formula (4.10) that

$$\frac{(1 - \theta_1)z^{\theta_2}}{h(\check{\varphi})} = \frac{-\theta_1}{\theta_2 - \theta_1} \left(\frac{z}{\check{\varphi}}\right)^{\theta_2}.$$

By (2.12) the sum of the left-hand sides of the last two formulas is $h(z)/h(\check{\varphi})$, while by (4.2) the sum of the right-hand sides is $g(z/\check{\varphi})$. Thus, we have

$$\frac{h(z)}{h(\check{\varphi})} = g\left(\frac{z}{\check{\varphi}}\right), \quad z > 0. \tag{4.11}$$

From (4.11) and (2.17) we obtain the alternative expression for the price of the perpetual dynamic fund protection option,

$$V(s_1, s_2) = \begin{cases} s_2 & \text{if } \dfrac{s_1}{s_2} \le \check{\varphi} \\[2mm] s_2\, g\left(\dfrac{s_1}{\check{\varphi} s_2}\right) & \text{if } \dfrac{s_1}{s_2} > \check{\varphi} \end{cases}. \tag{4.12}$$

This formula can be compared directly with (4.1). We see that the differences of the prices is

$$V(s_1, s_2) - W(s_1, s_2) = \begin{cases} 0 & \text{if } \dfrac{s_1}{s_2} \leq \check{\varphi} \\[2ex] s_2\left[g\left(\dfrac{s_1}{\check{\varphi}s_2}\right) - 1\right] & \text{if } \check{\varphi} < \dfrac{s_1}{s_2} \leq \check{b}. \\[2ex] s_2\left[g\left(\dfrac{s_1}{\check{\varphi}s_2}\right) - g\left(\dfrac{s_1}{\check{b}s_2}\right)\right] & \text{if } \check{b} < \dfrac{s_1}{s_2} \leq 1 \end{cases} \tag{4.13}$$

Both differences in the second and third lines are strictly positive. To verify this, observe that $g(x)$, $x > 0$, as defined by (4.2), is a positive convex function, with minimum value 1 attained for $x = 1$.

There is an ordering between the optimal exercise times: If the perpetual dynamic fund protection option and the perpetual maximum option are exercised optimally, the former is always exercised after the latter. To show this, suppose that it has not been optimal to exercise the maximum option by time t, that is,

$$\check{b} < \frac{S_1(\tau)}{S_2(\tau)} < \check{c}, \quad \text{for } 0 \leq \tau \leq t. \tag{4.14}$$

It follows from this, (1.1), and $\check{c} > 1$ that

$$F(\tau) < \check{c}S_2(\tau), \quad \text{for } 0 \leq \tau \leq t, \tag{4.15}$$

and, hence,

$$\frac{S_1(\tau)}{F(\tau)} > \frac{S_1(\tau)}{\check{c}S_2(\tau)} > \frac{\check{b}}{\check{c}} = \check{\varphi}, \quad \text{for } 0 \leq \tau \leq t \tag{4.16}$$

by (4.14) and (4.5). This shows that the dynamic fund protection option has indeed not been exercised by time t.

Last but not least, there is a surprisingly direct connection between the price of the perpetual dynamic fund protection option and that of the perpetual maximum option.

Theorem

For $\check{\varphi} \leq s_1/s_2 \leq 1$,

$$V(s_1, s_2) = W(\check{c}s_1, s_2). \tag{4.17}$$

Despite its formal simplicity, this result does not seem to have an interpretation. For its proof, we observe that the condition

$$\check{\varphi} \leq s_1/s_2 \leq 1$$

is equivalent to

$$\check{b} \leq \check{c}s_1/s_2 \leq \check{c}$$

by (4.5). Hence, with this condition, it follows from (4.12), (4.5), and (4.1) that

$$V(s_1, s_2) = s_2 g\left(\frac{s_1}{\check{\varphi}s_2}\right) = s_2 g\left(\frac{\check{c}s_1}{\check{b}s_2}\right) = W(\check{c}s_1, s_2),$$

which is (4.17). In the particular case that $s_1 = s_2 = s$, formula (4.17) becomes

$$V(s, s) = s\check{c}. \tag{4.18}$$

Thus, the quantity $\check{c}$ can be interpreted as the price of the perpetual dynamic fund protection option when $s_1 = s_2 = 1$. We shall refer to this result at the end of the next section.

Note that from (4.17) and $\check{c} > 1$, we have

$$V(s_1, s_2) > W(s_1, s_2), \quad \check{\varphi} \leq s_1/s_2 \leq 1, \tag{4.19}$$

illustrating the general fact that the dynamic fund protection option is more expensive than the maximum option.

5. SMOOTH PASTING CONDITION AND REPLICATING PORTFOLIO

Because an explicit expression for the price of the perpetual dynamic fund protection option is available, it can be used to illustrate two general concepts in the theory of option pricing. The first concept is the *smooth pasting condition* or the *principle of smooth fit.* In the finance literature, it is also called the *high contact condition,* a term coined by the Nobel laureate Paul Samuelson (1965). Brekke and Øksendal (1991) proved that, under weak conditions, a possible solution to an optimal stopping problem satisfying the smooth pasting condition is, in fact, an optimal solution to the problem. Shepp and Shiryaev (1993, p. 636) pointed out that the smooth pasting condition was discovered by the great Russian mathematician A. N. Kolmogorov in the 1950s, and it was later independently found by H. Chernoff in the United States and by D. V. Lindley in Great Britain. A brief history of the smooth pasting condition can be found in Dubins, Shepp, and Shiryaev (1993, p. 238); see also Peskir (2001, p. 4).
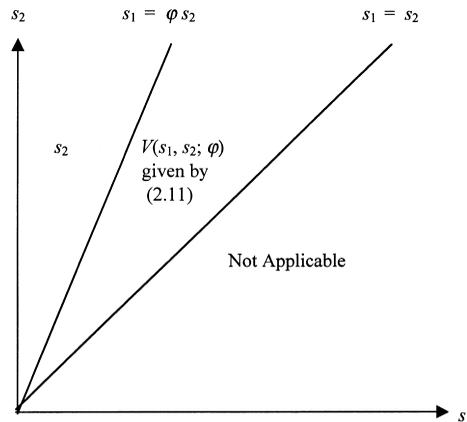
For each φ, $0 < \varphi < 1$, the exercise strategy T_φ gives rise to a value function

$$\begin{cases} V(s_1, s_2; \varphi) & \text{if } \varphi < \dfrac{s_1}{s_2} \leq 1 \\ s_2 & \text{if } 0 < \dfrac{s_1}{s_2} \leq \varphi \end{cases}, \tag{5.1}$$

which is illustrated in Figure 3.

The price function (2.17) is (5.1), with $\varphi = \check{\varphi}$. It follows from (2.14) that the value function (5.1) is continuous along the junction $s_1 = \varphi s_2$. The smooth pasting condition states that the optimal $\check{\varphi}$ is such that the value function (5.1) is continuously differentiable. In other words, $\check{\varphi}$ is such that the gradient of the price function (2.17) is continuous along the junction $s_1 = \check{\varphi} s_2$:

Figure 3
Value Function of Exercise Strategy T_φ

$$\left.\frac{\partial V(s_1, s_2; \check{\varphi})}{\partial s_1}\right|_{s_1=\check{\varphi}s_2} = 0, \tag{5.2}$$

$$\left.\frac{\partial V(s_1, s_2; \check{\varphi})}{\partial s_2}\right|_{s_1=\check{\varphi}s_2} = 1. \tag{5.3}$$

To verify this, we differentiate (2.11) to obtain

$$\frac{\partial V(s_1, s_2; \varphi)}{\partial s_1} = \frac{h'(s_1/s_2)}{h(\varphi)} \tag{5.4}$$

and

$$\frac{\partial V(s_1, s_2; \varphi)}{\partial s_2} = \frac{h(s_1/s_2) - (s_1/s_2)h'(s_1/s_2)}{h(\varphi)}. \tag{5.5}$$

Hence,

$$\left.\frac{\partial V(s_1, s_2; \varphi)}{\partial s_1}\right|_{s_1=\varphi s_2} = \frac{h'(\varphi)}{h(\varphi)} \tag{5.6}$$

and

$$\left.\frac{\partial V(s_1, s_2; \varphi)}{\partial s_2}\right|_{s_1=\varphi s_2} = 1 - \frac{\varphi h'(\varphi)}{h(\varphi)}. \tag{5.7}$$

Now it is obvious that (5.6) has the value 0, and (5.7) the value 1, if and only if $h'(\varphi) = 0$; that is, $\varphi = \check{\varphi}$ as given by (2.16). In other words, the four conditions (5.2), (5.3), (2.15), and (2.16) are equivalent.

The second concept we want to illustrate is the concept of the *replicating portfolio.* For exercise strategy T_φ, the payoff is $F(T_\varphi)$ at time T_φ. At least in theory, it is possible to replicate this payoff with a self-financing portfolio that has value $V(S_1(t), S_2(t); \varphi)$ at time t, $0 \le t \le T_\varphi$, by continuously adjusting the allocation of the assets (including the stock dividends) to stock 1, stock 2, and the risk-free investment. For ease of notation, let us discuss the composition of the replicating portfolio at time $t = 0$. According to formula (8.35) of Gerber and Shiu (1996b), the amount of stock j invested in the replicating portfolio must be

$$s_j \frac{\partial V(s_1, s_2; \varphi)}{\partial s_j}, \quad j = 1, 2, \tag{5.8}$$

and the complement is invested in the risk-free asset. It follows from (5.4), (5.5), and (2.11) that

$$s_1 \frac{\partial V(s_1, s_2; \varphi)}{\partial s_1} + s_2 \frac{\partial V(s_1, s_2; \varphi)}{\partial s_2} = V(s_1, s_2; \varphi). \tag{5.9}$$

(Note that this formula is also an immediate consequence of Euler's theorem for homogeneous functions of degree one.) Formula (5.9) shows that the assets are exclusively invested in stocks of types 1 and 2, and nothing is invested in the risk-free asset. This suggests that $V(s_1, s_1; \varphi)$ does not depend on the value of the risk-free force of interest r. At first sight, this comes as a surprise, because r shows up in (2.13). However, if we apply (2.7) to (2.13), then r disappears in the quadratic equation that defines θ_1 and θ_2. Hence, $V(s_1, s_1; \varphi)$ is indeed independent of r.

The replicating portfolio is quite simple when $s_1 = s_2$. Then, according to (3.14), no funds are invested in stock 2. (The same conclusion can be reached by setting $s_1 = s_2$ in (5.5), and observing that $h'(1) = h(1) = \theta_2 - \theta_1$.) Hence, the total amount,

$$V(s_1, s_1; \varphi) = V(1, 1; \varphi)s_1,$$

is invested in stock 1. These findings are plausible; when $s_1 = s_2$, the guarantee, which is in terms of $\{S_1(t)\}$, is used immediately.

For $s_1 = \varphi s_2$, we have

$$V(s_1, s_2; \varphi) = s_2$$

by (2.14). For the optimal exercise strategy and for $s_1 = \check{\varphi} s_2$, the replicating portfolio consists of exactly one unit of stock 2, because of (5.9), (5.2), and (5.3).

The preceding analysis is readily adapted to future times t. If the option has not been exercised by time t, it suffices to replace s_1 by $S_1(t)$ and s_2 by $F(t)$. For exercise strategy T_φ, each time $F(t)$ falls to the level of $S_1(t)$, the replicating portfolio consists of exactly $V(1, 1; \varphi)$ unit(s) of stock 1. For the optimal exercise strategy, the option is exercised when $F(t)$ rises to $S_1(t)/\check{\varphi}$; at this moment, the replicating portfolio consists of exactly $n(t)$ units of stock 2, with $n(t)$ being defined by (2.5). Note that $V(1, 1; \varphi) = h(1)/h(\varphi)$ by (3.15); in particular, $V(1, 1; \check{\varphi}) = V(1, 1) = \check{c}$ by (4.18) with $s = 1$.

6. Dynamic Fund Protection with an Exponential Guaranteed Level

In this and the following section, we consider two special cases. In each, the price of one of the two "stocks" follows a geometric Brownian motion, while the other is deterministic (exponential or constant).

Let $\{S(t)\}$ be the price process of a stock,

$$S(t) = S(0)e^{X(t)}, \quad t \geq 0. \tag{6.1}$$

Here $\{X(t)\}$ is a Brownian motion (Wiener process), with instantaneous variance σ^2 and drift

$$\mu = r - \frac{\sigma^2}{2} - \zeta, \tag{6.2}$$

where the positive constant ζ can be interpreted as the dividend-yield rate. We consider the dynamic fund protection that was introduced by Gerber and Pafumi (2000), with a deterministic guaranteed level $Ke^{\gamma t}$ at time t ($0 < K \leq S(0)$, $-\infty < \gamma < r$). However, there is now a withdrawal feature built into this protection: If the option owner chooses to exercise the option at time t, he or she obtains the amount

$$F(t) = S(t) \max\left\{1, \max_{0 \leq \tau \leq t} \frac{Ke^{\gamma \tau}}{S(\tau)}\right\}. \tag{6.3}$$

An example for the guaranteed level function is

$$Ke^{\gamma t} = S(0)0.9(1.03)^t,$$

which may arise from nonforfeiture requirements.

What is the price of this perpetual option, and what is the optimal exercise strategy? This problem can be viewed as a special case of the general problem that has been treated in Sections 2, 3, and 5. To see this, it suffices to set

$$S_1(t) = Ke^{\gamma t}, \quad t \geq 0, \tag{6.4a}$$

$$S_2(t) = S(t), \quad t \geq 0. \tag{6.4b}$$

Hence $\sigma_1 = 0$, $\mu_1 = \gamma$, $\zeta_1 = r - \gamma$, and $\sigma_2 = \sigma$, $\mu_2 = \mu$, $\zeta_2 = \zeta$. It follows from (2.17) that the price of the perpetual option is

$$V(K, s) = \begin{cases} \dfrac{h(K/s)}{h(\check{\phi})} \, s & \text{if } \check{\phi} < \dfrac{K}{s} \le 1 \\ s & \text{if } 0 < \dfrac{K}{s} \le \check{\phi} \end{cases}, \tag{6.5}$$

where $s = S(0)$. According to (2.13), θ_1 and θ_2 are now the solution of the quadratic equation

$$-r + \gamma\theta + \mu(1 - \theta) + \frac{\sigma^2}{2}(1 - \theta)^2 = 0. \tag{6.6}$$

The optimal exercise of the option takes place at the first time when the ratio $Ke^{\gamma t}/F(t)$ falls to the level $\check{\phi}$, with $\check{\phi}$ given by formula (2.16). By (6.2) we can rewrite (6.6) as

$$\frac{\sigma^2}{2}\theta^2 - \left(r + \frac{\sigma^2}{2} - \gamma - \zeta\right)\theta - \zeta = 0. \tag{6.7}$$

It follows that θ_1 and θ_2—and, with that, the price of the perpetual option—depend on r and γ only through their difference.

7. INDEXED RUSSIAN OPTIONS

In Section 6 we looked at the special case where $S_1(t)$ is an exponential function. In this section we reverse the roles. Let

$$S_1(t) = S(t), \quad t \ge 0, \tag{7.1a}$$

$$S_2(t) = me^{\gamma t}, \quad t \ge 0, \tag{7.1b}$$

where $\{S(t)\}$ is the price process of a stock as in Section 6, $m \ge S(0)$, and $-\infty < \gamma \le r$. Here $\sigma_1 = \sigma$, $\mu_1 = \mu$, $\zeta_1 = \zeta$, and $\sigma_2 = 0$, $\mu_2 = \gamma$, $\zeta_2 = r - \gamma$. Formula (1.1) boils down to

$$F(t) = \max\{me^{\gamma t}, \max_{0 \le \tau \le t} e^{\gamma(t-\tau)}S(\tau)\}. \tag{7.2}$$

An option, whose owner can get this amount at a time t of his or her choice, is called an *indexed Russian option.* Shepp and Shiryaev (1993) studied the case $\gamma = 0$; with m interpreted as the maximal stock price of the past, $F(t)$ is the observed maximal stock price up to time t. They coined the term "Russian option" in honor of A. N. Kolmogorov, who first enunciated the smooth pasting condition. Other papers with discussions on Russian options are Duffie and Harrison (1993), Gerber, Michaud, and Shiu (1995), Gerber and Shiu (1994, 1996a), Guo (2002), Guo and Shepp (2001), Kallianpur (1998), Kramkov and Shiryaev (1994), and Shepp and Shiryaev (1994). Expositions on Russian options can also be found in books such as Kwok (1998), Panjer et al. (1998), and Shiryaev (1999).

From (2.17) we see that, with $s = S(0)$, the price of the indexed Russian option is

$$V(s, m) = \begin{cases} \dfrac{h(s/m)}{h(\check{\phi})} \, m & \text{if } \check{\phi}m < s \le m \\ m & \text{if } s \le \check{\phi}m \end{cases}. \tag{7.3}$$

According to formula (2.13), θ_1 and θ_2 are now the solution of the quadratic equation

$$-r + \mu\theta + \gamma(1 - \theta) + \frac{\sigma^2}{2}\theta^2 = 0. \tag{7.4}$$

The optimal strategy is to exercise the indexed Russian option at the first time when the ratio $S(t)/F(t)$ falls to the level $\check{\phi}$, with $\check{\phi}$ given by formula (2.16). By (6.2) we can rewrite (7.4) as

$$\frac{\sigma^2}{2}\theta^2 + \left(r - \gamma - \frac{\sigma^2}{2} - \zeta\right)\theta - (r - \gamma) = 0. \tag{7.5}$$

This shows that the price of the indexed Russian option depends on r and γ only through their difference.

The formula given by Shepp and Shiryaev (1993) is

$$V(s, m) = \begin{cases} mg\left(\dfrac{s}{\check{\phi}m}\right) & \text{if } \check{\phi}m < s \leq m \\ m & \text{if } s \leq \check{\phi}m \end{cases}, \tag{7.6}$$

with $g(x)$ defined by (4.2). Formula (7.6) follows from (4.12).

8. GENERALIZED PERPETUAL LOOKBACK PUT OPTIONS

As mentioned in the last section, with $\gamma = 0$ and with m interpreted as the maximal stock price of the past, $F(t)$ defined by (7.2) is the observed maximal stock price up to time t, $t \geq 0$. A *lookback put* option is an option with payoff

$$[F(t) - S(t)]_+ = F(t) - S(t), \tag{8.1}$$

with t being the exercise time. We now consider a generalization of the perpetual lookback put option and of the perpetual dynamic protection option. Let κ be a constant between 0 and 1. Consider an option that pays its holder

$$F(t) - \kappa S_1(t), \tag{8.2}$$

if the holder chooses to exercise it at time t. In (8.2), $F(t)$ is defined by (1.1). We continue to assume (2.2).

Again, we have to consider only exercise strategies of the form (2.9). For each strategy T_φ, let

$$V(s_1, s_2; \varphi, \kappa) = E(e^{-rT_\varphi}[F(T_\varphi) - \kappa S_1(T_\varphi)]), \quad \varphi s_2 \leq s_1 \leq s_2 \tag{8.3}$$

denote its value, where $s_j = S_j(0)$, $j = 1, 2$. Since

$$F(T_\varphi) - \kappa S_1(T_\varphi) = (1 - \kappa\varphi)F(T_\varphi),$$

we have

$$V(s_1, s_2; \varphi, \kappa) = (1 - \kappa\varphi)V(s_1, s_2; \varphi) = (1 - \kappa\varphi)\frac{h(s_1/s_2)}{h(\varphi)}s_2 \tag{8.4}$$

by (2.11) and (2.12). Let $\widehat{\varphi}$ be the number that maximizes the expression

$$\frac{1 - \kappa\varphi}{h(\varphi)}. \tag{8.5}$$

Then the time-0 price of this option is

$$\begin{cases} \dfrac{h(s_1/s_2)}{h(\widehat{\varphi})}(1 - \kappa\widehat{\varphi})s_2 & \text{if } \widehat{\varphi} < \dfrac{s_1}{s_2} \leq 1 \\ (1 - \kappa\widehat{\varphi})s_2 & \text{if } 0 < \dfrac{s_1}{s_2} \leq \widehat{\varphi} \end{cases}. \tag{8.6}$$

The above is a generalization of Section 10.12 of Panjer et al. (1998). To understand the smooth pasting condition in this case, one will find Exercise 10.31 and Figure 10.6 of Panjer (1998) useful. Also, $\widehat{\varphi}$ satisfies the equation

$$\kappa = \frac{h'(\widehat{\varphi})}{(\theta_1 - 1)(\theta_2 - 1)(\widehat{\varphi}^{\theta_1} - \widehat{\varphi}^{\theta_2})}, \tag{8.7}$$

generalizing Exercise 10.32 of Panjer (1998).

APPENDIX

The purpose of this Appendix is to explain why it suffices to consider exercise strategies of the form (2.9). Mathematically, the maximization problem is an *optimal stopping* problem for a stationary Markov process with a stationary payoff function. The *optimal continuation region* is

$$C = \{(s_1, s_2)|V(s_1, s_2) > s_2\}.$$

Then the optimal exercise strategy is to exercise the option at the first time t when $(S_1(t), F(t))$ is not in the region C. Note that $V(s_1, s_2)$ is a homogeneous function of degree one. Thus,

$$V(s_1, s_2) = s_2 V(s_1/s_2, 1).$$

It follows that

$$C = \{(s_1, s_2)|V(s_1/s_2, 1) > 1\}.$$

Because $V(z, 1)$ is a nondecreasing function of z, $z > 0$, we conclude that

$$C = \left\{ (s_1, s_2) \left| \frac{s_1}{s_2} > \varphi \right. \right\}$$

for some number φ. Hence the optimal exercise strategy is indeed of the form (2.9).

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Discussions

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Professors Gerber and Shiu have done a very fine job of evaluating the value of the dynamic fund protection option with perpetual life, and relating the fund protection option with the perpetual maximum option and the Russian option (perpetual American lookback option). In particular, they have performed a detailed and insightful analysis of the optimal exercise strategy of these options.

The two general concepts in arbitrage option pricing theory—smooth pasting condition and replicating portfolio—are demonstrated as part of the solution procedure. In this discussion, we would like to derive the price functions of the dynamic fund protection option and maximum option by relating them to the value of a protection fund with rights to reset to the value of another guaranteed fund.

We consider a primary fund with value process S_2^t, which is protected with reference to another (guaranteed) fund with value process S_1^t. The holder of the primary fund has the right to reset the value to that of the guaranteed fund upon exercising the reset right. The number of resets allowed can be finite or infinite. Further, the protection fund is assumed to have perpetual life and withdrawal right. With an

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infinite number of resets, the holder should always exercise the reset right whenever the value of the primary fund falls to the value of the guaranteed fund. This reset strategy is exactly the same as the perpetual dynamic fund protection option considered by Professors Gerber and Shiu.

When the holder is allowed to reset only once, the perpetual protection fund is equivalent to the perpetual maximum option. This is because the fund becomes S_2^t upon withdrawal and S_1^t upon reset, corresponding to the payoff that takes the maximum of S_1^t and S_2^t. In this discussion, we would like to obtain the price formula of such a perpetual protection fund with withdrawal right and n reset rights. By setting $n = 1$ and $n \to \infty$, we recover the price functions of the perpetual maximum option and the dynamic fund protection option, respectively. We also discuss the monotonic properties of the value functions and the threshold values at which the holder should withdraw or reset.

With both withdrawal and reset rights, the option pricing model has two-sided thresholds. Let S_2 and S_1 denote the values of the primary fund and guaranteed fund, respectively, at the current time (taken to be the zero). Under the Black-Scholes risk-neutral valuation framework, the value processes are assumed to follow the lognormal processes

$$\frac{dS_i^t}{S_i^t} = (r - \zeta_i)dt + \sigma_i dZ_i, \quad i = 1, 2, \tag{D1}$$

where r is the riskless interest rate, ζ_i is the dividend yield of fund i, σ_i is the volatility parameter of fund i, and dZ_i is the standard Wiener process. We assume $dZ_1 dZ_2 = \rho dt$, where ρ is the correlation coefficient between S_1^t and S_2^t. Let $V_n(S_1, S_2)$ denote the value of the perpetual protection fund with n reset rights and withdrawal right. In our pricing formulation, we take advantage of the linear homogeneity property of $V_n(S_1, S_2)$, and accordingly, define

$$W_n(x) = \frac{V_n(S_1, S_2)}{S_2}, \quad x = \frac{S_1}{S_2}. \tag{D2}$$

This corresponds to the choice of S_2 as the *numeraire*. In the continuation region, the governing equation for $W_n(x)$ takes the form

$$\frac{\sigma^2}{2} x^2 \frac{d^2 W_n}{dx^2} + (\zeta_1 - \zeta_2) x \frac{dW_n}{dx} - \zeta_2 W_n = 0, \quad x_n^w < x < x_n^r, \tag{D3}$$

where $\sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$, x_m^w and x_n^r are the threshold values for x at which the holder should optimally withdraw and reset, respectively. The boundary conditions are prescribed as follows:

$$W_n(x_n^w) = 1 \quad \text{and} \quad W_n'(x_n^w) = 0, \tag{D4a}$$

$$W_n(x_n^r) = x_n^r W_{n-1}(1) \quad \text{and} \quad W_n'(x_n^r) = W_{n-1}(1). \tag{D4b}$$

Upon withdrawal, $V_n(S_1, S_2)$ becomes S_2 and so we have $W_n(x_n^w) = 1$. When the holder resets at $x = x_n^r$, the option writer has to supply enough funding to increase the number of units of the primary fund such that the new fund value equals S_1. The corresponding number of units equals x_n^r, which is the ratio of the fund values at the reset threshold x_n^r. Subsequently, the protection fund has one reset right less and x becomes 1 since the values of the new "upgraded" fund and guaranteed fund are identical upon reset. Hence, the value of the protection fund at reset threshold becomes $x_n^r W_{n-1}(1)$. When $n = 0$, there is no reset right so we have $W_0(x) = 1$ for all values of x. The derivative conditions at x_n^w and x_n^r represent optimality conditions at the withdrawal and reset thresholds, respectively.

Consider the limiting case $n \to \infty$, where we have $W_\infty(x_\infty^r) = x_\infty^r W_\infty(1)$. The equation is seen to be satisfied by $x_\infty^r = 1$ (for a more rigorous justification, see Equation D13). This represents immediate reset whenever S_2^t falls to S_1^t, given that the holder has an infinite number of reset rights. Furthermore, the corresponding derivative condition becomes $W_\infty'(1) = W_\infty(1)$, which is equivalent to Equation (3.14) in Gerber-Shiu's paper (they argue that the value of $V(S_1, S_2)$ is unaffected by marginal changes in S_2 when S_2 is "close" to S_1). We would like to obtain closed-form solutions for x_n^r, x_n^w and $W_n(x)$. Since the

protection fund with more outstanding reset rights should be more expensive, we have the obvious monotonic properties on the price functions:

$$W_1(x) < W_2(x) < \cdots < W_\infty(x), \tag{D5a}$$

from which we deduce the following monotonic properties of the threshold values:

$$x_1^w > x_2^w > \cdots > x_\infty^w, \tag{D5b}$$

$$x_1^r > x_2^r > \cdots > x_\infty^r = 1. \tag{D5c}$$

When $n = 1$, Gerber and Shiu obtained the solution for $W_1(x)$ in terms of the function

$$g(z) = \frac{\theta_2 z^{\theta_1} - \theta_1 z^{\theta_2}}{\theta_2 - \theta_1}, \quad z > 0, \ \theta_1 \leq 0 \text{ and } \theta_2 \geq 1, \tag{D6}$$

where θ_1 and θ_2 are the two roots of the auxiliary equation:

$$\frac{\sigma^2}{2} \theta(\theta - 1) + (\zeta_2 - \zeta_1)\theta - \zeta_2 = 0. \tag{D7}$$

The function $g(z)$ satisfies the governing differential Equation (D3) and the boundary conditions: $g(1) = 1$ and $g'(1) = 0$. Suppose we set $W_1(x) = g(x/x_1^w)$ so that $W_1(x_1^w) = 1$ and $W_1'(x_1^w) = 0$ are automatically satisfied. The other two boundary conditions, $W_1(x_1^r) = x_1^r$ and $W_1'(x_1^r) = 1$, lead to the following pair of equations for the determination of x_1^w and x_1^r:

$$\frac{1}{\theta_2 - \theta_1} \left[\theta_2 \left(\frac{x_1^r}{x_1^w} \right)^{\theta_1} - \theta_1 \left(\frac{x_1^r}{x_1^w} \right)^{\theta_2} \right] = x_1^r, \tag{D8a}$$

$$\frac{\theta_1 \theta_2}{\theta_2 - \theta_1} \left[\left(\frac{x_1^r}{x_1^w} \right)^{\theta_1} - \left(\frac{x_1^r}{x_1^w} \right)^{\theta_2} \right] = x_1^r. \tag{D8b}$$

Solving the above equations, we obtain

$$x_1^w = \left(\frac{-\theta_1}{1 - \theta_1} \right)^{(1-\theta_1)/(\theta_2-\theta_1)} \left(\frac{\theta_2}{\theta_2 - 1} \right)^{(\theta_2-1)/(\theta_2-\theta_1)}, \tag{D9a}$$

$$x_1^r = \left(\frac{-\theta_1}{1 - \theta_1} \right)^{-\theta_1/(\theta_2-\theta_1)} \left(\frac{\theta_2}{\theta_2 - 1} \right)^{\theta_2/(\theta_2-\theta_1)}. \tag{D9b}$$

By setting

$$x_1^r = W_{n-1}(1)x_n^r \quad \text{and} \quad x_1^w = W_{n-1}(1)x_n^w \tag{D10}$$

in Equations (D8a, b) and considering the function $g(x/x_n^w)$, we observe that

$$g\left(\frac{x_n^r}{x_n^w} \right) = x_n^r W_{n-1}(1) \quad \text{and} \quad g'\left(\frac{x_n^r}{x_n^w} \right) = W_{n-1}(1). \tag{D11}$$

Hence, we deduce that

$$W_n(x) = g\left(\frac{x}{x_n^w} \right), \quad x_n^w < x < x_n^r. \tag{D12}$$

Once we know x_1^w, x_1^r and $W_1(x)$, we can compute $W_1(1) = g(1/x_1^w)$; then we apply the recursive relations (D10) to obtain x_2^w and x_2^r; also $W_2(x) = g(x/x_2^w)$ and $W_2(1) = g(1/x_2^w)$, and so forth. From the

recursive relations (D10) and the monotonic properties on $W_n(x)$ in Equation (D5a), we deduce immediately the monotonic properties on x_n^w and x_n^r (see Equations D5b, c).

Consider the limiting case where $n \to \infty$; the boundary conditions (D4b) become

$$W_\infty(x_\infty^r) = x_\infty^r W_\infty(1) \quad \text{and} \quad W_\infty'(x_\infty^r) = W_\infty(1). \tag{D13}$$

By virtue of the monotonic increasing property of $W_\infty'(x)$, the curve of $y = W_\infty(x)$ and the line $y = W_\infty(1)x$ can intersect at only one point, namely, $x = 1$. Hence, the equation $W_\infty(x_\infty^r) = x_\infty^r W_\infty(1)$ can have the unique root, $x_\infty^r = 1$. The other condition becomes $W_\infty'(1) = W_\infty(1)$. Hence, the governing equation for the value of the perpetual fund protection option, which is equal to $W_\infty(x)$, is given by

$$\frac{\sigma^2}{2} x^2 \frac{d^2 W_\infty}{dx^2} + (\zeta_1 - \zeta_2)x \frac{dW_\infty}{dx} - \zeta_2 W_\infty = 0, \quad x_\infty^w < x < 1, \tag{D14}$$

subject to the auxiliary conditions:

$$W_\infty(x_\infty^w) = 1 \quad \text{and} \quad W_\infty'(x_\infty^w) = 0, \tag{D15a}$$

$$W_\infty'(1) = W_\infty(1). \tag{D15b}$$

The solution to $W_\infty(x)$ is easily seen to be

$$W_\infty(x) = \frac{h(x)}{h(x_\infty^w)}, \quad x_\infty^w < x < 1, \tag{D16}$$

where

$$h(x) = (\theta_2 - 1)x^{\theta_1} - (\theta_1 - 1)x^{\theta_2}, \quad x > 0. \tag{D17}$$

Note that $h(x)$ satisfies Equation (D14) and the Robin boundary condition (D15b). The boundary condition $W_\infty(x_\infty^w) = 1$ is satisfied by the inclusion of the multiplicative factor $1/h(x_\infty^w)$ in $W_\infty(x)$. The optimality condition, $W_\infty'(x_\infty^w) = 0$, gives the following algebraic equation for x_∞^w:

$$h'(x_\infty^w) = \theta_1(\theta_2 - 1)(x_\infty^w)^{\theta_1} - \theta_2(\theta_1 - 1)(x_\infty^w)^{\theta_2} = 0. \tag{D18}$$

Alternatively, from the recursive relations (D10), we deduce that (see Equation 4.5 in Gerber-Shiu's paper)

$$\frac{x_1^w}{x_1^r} = \frac{x_\infty^w}{x_\infty^r} = x_\infty^w. \tag{D19}$$

Also, from Equations (D12) and (D16), we obtain another relation (see Equations 4.11 and 4.17 in Gerber-Shiu's paper)

$$W_\infty(x) = g\left(\frac{x}{x_\infty^w}\right) = \frac{h(x)}{h(x_\infty^w)}, \quad x_\infty^w < x < 1. \tag{D20}$$

In Figure 1, we show the plots of the price functions $W_n(x)$ for $n = 1, 2, 3$, and ∞. The values of the price functions increase monotonically with increasing numbers of reset rights and always stay above 1. We also plot the threshold values, x_n^w and x_n^r, against $1/n$ in Figure 2. The monotonic properties on x_n^w and x_n^r as stated in Equations (D5b, c) are verified. In particular, we observe that x_n^r tends to 1 as $n \to \infty$.

In summary, we have illustrated that the maximum call and the dynamic protection fund option correspond to the protection fund with rights to reset to a reference guaranteed fund once and an infinite number of times, respectively. We obtain the closed-form formula for the price function of the

Figure 1

Plots of the Price Functions of the Protection Fund $W_n(x)$ Against x, with Varying Number of Reset Rights n.
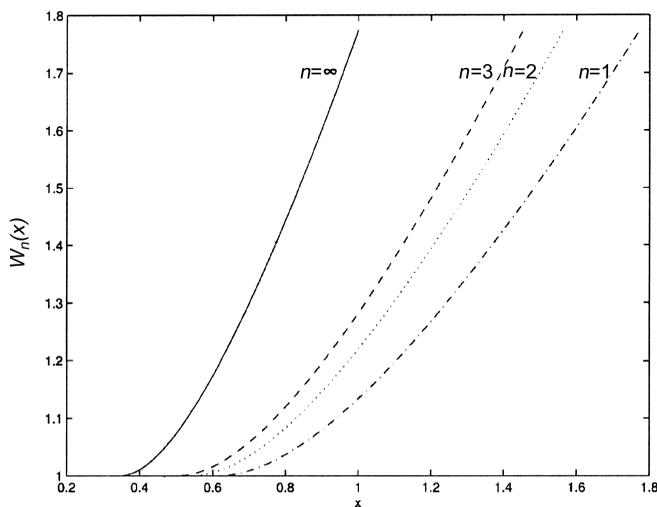
protection fund with n reset rights. With a finite number of resets, there are two threshold values, an upper threshold for reset and a lower threshold for withdrawal. When an infinite number of resets is allowed, we prove mathematically that the holder exercises the reset right whenever the value of the protection fund falls to the value of the guaranteed fund.

Figure 2

Threshold Values, x_n^w and x_n^r, at Which the Protection Fund Holder Should Optimally Withdraw and Reset, Respectively, Against the Reciprocal of the Number of Reset Rights, $1/n$.
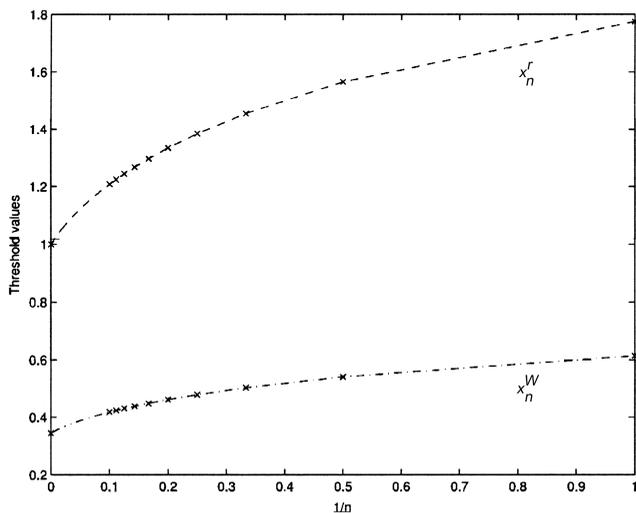
JÉRÔME PANSERA*

Professors Gerber and Shiu are to be congratulated for this excellent and very instructive paper. The aim of this discussion is to provide an alternative derivation for the price of the perpetual dynamic fund protection option (that is, 2.17 or, equivalently, 4.12 in the Gerber and Shiu paper) by a change of *numéraire.* When using the second stock as the *numéraire,* the problem simplifies to a one-stock situation and (4.12) then follows directly from the price of the Russian option, given by Shepp and Shiryaev (1993). Moreover, with another change of *numéraire,* it is also possible to retrieve (2.17) from (6.5). Thus, by changing *numéraire,* one can recover the price of the perpetual dynamic fund protection option from the two special cases presented in Sections 6 and 7 of the paper.

Using the Bond as the *Numéraire*

The financial market considered in the paper consists of three assets: one risk-free security (which we call the bond, for simplicity) and two stocks. The time-*t* prices of these three assets are given, respectively, by

$$B(t) = e^{rt}, \tag{D1}$$

$$S_1(t) = S_1(0)e^{X_1(t)}, \tag{D2}$$

$$S_2(t) = S_2(0)e^{X_2(t)}, \tag{D3}$$

where $\{X_1(t), X_2(t)\}$ is a bivariate Brownian motion with instantaneous variances σ_1^2 and σ_2^2, correlation ρ, and drifts μ_1 and μ_2 under the risk-neutral measure Q.

Note that the two processes S_1 and S_2 do not completely describe the dynamics of a tradable asset since the two stocks pay dividends. (We assume, as done in the paper, that $\zeta_1 = r - \mu_1 - \sigma_1^2/2 > 0$ and $\zeta_2 = r - \mu_2 - \sigma_2^2/2 > 0$.) The standard procedure to overcome this problem is to introduce the two "dividend-reinvested" processes

$$\tilde{S}_1(t) := e^{\zeta_1 t}S_1(t),$$

$$\tilde{S}_2(t) := e^{\zeta_2 t}S_2(t).$$

Concretely, $\tilde{S}_i(t)$ is the time-*t* value of the self-financing portfolio obtained by reinvesting in stock i its dividend payments. For more details, see, for instance, Baxter and Rennie (1996, Section 4.2) or Musiela and Rutkowski (1997, Section 6.2.1).

It is well-known that the risk-neutral measure Q, used throughout the paper, possesses the following two properties.
(a) First, the time-0 value of the American claim $F(T)$ is given by

$$\sup_{T \in \mathcal{T}} E^Q\left[\frac{F(T)}{B(T)}\right], \tag{D4}$$

where $\mathcal{T}$ is the set of all finite stopping times.[1]
(b) Second, the "bond-discounted" tradable assets

$$\tilde{S}_1^*(t) := \tilde{S}_1(t)/B(t), \tag{D5}$$
$$\tilde{S}_2^*(t) := \tilde{S}_2(t)/B(t) \tag{D6}$$

are Q-martingales.

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[1] We may restrict our attention to finite stopping times for the following reason. As explained in the paper, $\zeta_1 > 0$ and $\zeta_2 > 0$ imply $0 < \breve{\varphi} < 1$. Then, using the second Borel-Cantelli Lemma, one can show that the optimal exercise time (2.9) is finite (almost surely). For example, see the proof of Equation (42) in Shiryaev (1999, Chapter VIII, §2d).

Note that the bond price B appears as the denominator in (D4), (D5), and (D6). This is because the process B is implicitly taken as the *numéraire* in the paper; the measure Q is tied to this specific choice of *numéraire.*

Using $\tilde{S}_2$ as the *Numéraire*

Let us now use the process $\tilde{S}_2$ as the *numéraire,* instead of B. By analogy with the above, we introduce the two processes

$$B^{**}(t) := B(t)/\tilde{S}_2(t),$$

$$\tilde{S}_1^{**}(t) := \tilde{S}_1(t)/\tilde{S}_2(t).$$

The following lemma completes the analogy.

Lemma

There exists a probability measure R such that

(a) The time-0 value of the American claim $F(T)$ is given by

$$\tilde{S}_2(0) \sup_{T \in \mathcal{T}} E^R\left[\frac{F(T)}{\tilde{S}_2(T)}\right]. \tag{D7}$$

(b) B^{**} and $\tilde{S}_1^{**}$ are R-martingales.
(c) More precisely,

$$B^{**}(t) = B^{**}(0)e^{X_0^{**}(t)}, \tag{D8}$$

$$\tilde{S}_1^{**}(t) = \tilde{S}_1^{**}(0)e^{\tilde{X}_1^{**}(t)}, \tag{D9}$$

where $\{X_0^{**}(t), \tilde{X}_1^{**}(t)\}$ is a bivariate Brownian motion with instantaneous variances σ_2^2 and $\sigma^2 := \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$, correlation $(\sigma_2 - \rho\sigma_1)/\sigma$, and drifts $-\sigma_2^2/2$ and $-\sigma^2/2$ under the measure R.

This lemma is just an example of the general technique known as a change of *numéraire.* A standard reference on the subject is Geman, El Karoui, and Rochet (1995). Very readable introductions are provided by Baxter and Rennie (1996, Section 6.4) and Björk (1998, Chapter 19). Among the actuarial literature, Gerber and Shiu's (1994, 1996) factorization formula, based on Esscher transforms, also has interesting connections with changes of *numéraire* (see especially Section 10 of their second paper).
Let us only outline a sketch of proof while omitting some technical details, by giving appropriate references where needed.

PROOF

For every $t < \infty$, let

$$L_t = \frac{\tilde{S}_2(t)}{S_2(0)B(t)} \tag{D10}$$

and introduce,[2] by means of its Radon-Nikodým derivative with respect to Q, the probability measure R:

$$\left.\frac{dR}{dQ}\right|_{\mathcal{F}_t} = L_t.$$

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[2] Because we are dealing with an infinite time horizon, there are some technical difficulties here. In particular, the two measures Q and R may not be equivalent, but merely locally equivalent (i.e., $Q \sim R$ only on $\mathcal{F}_t$ for $t < \infty$). This is why we chose to restrict our attention to finite stopping times. See Karatzas and Shreve (1998, Section 1.7) and Shiryaev (1999, Chapter V, §3a).

(a) One can show that, for $T \in \mathcal{T}$ and $G \in \mathcal{F}_T$,

$$\mathrm{E}^Q[1_G L_T] = \mathrm{E}^R[1_G], \tag{D11}$$

where 1_G is the indicator function of the event G. Then, by taking the limit of a sequence of $\mathcal{F}_T$-measurable simple functions, (D11) extends from indicator functions to nonnegative $\mathcal{F}_T$-measurable random variables. In particular, for every $T \in \mathcal{T}$,

$$\mathrm{E}^Q\left[\frac{F(T)}{B(T)}\right] = \tilde{S}_2(0)\mathrm{E}^R\left[\frac{F(T)}{\tilde{S}_2(T)}\right], \tag{D12}$$

which shows that (D4) = (D7), after taking the supremum. See Shiryaev (1999, Chapter VIII), where this technique is used, also for perpetual American options, in §2a.6 and §2d.2.

Equation (D12) has the following financial interpretation. There are two economies: one with *numéraire B*, the other with *numéraire $\tilde{S}_2$*. For a fixed $T \in \mathcal{T}$, the left-hand side of (D12) represents the time-0 price in the first economy of a European option with payoff $F(T)$ (where the exercise time is determined by the stopping time T, not by the option owner), while the right-hand side is the time-0 price in the second economy of the same option. These two prices agree because, if a claim is attainable in one economy, it must be attainable in the other, using the same hedging portfolio (renormalized in the appropriate *numéraire*). This intuitively clear *numéraire invariance* result is often stated for basic European options only (where T is a deterministic time), but the result also holds in a much broader context, including American options with path-dependent payoffs, as needed here. See Detemple (2001, Section 8), especially Remark 20 (iii).

(b) If Z is a stochastic process, then, from Bayes' Rule,

$$\mathrm{E}^Q\left[\frac{Z(t)/B(t)}{Z(s)/B(s)} \,\bigg|\, \mathcal{F}_s\right] = 1 \quad \text{for} \quad s \leq t$$

if and only if

$$\mathrm{E}^R\left[\frac{Z(t)/\tilde{S}_2(t)}{Z(s)/\tilde{S}_2(s)} \,\bigg|\, \mathcal{F}_s\right] = 1 \quad \text{for} \quad s \leq t.$$

That is, Z/B is a Q-martingale if and only if $Z/\tilde{S}_2$ is an R-martingale. Since $1 = B/B$, $\tilde{S}_1^* = \tilde{S}_1/B$, and $\tilde{S}_2^* = \tilde{S}_2/B$ are Q-martingales, the corresponding ratios $B^{**} = B/\tilde{S}_2$, $\tilde{S}_1^{**} = \tilde{S}_1/\tilde{S}_2$, and $1 = \tilde{S}_2/\tilde{S}_2$ are R-martingales. This proves (b).

(c) By definition chasing,

$$B^{**}(t) = B^{**}(0) \exp[(r - \zeta_2)t - X_2(t)], \tag{D13}$$
$$\tilde{S}_1^{**}(t) = \tilde{S}_1^{**}(0) \exp[(\zeta_1 - \zeta_2)t + X_1(t) - X_2(t)], \tag{D14}$$

where $\{X_1(t), X_2(t)\}$ is a bivariate Brownian motion with instantaneous variances σ_1^2 and σ_2^2, correlation ρ, and drifts $\mu_1 + \rho\sigma_1\sigma_2$ and $\mu_2 + \sigma_2^2$ under R (by the Cameron-Martin-Girsanov Theorem and the change of measure D10). Letting X_0^{**} and $\tilde{X}_1^{**}$ be the Brownian motions inside the square brackets of (D13) and (D14), respectively, completes the proof. $\square$

Now, let us see how the lemma simplifies the valuation of the perpetual dynamic fund protection option. The time-0 price of this American claim is given by (2.8) in the paper:

$$V = \sup_{T \in \mathcal{T}} \mathrm{E}^Q\left[\frac{S_2(T) \max\left\{1, \max_{0 \leq t \leq T} \dfrac{S_1(t)}{S_2(t)}\right\}}{B(T)}\right]. \tag{D15}$$

This formula is of the form (D4). If we express the price using (D7) instead, we get

$$V = \tilde{S}_2(0) \sup_{T \in \mathscr{T}} \mathrm{E}^R \left[\frac{S_2(T) \max\left\{1, \max_{0 \le t \le T} \frac{S_1(t)}{S_2(t)}\right\}}{\tilde{S}_2(T)} \right]$$

$$= \tilde{S}_2(0) \sup_{T \in \mathscr{T}} \mathrm{E}^R \left[e^{-\zeta_2 T} \max\left\{1, \max_{0 \le t \le T} \frac{S_1(t)}{S_2(t)}\right\} \right]. \tag{D16}$$

By contrast with (D15), we see that $S_2(T)$ is no longer present inside the expectation of (D16); this is the whole point of the change of *numéraire*. There is still $S_2(t)$ in the denominator though, but this is not really a problem because

$$\frac{S_1(t)}{S_2(t)} = e^{(\zeta_2 - \zeta_1)t} \tilde{S}_1^{**}(t)$$

$$= \tilde{S}_1^{**}(0) e^{X(t)},$$

where X is a Brownian motion with instantaneous variance σ^2 and drift $\mu := \zeta_2 - \zeta_1 - \sigma^2/2$ under R (by part (c) of the lemma). That is, the ratio $\{S(t) := S_1(t)/S_2(t)\}$ follows a geometric Brownian motion and may, thus, be interpreted as the price process S of a single stock.

After this change of *numéraire,* the problem is then reduced to a one-stock situation. The supremum in (D16) can now be evaluated by Shepp and Shiryaev's (1993) result, given in Section 7 of the paper, by taking $m = 1$, $\gamma = 0$, and $r = \zeta_2$. Explicitly, with these parameters, (7.6) of the paper becomes

$$\sup_{T \in \mathscr{T}} \mathrm{E}^R[e^{-\zeta_2 T} \max\{1, \max_{0 \le t \le T} S(t)\}] = \begin{cases} 1 & \text{if } S(0) \le \check{\phi}, \\ g\left(\dfrac{S(0)}{\check{\phi}}\right) & \text{if } \check{\phi} < S(0) \le 1, \end{cases} \tag{D17}$$

while (7.4) reduces to

$$0 = -\zeta_2 + \mu\theta + \frac{\sigma^2}{2}\theta^2. \tag{D18}$$

In (D17), the function g is defined by (4.2) of the paper and the constant $\check{\phi}$, by (2.16). Both g and $\check{\phi}$ depend on θ_1 and θ_2, which are the two solutions of Equation (D18). Substituting (D17) in (D16) (in other words, multiplying (D17) by $\tilde{S}_2(0) = S_2(0)$) gives the time-0 price of the perpetual dynamic fund protection option.

Finally, let us verify that this result agrees with that of the paper. It is straightforward to see that (2.13) is equivalent to (D18).[3] That is, θ_1 and θ_2 are defined in the same way by (2.13) and by (D18). Consequently, the price found above agrees with (4.12) because $S(0)$ (in our notation) corresponds to s_1/s_2 (in Gerber and Shiu's notation). Also, note that the optimal exercise time for (D15) or (D16), as given by (2.9),

$$T_{\check{\phi}} = \min\{t \,|\, S_1(t) = \check{\phi} F(t)\},$$

---

[3] A quick check is the following. By taking $\theta = 0$, 1, and 1/2 in the right-hand sides of (2.13) and (D18) one gets, respectively, $-\zeta_2$, $-\zeta_1$, and $-\zeta_1/2 - \zeta_2/2 - \sigma^2/8$. Since both right-hand sides are quadratic polynomials and since they agree on three different points, they must be the same.

depends on S_1 and S_2 only through their ratio, and translates into

$$T_{\breve{\phi}} = \min\{t|S(t) = \breve{\phi} \max\{1, \max_{0 \le \tau \le t} S(\tau)\}\},$$

for (D17).

In the above derivation, the dynamics of the process $\tilde{X}_1^{**}$ were used, but not the ones of the process X_0^{**}; (D8) was stated only for completeness in part (c) of the lemma.

Using $\tilde{S}_1$ as the *Numéraire*

Let us point out that the price of the perpetual dynamic fund protection option can also be retrieved from (6.5) of the paper, this time using $\tilde{S}_1$ as the *numéraire.* With this choice of *numéraire,* (D15) becomes

$$V = \tilde{S}_1(0) \sup_{T \in \mathcal{T}} E^U \left[\frac{S_2(T) \max\left\{1, \max\limits_{0 \le t \le T} \frac{S_1(t)}{S_2(t)}\right\}}{\tilde{S}_1(T)} \right]$$

$$= \tilde{S}_1(0) \sup_{T \in \mathcal{T}} E^U \left[e^{-\zeta_1 T} S(T) \max\left\{1, \max_{0 \le t \le T} \frac{1}{S(t)}\right\} \right], \tag{D19}$$

with

$$\left. \frac{dU}{dQ} \right|_{\mathcal{F}_t} = \frac{\tilde{S}_1(t)}{S_1(0)B(t)}$$

and S redefined by

$$S(t) := \frac{S_2(t)}{S_1(t)}$$

$$= S(0)e^{X(t)},$$

where X is a Brownian motion with instantaneous variance σ^2 and drift $\zeta_1 - \zeta_2 - \sigma^2/2$ under the new probability measure U (by the Cameron-Martin-Girsanov Theorem).

The right-hand side of (D19) can be evaluated by Section 6 of the paper, by taking $K = 1$, $\gamma = 0$, and $r = \zeta_1$. Explicitly, (6.5) of the paper yields

$$V = \begin{cases} S_2(0) & \text{if } S(0) \le \breve{\phi}, \\ S_2(0) \dfrac{h(S_1(0)/S_2(0))}{h(\breve{\phi})} & \text{if } \breve{\phi} < S(0) \le 1, \end{cases} \tag{D20}$$

with h and $\breve{\phi}$ depending on θ_1 and θ_2, the two solutions of (6.6) in the paper:

$$0 = -\zeta_1 + \left(\zeta_1 - \zeta_2 - \frac{\sigma^2}{2}\right)\theta + \frac{\sigma^2}{2}(1 - \theta)^2. \tag{D21}$$

It is then only a matter of elementary algebra to confirm that (D20) agrees with (2.17) of the paper, by checking that (D21) is equivalent to (D18).

ACKNOWLEDGMENTS

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CARISA K.W. YU*

I wish to congratulate Professors Gerber and Shiu for this interesting paper, which studies the pricing of dynamic fund protection without a maturity date. The authors explain the general problem in the paper in terms of stocks. They give closed-form expressions for the optimal exercise strategy and the price of the perpetual dynamic fund protection option. In Section 4, they compare the price of the perpetual dynamic fund protection option with that of the perpetual maximum option.

I am especially interested in Section 4 of the paper. In this discussion, I shall show how to derive the formulas for the two endpoints of the optimal continuation (nonexercise) interval, as given by formulas (4.3) and (4.4) of the paper. For simplicity, I shall consider the case of one stock.

THE OPTIMAL EXERCISE STRATEGY

For the American option, an *optimal exercise strategy* is a *stopping time T* for which the maximum value of the expected discounted payoff is attained. (The expectation is taken with respect to the risk-neutral measure.) For some perpetual options, the optimization problem can be simplified as the problem of determining the optimal value of one or two parameters.

Consider a perpetual American option with payoff function

$$\Pi(z) = \max(K, z), \quad z \geq 0, \tag{D1}$$

where $K > 0$ is the guaranteed price. Consider option-exercise strategies of the form:

$$T_{u,v} = \min\{t \mid S(t) = u \text{ or } S(t) = v\}, \tag{D2}$$

with $0 < u \leq s = S(0) \leq v$. The strategy $T_{u,v}$ is to exercise the option as soon as the stock price rises to the level v or falls to the level u for the first time; the value of this strategy is

$$V(s; u, v) = E[e^{-rT_{u,v}} \Pi(S(T_{u,v})) \mid S(0) = s], \quad 0 < u \leq s \leq v. \tag{D3}$$

Following Section 10.10 in Panjer et al. (1998), we express formula (D3) as

$$V(s; u, v) = \Pi(u)A(s; u, v) + \Pi(v)B(s; u, v), \quad 0 < u \leq s \leq v, \tag{D4}$$

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where

$$A(s; u, v) = \mathrm{E}[e^{-rT_{u,v}}I(S(T_{u,v}) = u)|S(0) = s] = \frac{v^{\theta_2}s^{\theta_1} - v^{\theta_1}s^{\theta_2}}{v^{\theta_2}u^{\theta_1} - v^{\theta_1}u^{\theta_2}}, \tag{D5}$$

and

$$B(s; u, v) = \mathrm{E}[e^{-rT_{u,v}}I(S(T_{u,v}) = v)|S(0) = s] = \frac{s^{\theta_2}u^{\theta_1} - s^{\theta_1}u^{\theta_2}}{v^{\theta_2}u^{\theta_1} - v^{\theta_1}u^{\theta_2}}. \tag{D6}$$

Here θ_1 and θ_2 are the solution of the quadratic equation

$$\frac{\sigma^2}{2}\theta^2 + \left(r - \frac{\sigma^2}{2} - \zeta\right)\theta - r = 0, \tag{D7}$$

with $\gamma = 0$, (7.5) of the paper is the same as (D7).

The problem is to find $\breve{u}$ and $\breve{v}$, the values of u and v that maximize $V(s; u, v)$. Then $V(s; \breve{u}, \breve{v})$, $\breve{u} \leq s = S(0) \leq \breve{v}$, is the price of the perpetual American option. The optimal value $\breve{u}$ and $\breve{v}$ are obtained from the first-order conditions:

$$V_u(s; \breve{u}, \breve{v}) = 0, \tag{D8}$$

$$V_v(s; \breve{u}, \breve{v}) = 0. \tag{D9}$$

Exercise 10.27 in Panjer et al. (1998) shows that (D8) and (D9) are equivalent to the *high contact* or *smooth pasting* conditions:

$$V_s(\breve{u}; \breve{u}, \breve{v}) = \Pi'(\breve{u}), \tag{D10}$$

$$V_s(\breve{v}; \breve{u}, \breve{v}) = \Pi'(\breve{v}). \tag{D11}$$

DERIVATION OF $\breve{u}$ AND $\breve{v}$

With $\breve{u} < K < \breve{v}$, it follows from (D1) that

$$\Pi'(\breve{u}) = 0, \tag{D12}$$

$$\Pi'(\breve{v}) = 1. \tag{D13}$$

Thus, conditions (D10) and (D11) become

$$V_s(\breve{u}; \breve{u}, \breve{v}) = 0, \tag{D14}$$

$$V_s(\breve{v}; \breve{u}, \breve{v}) = 1. \tag{D15}$$

Combining (D5) and (D6) as a matrix equation

$$\begin{pmatrix} A(s; u, v) \\ B(s; u, v) \end{pmatrix} = \begin{pmatrix} u^{\theta_1} & v^{\theta_1} \\ u^{\theta_2} & v^{\theta_2} \end{pmatrix}^{-1} \begin{pmatrix} s^{\theta_1} \\ s^{\theta_2} \end{pmatrix}, \tag{D16}$$

we can rewrite (D4) as

$$V(s; u, v) = (\Pi(u) \quad \Pi(v)) \begin{pmatrix} u^{\theta_1} & v^{\theta_1} \\ u^{\theta_2} & v^{\theta_2} \end{pmatrix}^{-1} \begin{pmatrix} s^{\theta_1} \\ s^{\theta_2} \end{pmatrix}. \tag{D17}$$

Applying (D17) to (D14) yields

$$(K \quad \breve{v}) \begin{pmatrix} \breve{u}^{\theta_1} & \breve{v}^{\theta_1} \\ \breve{u}^{\theta_2} & \breve{v}^{\theta_2} \end{pmatrix}^{-1} \begin{pmatrix} \theta_1\breve{u}^{\theta_1-1} \\ \theta_2\breve{u}^{\theta_2-1} \end{pmatrix} = 0, \tag{D18}$$

or

$$(K \quad \breve{v})\begin{pmatrix} \breve{u}^{\theta_1} & \breve{v}^{\theta_1} \\ \breve{u}^{\theta_2} & \breve{v}^{\theta_2} \end{pmatrix}^{-1}\begin{pmatrix} \theta_1 \breve{u}^{\theta_1} \\ \theta_2 \breve{u}^{\theta_2} \end{pmatrix} = 0. \tag{D19}$$

Similarly, it follows from (D15) and (D17) that

$$(K \quad \breve{v})\begin{pmatrix} \breve{u}^{\theta_1} & \breve{v}^{\theta_1} \\ \breve{u}^{\theta_2} & \breve{v}^{\theta_2} \end{pmatrix}^{-1}\begin{pmatrix} \theta_1 \breve{v}^{\theta_1-1} \\ \theta_2 \breve{v}^{\theta_2-1} \end{pmatrix} = 1, \tag{D20}$$

or

$$(K \quad \breve{v})\begin{pmatrix} \breve{u}^{\theta_1} & \breve{v}^{\theta_1} \\ \breve{u}^{\theta_2} & \breve{v}^{\theta_2} \end{pmatrix}^{-1}\begin{pmatrix} \theta_1 \breve{v}^{\theta_1} \\ \theta_2 \breve{v}^{\theta_2} \end{pmatrix} = \breve{v}. \tag{D21}$$

Since

$$\begin{pmatrix} \breve{u}^{\theta_1} & \breve{v}^{\theta_1} \\ \breve{u}^{\theta_2} & \breve{v}^{\theta_2} \end{pmatrix}^{-1}\begin{pmatrix} \breve{v}^{\theta_1} \\ \breve{v}^{\theta_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{D22}$$

we see that

$$(K \quad \breve{v})\begin{pmatrix} \breve{u}^{\theta_1} & \breve{v}^{\theta_1} \\ \breve{u}^{\theta_2} & \breve{v}^{\theta_2} \end{pmatrix}^{-1}\begin{pmatrix} \breve{v}^{\theta_1} \\ \breve{v}^{\theta_2} \end{pmatrix} = \breve{v}. \tag{D23}$$

Subtracting (D19) from (D21) yields

$$(K \quad \breve{v})\begin{pmatrix} \breve{u}^{\theta_1} & \breve{v}^{\theta_1} \\ \breve{u}^{\theta_2} & \breve{v}^{\theta_2} \end{pmatrix}^{-1}\begin{pmatrix} (1 - \theta_1)\breve{v}^{\theta_1} \\ (1 - \theta_2)\breve{v}^{\theta_2} \end{pmatrix} = 0. \tag{D24}$$

We can combine (D18) and (D22) as

$$(K \quad \breve{v})\begin{pmatrix} \breve{u}^{\theta_1} & \breve{v}^{\theta_1} \\ \breve{u}^{\theta_2} & \breve{v}^{\theta_2} \end{pmatrix}^{-1}\begin{pmatrix} \theta_1 \breve{u}^{\theta_1} & (1 - \theta_1)\breve{v}^{\theta_1} \\ \theta_2 \breve{u}^{\theta_2} & (1 - \theta_2)\breve{v}^{\theta_2} \end{pmatrix} = (0 \quad 0). \tag{D25}$$

Thus, the determinant of the matrix

$$\begin{pmatrix} \theta_1 \breve{u}^{\theta_1} & (1 - \theta_1)\breve{v}^{\theta_1} \\ \theta_2 \breve{u}^{\theta_2} & (1 - \theta_2)\breve{v}^{\theta_2} \end{pmatrix} \tag{D26}$$

must be zero, yielding the formula

$$\frac{\breve{u}}{\breve{v}} = \left(\frac{-\theta_1(\theta_2 - 1)}{\theta_2(1 - \theta_1)}\right)^{1/(\theta_2-\theta_1)}, \tag{D27}$$

which is denoted as $\breve{\varphi}$ in the paper.

We now determine $\breve{v}$ in terms of K, θ_1 and θ_2. Replacing $\breve{u}$ in (D22) by the product $\breve{\varphi}\breve{v}$, we see that formula (D22) is identical to

$$(K \quad \breve{v})\begin{pmatrix} \breve{\varphi}^{\theta_1} & 1 \\ \breve{\varphi}^{\theta_2} & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 - \theta_1 \\ 1 - \theta_2 \end{pmatrix} = 0. \tag{D28}$$

The inverse of the matrix

$$\begin{pmatrix} \breve{\varphi}^{\theta_1} & 1 \\ \breve{\varphi}^{\theta_2} & 1 \end{pmatrix} \tag{D29}$$

is

$$\begin{pmatrix} 1 & -1 \\ -\breve{\varphi}^{\theta_2} & \breve{\varphi}^{\theta_1} \end{pmatrix}, \tag{D30}$$

divided by its determinant. Hence (D25) is equivalent to

$$(K \quad \breve{v}) \begin{pmatrix} 1 & -1 \\ -\breve{\varphi}^{\theta_2} & \breve{\varphi}^{\theta_1} \end{pmatrix} \begin{pmatrix} 1 - \theta_1 \\ 1 - \theta_2 \end{pmatrix} = 0, \tag{D31}$$

from which it follows that

$$\frac{\breve{v}}{K} = \frac{(1 - \theta_1) - (1 - \theta_2)}{(1 - \theta_1)\breve{\varphi}^{\theta_2} - (1 - \theta_2)\breve{\varphi}^{\theta_1}} = \frac{\theta_2 - \theta_1}{\breve{\varphi}^{\theta_1}[(1 - \theta_1)\breve{\varphi}^{\theta_2 - \theta_1} - (1 - \theta_2)]}. \tag{D32}$$

Replacing $\breve{\varphi}$ in (D26) by the right-hand side of (D24) and simplifying, we obtain

$$\frac{\breve{v}}{K} = \left(\frac{-\theta_1}{1 - \theta_1}\right)^{-\theta_1/(\theta_2 - \theta_1)} \left(\frac{\theta_2}{\theta_2 - 1}\right)^{\theta_2/(\theta_2 - \theta_1)}, \tag{D33}$$

which is denoted as $\breve{c}$ in the paper.

Finally, it follows from (D27) and (D24) that

$$\frac{\breve{u}}{K} = \frac{\breve{v}}{K} \frac{\breve{u}}{\breve{v}} = \left(\frac{-\theta_1}{1 - \theta_1}\right)^{(1 - \theta_1)/(\theta_2 - \theta_1)} \left(\frac{\theta_2}{\theta_2 - 1}\right)^{(\theta_2 - 1)/(\theta_2 - \theta_1)}, \tag{D34}$$

which is denoted as $\breve{b}$ in the paper.

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AUTHORS' REPLY

We are very grateful to have received three discussions that greatly enhance the value of our paper. Professor Kwok and Mr. Chu present a brilliant idea in their discussion. By introducing the right to a finite number of resets, they build a bridge between the perpetual maximum option and the perpetual dynamic fund protection option. In particular, this shows why the pricing of the latter can be reduced to the pricing of the former, providing a valuable insight for formula (4.17) and explaining much of Section 4 of our paper. For actuaries interested in implementing dynamic fund protection in their product design, this is an elegant and practical approach.

It is perhaps useful for us to explain the bridge between these two options in terms of the probabilistic approach in our paper and without a *change of numeraire.* As in our paper, let

$$W(s_1, s_2) = \sup_T E[e^{-rT} \max(S_1(T), S_2(T))|S_1(0) = s_1, S_2(0) = s_2], s_1 > 0, s_2 > 0, \tag{R1}$$

denote the time-0 price of the perpetual maximum option. Formulas (4.1)–(4.4) of our paper show how the price can be expressed in terms of the function $g(x)$ and the endpoints $\breve{b}$ and $\breve{c}$ of the optimal nonexercise interval. Ms. Yu has kindly provided a detailed derivation of these two threshold values in the case of one stock.

As in Kwok and Chu's discussion, let $V_n(s_1, s_2)$ denote the price of the perpetual option with up to n resets. The word "perpetual" means that the option has no fixed expiration date. The holder of the

option can cash in or withdraw at any time until the n-th reset. Immediately after the n-th reset, the optionholder must withdraw. The option expires upon withdrawal. Now,

$$V_1(s_1, s_2) = W(s_1, s_2),$$

and, for $n = 1, 2, 3, \ldots,$

$$V_{n+1}(s_1, s_2) = \sup_T \mathrm{E}[e^{-rT} \max(V_n(S_1(T), S_1(T)), S_2(T))|S_1(0) = s_1, S_2(0) = s_2]. \quad \text{(R2)}$$

Using the homogeneity property of the function $V_n(s_1, s_2)$ and the notation

$$v_n = V_n(1, 1), \quad n = 1, 2, 3, \ldots, \quad \text{(R3)}$$

we can rewrite the recursive equation (R2) as

$$V_{n+1}(s_1, s_2) = \sup_T \mathrm{E}[e^{-rT} \max(v_n S_1(T), S_2(T))|S_1(0) = s_1, S_2(0) = s_2]$$

$$= \sup_T \mathrm{E}[e^{-rT} \max(S_1(T), S_2(T))|S_1(0) = v_n s_1, S_2(0) = s_2]. \quad \text{(R4)}$$

Comparing (R4) with (R1), we obtain the *key formula*

$$V_{n+1}(s_1, s_2) = W(v_n s_1, s_2), \quad n = 1, 2, 3, \ldots. \quad \text{(R5)}$$

Our v_n is $W_n(1)$ in Kwok and Chu's discussion. It is obvious that

$$v_1 \leq v_2 \leq v_3 \leq \ldots. \quad \text{(R6)}$$

We shall show that the sequence $\{v_n\}$ is bounded above by $\check{c}$ and that

$$\lim_{n \to \infty} v_n = \check{c}. \quad \text{(R7)}$$

Via (4.1) of our paper, formula (R5) reveals the optimal strategy for the holder of the perpetual option with up to $n + 1$ resets: As long as the stock-price ratio $S_1(t)/S_2(t)$ is between $\check{b}/v_n$ and $\check{c}/v_n$, no action should be taken. If the ratio first attains the lower endpoint of the interval, $\check{b}/v_n$, say at time T, the amount $S_2(T)$ should be withdrawn (and the perpetual option expires). If, on the other hand, the ratio first attains the upper endpoint of the interval, $\check{c}/v_n$, say at time T, the first reset should be implemented, so that $S_2(T)$ is replaced by $S_1(T)$; thereafter the optimal strategy for a perpetual option with up to n resets is applied. Kwok and Chu use the symbols $x^{\widetilde{w}}_{n+1}$ and x^r_{n+1} for our $\check{b}/v_n$ and $\check{c}/v_n$, respectively.

To examine the convergence of the sequence $\{v_n\}$, we introduce the function

$$k(x) = g\left(\frac{x}{\check{b}}\right) = \frac{\theta_2 \left(\dfrac{x}{\check{b}}\right)^{\theta_1} - \theta_1 \left(\dfrac{x}{\check{b}}\right)^{\theta_2}}{\theta_2 - \theta_1}, \quad x > 0, \quad \text{(R8)}$$

and rewrite formula (4.1) of our paper as

$$W(s_1, s_2) = \begin{cases} s_2 & \text{if } \dfrac{s_1}{s_2} \leq \check{b} \\[2ex] s_2 k\left(\dfrac{s_1}{s_2}\right) & \text{if } \check{b} < \dfrac{s_1}{s_2} \leq \check{c} \\[2ex] s_1 & \text{if } \dfrac{s_1}{s_2} \geq \check{c} \end{cases} \quad \text{(R9)}$$

for $s_1 > 0, s_2 > 0.$

Then,

$$v_1 = W(1, 1) = k(1), \tag{R10}$$

and, by (R5),

$$v_{n+1} = W(v_n, 1) = k(v_n), \quad n = 1, 2, 3, \ldots . \tag{R11}$$

We remark that the function $W_n(x)$ in Kwok and Chu's discussion is our $k(v_{n-1}x)$.

The price function $W(s_1, s_2)$ is continuous and its gradient is also continuous (the smooth-pasting condition). Considering $s_1/s_2 = \breve{b}$ and $s_1/s_2 = \check{c}$, we find that

$$k(\breve{b}) = 1, \tag{R12}$$

$$k(\check{c}) = \check{c}, \tag{R13}$$

$$k'(\breve{b}) = 0, \tag{R14}$$

$$k'(\check{c}) = 1. \tag{R15}$$

Because both x^{θ_1} and x^{θ_2} are convex functions, the function $k(x)$ is also convex. From this and (R14), we have $k'(x) > 0$ if and only if $x > \breve{b}$.

We now show by induction that the sequence $\{v_n\}$ is bounded above by $\check{c}$. Because $1 < \check{c}$ and $k(x)$ is an increasing function for $x > \breve{b}$ (and $\breve{b} < 1$), we see that $k(1) < k(\check{c})$, or

$$v_1 < \check{c}$$

by (R10) and (R13). For the induction step, suppose that $v_n < \check{c}$. Again, because $k(x)$ is an increasing function for $x > \breve{b}$, we have $k(v_n) < k(\check{c})$, or

$$v_{n+1} < \check{c}$$

by (R11) and (R13). Thus, the increasing sequence $\{v_n\}$ is bounded above by $\check{c}$.

It follows that $\{v_n\}$ has a limit. What is it? Because k is a continuous function, the limit is a *fixed point* of k. From (R13), we know that $\check{c}$ is a fixed point of k. From (R15) and the convexity of k, we see that there are no other fixed points. Thus, the limit of the increasing sequence $\{v_n\}$ is $\check{c}$. Therefore, we have proved (R7).

It is instructive to take the limit $n \to \infty$ in the key formula (R5). Then we get

$$\lim_{n\to\infty} V_n(s_1, s_2) = W(\check{c}s_1, s_2), \tag{R16}$$

which, if $s_1 < s_2$, is $V(s_1, s_2)$ by formula (4.17) of our paper. Thus,

$$V(s_1, s_2) = \lim_{n\to\infty} V_n(s_1, s_2), \quad s_1 < s_2. \tag{R17}$$

This shows that the price of the perpetual dynamic fund protection option can indeed be obtained as the limit of the prices of the perpetual option with resets. As a check, we note that a special case of (R17) is

$$V(s, s) = \lim_{n\to\infty} V_n(s, s) = s \lim_{n\to\infty} V_n(1, 1) = s \lim_{n\to\infty} v_n = s\check{c}.$$

which is the same as (4.18) of our paper.

Mr. Pansera's discussion shows that the formulas for two stocks can be obtained from those in a one-stock model by the change of *numeraire* method. Using stock 2 as a *numeraire* is also a key step in the approach by Professor Kwok and Mr. Chu. Thus, although Ms. Yu has only derived the optimal threshold values for the case of one stock, her results are readily generalized to the two-stock case.

In conclusion, we thank the discussants for their thought-provoking contributions. They have furthered our understanding of dynamic fund protection.