

Optimal consumer behaviour in a jump-diffusion environment*

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Abstract

Traditional Merton-type optimal portfolio and consumption models are based on diffusion models for risky securities. Recently, modellers have begun to extend this framework to allow for jumps in underlying security prices.

By extending a model of Richard (1975), this paper examines the question of lifetime personal financial planning—how should individuals determine their optimal consumption, portfolio selection and life insurance/annuity needs? It does so in a richer security market setting, incorporating jumps. The paper uses the Markov chain approximation method of Kushner (1977) to determine numerical results for the model.

In contrast to the no-jump setting, we find all control variables attain lower values in the incomplete market setting of jump-diffusing security prices. More significant effects occur in a low return economic environment. Analysis of the expected dynamic path of consumption suggests this model of event risk does not cause sufficient concern to investors to alter the general pattern of lifetime consumption observed in the complete markets case.

Keywords: Optimal portfolio selection; stochastic control; jump-diffusion models; financial planning; Markov chain; life insurance; annuities.

Journal of Economic Literature Classification Numbers: C61, D14, D91, G11, G22, H55, J26.

1 Introduction

The optimal control problem we are concerned with here considers the selection of consumption, investment and life insurance/annuity purchase over an individual's lifetime. In doing so, we model market crashes as a jump process, with deterministic and constant arrival frequency and where the size of the jump is constant in terms of percentages of the market value. This process is used to extend a model first investigated by Richard (1975). Richard's model itself is an extension of the original Merton (1969) model in continuous time, extended to include considerations of human capital and life insurance and annuity purchase.

Our model is motivated by the recent trend in the literature of moving away from a perfect market setting towards one which is incomplete. Traditionally, Merton-type optimal consumption and investment models use a Wiener process for the modelling of the underlying security markets. Although it is a good approximation for stock market movements, there are important differences between such a model and reality. Cont and Tankov (2004), and the references cited therein, list the stylised facts exhibited by financial time series, and the shortcomings of modelling such series by using geometric Brownian motion. The authors also point out the advantages of Lévy process modelling: at a given time horizon, flexible modelling of the distribution of returns, particularly its tails. Time dependence in the returns data—modelling volatility clustering, for example—is more difficult to deal with in such a setting.

Nevertheless, Lévy processes have been widely adopted in modelling financial data. They have been harnessed in the pricing of contingent claims and in credit risk analysis (Chan 1999, Schönbucher 2002, Andersen & Andreasen 2000). Work has also begun in utilising Lévy processes in optimal consumption and investment modelling. Liu, Longstaff & Pan (2003) use the event risk framework of Duffie, Pan & Singleton (2000) to provide an analytic solution to the optimal portfolio problem, finding that jumps in price and volatility have important effects on investment behaviour. Emmer & Klüppelberg (2004) consider optimal portfolio selection where stock prices are modelled by exponential Lévy processes.

Our work differs from that of these authors in that we are also concerned with the implied consumption behaviour of individuals, and so are concerned with the intermediate consumption of agents—this control is omitted in the papers above. Recent work (Campbell & Viciara 2002, Gourinchas & Parker 2002, Purcal 2003) has shown a connection between the optimal lifetime consumption behaviour of agents in incomplete market models of optimal consumption and investment and empirical results of lifetime con-

sumption data. We are interested to know whether, in the incomplete market model we are dealing with here, such results are also present. In addition, our work is also distinguished by consideration of optimal life insurance and annuitisation decision making.

The paper is organised as follows. Section 2 details, interprets and extends the Richard model—the tool we use to analyse optimal demand for annuities. Section 3 explains the Markov chain approximation technique that was used to solve the stochastic optimal control we are dealing with. Section 4 reports the results of our modelling and analyses the findings while section 5 concludes.

2 Model

Building on Merton's (1969, 1971) treatment of optimal consumption and investment, Richard extended the model to include markets for life insurance and annuities.¹ Yaari's (1965) work established that, in the presence of actuarially fair versions of these markets, with information about the probability distributions of future lifetimes available as public information, a risk averse individual with no bequest motive would hold his assets (liabilities) as a life annuity (life insurance).

Richard models a multi-period utility maximising investor with objective²

$$\max E \left[\int_{\tau}^T U(C(t), t) dt + B(Z(T), T) \right], \quad (1)$$

where T is the investor's uncertain time of death, and U , C , Z and B are the investor's utility, consumption, legacy at death and utility from bequest. The investor is able to choose between two securities, one risky and one risk-free,

¹For an empirical treatment of the Richard model in a Japanese context, see Chuma (1994).

²One problem that has been discussed concerning this objective function (Borch 1990, pp. 257–260) has been that it does not allow for the spouse or beneficiary predeceasing the insured. For simplicity, we assume that in the event the spouse dies before the insured, the insured immediately finds someone whom he or she wishes to insure at the same amount. That is, we are assuming that the insured immediately re-marries on the death of his or her spouse—and that the new spouse is the same age as the previous spouse.

The resolution of this issue is not straightforward; Borch does not attempt it. We leave the issue for future research. However, it must be borne in mind that our solution results in over-insurance. We also note that in section 2.4 we are able to interpret the optimal consumption decision rule as choosing consumption such that total wealth provides a reversionary annuity. This suggests the matter can be resolved by resorting to the theory of ordered deaths (Bowers, Gerber, Hickman, Jones & Nesbitt 1997, chapter 17).

with the price of the risky asset, Q , following geometric Brownian motion

$$\frac{dQ(t)}{Q(t)} = \alpha dt + \sigma dq(t), \quad (2)$$

where $dq(t)$ is a Wiener increment.

The investor's change in wealth is given by the stochastic differential equation

$$\begin{aligned} dW(t) = & -C(t)dt - P(t)dt + Y(t)dt + rW(t)dt \\ & + (\alpha - r)\pi(t)W(t)dt + \sigma\pi(t)Wdq(t), \end{aligned} \quad (3)$$

where $P(t)$, $Y(t)$, $W(t)$ are, respectively, the investor's life insurance premium paid, income (assumed to be non-stochastic), and wealth at time t . From equation (2), the mean return on risky investment is α , with standard deviation σ , while the risk-free investment returns r ; the investor places a proportion π of wealth in the risky asset.

Richard's model necessarily incorporates the probability of death of an investor. Let the investor's age-at-death, X (a continuous random variable), have a cumulative distribution function given by $F(x)$ and probability density function of $f(x)$. The survival function, $S(x)$, given by $1 - F(x)$, yields the probability that the investor lives to age x .

The investor buys instantaneous term life insurance to the amount of $Z(t) - W(t)$, if this difference is positive. For life insurance, a premium of $P(t)$ is paid. If we denote the instantaneous probability of death³ by $\mu(t)$, then the amount of premium paid for actuarially fair insurance will be⁴

$$P(t) = \mu(t) [Z(t) - W(t)]. \quad (4)$$

Here Richard has extended Yaari's ideas in his formulation of insurance/annuity demand. In equation (4), should wealth levels lie below the desired bequest, then the individual must purchase insurance to cover this deficit.

Alternatively, any wealth excess to the desired bequest level will be annuitised. In this situation an instantaneous term life annuity contract is purchased, promising to pay the issuer $W(t) - Z(t)$ on death. In return, the consumer receives annuity payments at a rate of $P(t)$ p.a.

³Strictly speaking, this is the conditional instantaneous probability of death (the probability the investor dies at exact age x , having survived to that age) is given by $f(x)/S(x)$. It is known as the *force of mortality* by demographers and actuaries, or as the *hazard rate* or *intensity rate* by reliability theorists (Elandt-Johnson & Johnson 1980).

⁴Richard actually considered the more general case of there being some sort of 'loading' to mortality. This means mortality rates are increased to above their true levels, ensuring profitability for the life insurer. For present purposes, the simpler case of actuarial fairness is sufficient.

2.1 Extending Richard to incorporate jumps

Let us now let the the risky asset follow the jump-diffusion, and replace (2) with

$$\frac{dQ(t)}{Q(t)} = \alpha dt + \sigma dq(t) + \xi dJ, \quad (5)$$

where ξdJ describes the jump term. We assume a deterministic and constant jump intensity, λ . Thus, should a jump occur, its size is constant at ξ of market value. Further, we assume ξ is negative, and so we have an asymmetric jump component—thus we have incorporated market crashes into the Richard model.

In his pioneering work in introducing jump diffusions into financial economics, Merton (1976) included a jump term with random (normally distributed) jump size. Our approach to solving the stochastic optimal control problem can deal with such a model. In the interests of simplicity, however, we leave this task to future research.

2.2 Re-expressing the objective

The investor's problem is to solve equation (1), subject to budget constraint (3) and initial wealth condition $W(0) = W_0$, by optimal choice of controls C, π and Z . Utility, U , is assumed to be strictly concave in C and B is assumed strictly concave in Z . Equation (1) can be re-expressed as

$$J(W, \tau) = \max_{C, Z, \pi} E_{\tau} \int_{\tau}^{\omega} \frac{S(T)}{S(\tau)} \mu(T) \left[\int_{\tau}^T U(C(t), t) dt + B(Z(T), T) \right] dT, \quad (6)$$

where ω represents the limiting age of the underlying mortality table, i.e., one's age at death lies in the range from zero to ω . Equation (6) may be further simplified to the following:⁵

$$J(W, \tau) = \max_{C, Z, \pi} E_{\tau} \int_{\tau}^{\omega} \frac{S(T)}{S(\tau)} [\mu(T)B(Z(T), T) + U(C(T), T)] dT. \quad (7)$$

The Hamilton-Jacobi-Bellman equation can thus be shown to be

$$\begin{aligned} 0 = \max_{C, Z, \pi} \left\{ \mu(t)B(Z(t), t) + U(C(t), t) - (\mu(t) + \rho)J + J_t \right. \\ \left. + [\pi\alpha W + (1 - \pi)rW + Y - C - P]J_W + \frac{1}{2}\sigma^2\pi^2W^2J_{WW} \right. \\ \left. + \lambda[J(W - \pi W\xi) - J(W)] \right\}. \quad (8) \end{aligned}$$

⁵Proceed by using Fubini's theorem to swap the order of integration over the triangle $T \geq t, t \geq \tau$ in \mathbb{R}^2 . For a proof that Fubini's theorem can be used in stochastic integration see, for example, Protter (1990, theorem IV.45).

2.3 Deterministic labour income

We may eliminate $Y(t)$ in equations (3) and (8) as it is non-stochastic. Richard demonstrates that (8) is equivalent to an equation involving capitalised $Y(t)$. Let $b(t)$ be defined as the capitalised value of future income:

$$b(t) = \int_t^\omega Y(\theta) \frac{S(\theta)}{S(t)} e^{-r(\theta-t)} d\theta, \quad (9)$$

and further define *adjusted wealth* as

$$\tilde{W}(t) \equiv W(t) + b(t). \quad (10)$$

Implicitly we are treating capitalised income as a traded asset. That is, markets are complete to the extent that future income can be perfectly replicated by using traded securities. This manipulation of our stochastic control problem allows us to remove $Y(t)$ from (8) and substitute $\tilde{W}(t)$ for $W(t)$.

2.4 Numerical specification

A critical step in producing numerical results for the model is parameter choice. Although Richard has partially parameterised his model, by presenting a solution to the model for isoelastic utility and utility of bequest,

$$U(C(t), t) = h(t) \frac{C(t)^\gamma}{\gamma}, \quad \gamma < 1, h > 0, C > 0 \text{ and} \quad (11)$$

$$B(Z(t), t) = m(t) \frac{Z(t)^\gamma}{\gamma}, \quad \gamma < 1, h > 0, Z > 0 \quad (12)$$

key parts of the model remain unspecified. We follow Purcal (2003) for a plausible choice of $h(t)$ and $m(t)$, associated with the utility and bequest functions and introduced in equations (11) and (12).

We continue to use constant relative risk aversion utility as it is a standard benchmark in economics and readily facilitates comparison with earlier literature.

Bequest function

A plausible choice of $h(t)$ is $e^{-\rho t}$, where ρ is the rate of the investor's time preference. The choice of $m(t)$ is not clear, however. If $m(t)$ is set equal to $h(t)$ consideration of Richard's original optimal closed-form solution readily yields that such a model would produce optimal consumption and bequest amounts that are identical. This result isn't particularly appealing. Purcal

shows that a choice of $m(t) = e^{-\rho t} [2/3 \int_t^\omega (\hat{S}(\theta)/\hat{S}(t)) \exp(-r(\theta - t)) d\theta]$ will satisfy prevailing social norms that a reasonable value is left to a surviving spouse—here the implied bequest motive of the investor is to leave a term certain annuity to his surviving spouse, which pays $2/3 C^*(t)$ from the date of death to the limiting age of the mortality table.

Parameter values

The economic and financial data we use to parameterise the model are summarised in table 1. In table 2 we set out the parameters we adopt for the model. The values we adopt reflect real values of asset accumulation, hence α , the rate of return on the risky asset, is chosen as the real rate of return on the Nikkei: $(1.064/1.039) - 1 \approx 0.025$. The safe rate, r , is similarly chosen. We set rate of time preference equal to the safe rate.⁶

We adopt Japanese male population mortality given by the Ministry of Health and Welfare (1995) Japanese life table, number 18, excluding the effects of the Kōbe earthquake.⁷ Values of γ of -0.5 and -4 reflect an individual who is somewhat risk averse and quite risk averse, respectively.

3 Solution method

The numerical method adopted in solving the problem is as described in Kushner (2001), namely a Markov chain approximation. It is an “explicit” solution approach involving a finite difference approximation to the HJB equation (8), which cleverly makes use of the Markov property of the Lévy process, and derives a Markov chain that is locally consistent with the underlying continuous time stochastic process.

Firstly, consider the coefficient of J_W in (8). Partition the terms that make up this coefficient into a positive group, $d^+ = (\alpha - r)\tilde{\pi}\tilde{W} + r\tilde{W} + \mu\tilde{W}$, and a negative group $d^- = C + \mu Z$, where $d = d^+ + d^-$, d being the coefficient of J_W . Let us approximate the partial derivatives in equation (8) as follows:

$$f_t(x, t) \rightarrow \frac{f(x, t + \delta) - f(x, t)}{\delta}$$

⁶For a careful treatment of the issues of parameterising a life cycle model, see Engen, Gale, Uccello, Carroll & Laibson (1999). Within the life cycle consumption literature there has been considerable debate about the relationship between the rate of time preference and the discount rate and its implications for consumption growth. See, for example, Deaton (1992).

⁷This is a period table, not a cohort table. Mitchell, Poterba, Warshawsky & Brown (1999) argue persuasively that cohort tables should be used for annuity calculations. We leave the cohortisation of this table and re-analysis to future research.

Table 1 Summary of Japanese economic and financial market experience, 1970-2000. All series from DX Data (2000) CEIC database.

	Wages ^a	Prices ^b	Nikkei ^c	Bill Rate ^d
Mean	5.9%	3.9%	6.4%	4.7%
Volatility	2.2%	2.5%	19.9%	0.9%
Range	(-1.9%, 29.1%)	(-1.1%, 24.7%)	(-41.1%, 99.4%)	(0.5%, 12.2%)

^aJapanese nominal wage index, all industries, monthly, Jan. 1970–Jun. 2000

^bJapanese consumer price index, monthly, Jan. 1970–Jun. 2000

^cJapanese index of Nikkei stock average 225, monthly, Feb. 1970–Jul. 2000

^dJapanese three month Bank of Japan bill rate, monthly, Jan. 1970–Jun. 2000.

Table 2 Parameters used in numerical simulation of the model.

$\alpha = 0.025$	Mortality: JLT18 (male)
$r = 0.005$	$\omega = 110$
$\rho = 0.005$	$\gamma = -0.5$ or -4
$\sigma = 0.2$	

$$\begin{aligned}
 f_x(x, t) &\rightarrow \frac{f(x+h, t+\delta) - f(x, t+\delta)}{h} \quad \text{for } d^+ \\
 f_x(x, t) &\rightarrow \frac{f(x, t+\delta) - f(x-h, t+\delta)}{h} \quad \text{for } d^- \\
 f_{xx}(x, t) &\rightarrow \frac{f(x+h, t+\delta) + f(x-h, t+\delta) - 2f(x, t+\delta)}{h^2}
 \end{aligned}$$

and write (8) as follows, where $V(\cdot, \cdot)$ represents the solution to the finite difference equation:

$$\begin{aligned}
 0 = & \max_{C, Z, \pi} \{ \mu \phi B(Z) + U(C) - (\mu(t) + \rho)V(W, t) \\
 & + \frac{V(W+h, t+\delta) - V(W, t+\delta)}{h} d^+ \\
 & - \frac{V(W, t+\delta) - V(W-h, t+\delta)}{h} d^- + \frac{V(W, t+\delta) - V(W, t)}{\delta} \\
 & + \frac{1}{2} (\sigma \pi W)^2 \frac{V(W+h, t+\delta) - 2V(W, t+\delta) + V(W-h, t+\delta)}{h^2} \\
 & + \lambda [V(W + \pi W \xi, t + \delta) - V(W, t + \delta)] \} \quad (13)
 \end{aligned}$$

Equation (13) can be written as

$$\begin{aligned}
V(W, t) = \max_{C, Z, \pi} \frac{1}{1 + \mu_t \delta + \rho \delta} & \left\{ k(W(t)) \cdot \delta + \lambda \delta \cdot V(W + \pi W \xi, t + \delta) \right. \\
& + V(W, t + \delta) \cdot \left[1 - \frac{\delta}{h} d - \frac{\delta}{h^2} (\sigma \pi W)^2 - \lambda \delta \right] \\
& + V(W + h, t + \delta) \cdot \left[\frac{\delta}{h} d^+ + \frac{1}{2} \frac{\delta}{h^2} (\sigma \pi W)^2 \right] \\
& \left. + V(W - h, t + \delta) \cdot \left[\frac{\delta}{h} d^- + \frac{1}{2} \frac{\delta}{h^2} (\sigma \pi W)^2 \right] \right\} \quad (14)
\end{aligned}$$

or more conveniently as

$$\begin{aligned}
V(W, t) = \max_{C, Z, \pi} \frac{1}{1 + \mu_t \delta + \rho \delta} & \{ k(W(t)) \cdot \delta + \lambda \delta \cdot V_{i+i\pi W \xi}^{t+\delta} \\
& + V_{i+1}^{t+\delta} \cdot (1 - \lambda \delta) \cdot p_D^h(i, i+1) \\
& + V_{i-1}^{t+\delta} \cdot (1 - \lambda \delta) \cdot p_D^h(i, i-1) \\
& + V_i^{t+\delta} \cdot (1 - \lambda \delta) \cdot p_D^h(i, i) \} \quad (15)
\end{aligned}$$

where the $p_D^h(\cdot, \cdot)$ may be interpreted as transition probabilities of a Markov chain, locally consistent with equation (3), and given by⁸

$$\begin{aligned}
p_D^h(W, W + h) &= \frac{\delta}{h^2} \frac{1}{1 - \lambda \delta} \left\{ \frac{1}{2} (\sigma \pi W)^2 + h [(\alpha - r) \pi W + rW + \mu W] \right\}, \\
p_D^h(W, W - h) &= \frac{\delta}{h^2} \frac{1}{1 - \lambda \delta} \left\{ \frac{1}{2} (\sigma \pi W)^2 + h [C + \mu Z] \right\}, \text{ and} \\
p_D^h(W, W) &= 1 - p(W, W + h) - p(W, W - h).
\end{aligned}$$

The boundary condition is $V(W, \omega) = \phi(\omega) B(Z(\omega))$. Thus, the solution to the investor's stochastic control problem (1) is approximated by the solution to equation (15) as $h \rightarrow 0$ and $\delta \rightarrow 0$ together. The convergence of this approximation method has been established by viscosity solution techniques (Fitzpatrick & Fleming 1990, Fitzpatrick & Fleming 1991).

Equation (15) was solved on a grid by backward iteration, using a computer.

4 Results

The jump process we introduced models a market crash with unknown timing but constant jump intensity, and with a deterministic and fixed relative size

⁸We must, through choice of δ and h^2 , ensure $0 \leq p_D^h(\cdot, \cdot) \leq 1$.

Table 3 Numerical results of the Richard model, with no jumps. Values are taken in the middle of the wealth grid, at total wealth value of 10.0. Risky investment in expressed as a percentage of total wealth.

Age	value of HJB equation	Consumption	Optimal controls	
			Risky investment	Desired bequest
30	-264.47	0.1781	33.1%	7.5132
40	-218.49	0.2023	33.1%	7.6810
50	-173.67	0.2358	33.1%	7.8957
60	-131.18	0.2842	33.1%	8.1628
70	-92.73	0.3584	33.2%	8.4783
80	-59.15	0.4837	33.2%	8.8394
90	-32.21	0.7255	33.2%	9.10516
100	-11.99	1.4023	33.2%	9.0687

when jump occurs. We expect this added jump process will have a negative impact on all the controls, as well as on the expected wealth as compared to the standard Richard (1975) model. However, the size of the impacts on these state and controls variables, and their relationships with the jump size are unknown, and are the focus of this section.

Below we will examine the impact of different parameter settings (jump frequency, risk aversion, and return parameters) on the model, as well as the expected paths of its state and control variables. As a point of comparison, table 3 gives the values of the control variables of the original Richard model (no jumps), parameterised as per section 2.4.

4.1 Jump Frequency

Table 4 provides the levels of the control variables under all four different four jump frequencies: 0%, 1%, 5%, and 10% for an investor with wealth level of 10.0 at various ages.

In table 4, the values in the second column—expected number of market crashes every decade correspond to λ values of 0%, 1%, 5%, and 10%. The wealth level is set at 10.0, midpoint of our total wealth grid, that is the investor has a total wealth level of 10.0 at age 30, 50, 70, and 90. The market crash is set at ten percent of the market value, that is the relative size is fixed.

Firstly, we see that compared to the complete market results (no event risk), the state and control variables are all negatively affected by the introduction of the market crash component. Furthermore, the size of the negative

Table 4 Dynamics of the control variables and the value function for an investor under four different jump/market crash frequency setting at ages 30, 50, 70 and 90. The values of the control variable and value function are taken from our wealth grid at the midpoint, where wealth is 10.0.

Age	Expected no. of market crashes every decade	Value of HJB equation	Optimal controls		
			Consumption	Risky investment	Desired bequest
30	0.0	-264.47	0.17810	33.08%	7.5132
	0.1	-264.50	0.17809	31.34%	7.5126
	0.5	-265.32	0.17772	24.51%	7.4969
	1.0	-267.81	0.17661	16.15%	7.4503
50	0.0	-173.67	0.23579	33.13%	7.8957
	0.1	-173.68	0.23578	31.37%	7.8953
	0.5	-174.04	0.23545	24.52%	7.8845
	1.0	-175.12	0.23449	16.16%	7.8522
70	0.0	-92.73	0.35838	33.16%	8.4783
	0.1	-92.74	0.35837	31.42%	8.4781
	0.5	-92.83	0.35812	23.55%	8.4722
	1.0	-93.12	0.35738	16.17%	8.4545
90	0.0	-32.21	0.72548	33.17%	9.1052
	0.1	-32.21	0.72546	31.43%	9.1051
	0.5	-32.22	0.72531	24.56%	9.1031
	1.0	-32.25	0.72484	16.17%	9.0971

impact increases with market crash frequencies. This confirms our expectations. In the incomplete market setting, a risk averse investor would reduce his/her consumption and risky investment to accommodate the increased downside risk. Also, we can see that the reduction in desired bequest is less than that of consumption, in terms of percentage of the original level. This is again expected, as we defined bequest as the present value of future consumption streams, which is discounted by the investor's rate of time preference.

As we see from table 4, the consumption level as well as desired bequest are moderately affected by the market crash component. We can see the level of consumption only reduced from 0.23579 to 0.23449 as we moved from complete markets to a market crash component with frequency of 10%. The result suggests that a utility-maximising risk-averse investor should not be strongly influenced by market crashes when making decisions regarding future consumption. More significant is the impact of the jump process on the level of investment in risky assets. This reduction was expected, as risk-averse investors would find the original investment in the risky asset not as attractive, since it does not provide sufficient return for the increased level of downside risk. It is therefore interesting to see how the size of this reduction was affected by the different jump frequencies.

For the complete market case the Merton ratio (that is $(\alpha - r)/[(1 - \gamma)\sigma^2]$) equals one-third. Our result gives the proportion of total wealth invested in risky asset (π) as approximately 33%⁹ for an investor aged 30, in line with the Merton ratio. This level of investment in risky assets is reduced 1.7% to 31.34% at age 30 with $\lambda = 1\%$. This firstly does not appear to be a significant reduction in absolute value, however, it does represent a reduction of over 5% of the original proportion. When jump frequencies are set at 5% and 10%, the level of risky investment reduced to 24.5% and 16.15% respectively, corresponding to relative reductions of 26% and 51% from the original proportion. This high level of reduction in all jump frequencies can be explained in part by our return parameters. Here, we are assuming risky asset returns of 2.5% and a risk-free return of 0.5%. This is a low return setting, and any market crash would have a more significant impact than in a high return setting, which we treat in section 4.3 below.

Intuitively, our results conclude that when facing market crash uncertainties, investors should reduce their exposure to the risky asset by moving towards riskless asset. The level of consumption and desired bequest are less sensitive to market uncertainties, and are only moderately influenced.

Impact on Expected Wealth

As we have discussed, a utility-maximising investor's response to an increase in jump frequency is to alter his/her investment strategy between the risky and risk-free assets. This allows the investor to reduce portfolio exposure to the market crash risk while maintaining objectives for future consumption. However, such reduction in the level of risky assets has an impact on the expected wealth. Given the long time horizon of the problem, this change in the portfolio return would have a significant impact on the accumulated wealth level. In table 5, portfolio returns are provided for all four jump intensity settings over a 10-year period.

In the above table, the portfolio compositions are taken from table 6.2 at age 30. The expected accumulations are calculated using the expected returns of risky and risk free assets, including the impact of jumps.¹⁰ And the expected return on the portfolios are calculated multiplying the portfolio composition by each assets' expected return.

Despite the small scale of percentages, we do see a significant reduction in the portfolio return. By measuring the difference between the risk free

⁹In this section, all the percentages are taken for an investor with a wealth level of 10.0, which is at the half way point of our wealth grid.

¹⁰When the expected number of jumps in 10 years is less than one, the return over 100 years is calculated and hence taken to the power of 0.1.

Table 5 Expected returns on portfolios over a 10-year period.

Jump frequency	Portfolio composition		Expected accumulated value over 10 years	Expected total return over 10 years
0%	risky	33.08%	1.280	12.67%
	riskfree	66.92%	1.051	
1%	risky	31.34%	1.267	11.87%
	riskfree	68.66%	1.051	
5%	risky	24.51%	1.214	9.12%
	riskfree	75.49%	1.051	
10%	risky	16.15%	1.152	6.74%
	riskfree	83.85%	1.051	

return and the portfolio return, we see a reduction of over 50% from the original portfolio return by utility maximising decisions. Again, the low return settings have a impact on the significance of this reduction. The case of high return settings will be discussed later.

4.2 Risk aversion

The results above were for an investor with an risk aversion parameter of 1.5 (i.e., $\gamma = -0.5$). A natural question is then what would be the impact for a more risk-averse investor? Here, we discuss the impact of the jump process has on two investors, one with our risk aversion parameter of 1.5 as above, and one more risk-adverse investor with a risk aversion parameter of 5.0 (i.e., $\gamma = -4$).

Impact in complete market

Table 6 provides the dynamics of the control variables, at ages 30 through to 100, for two investors each with wealth level of 10, but with different risk aversion. The security market here is assumed to be complete, corresponding to the Richard model.

From the table, we see a difference of over 23% of total wealth invested in risky asset between the two individuals. This represents a reduction of over 70% moving from relative risk aversion of 1.5 to 5.0. This is approximately the same as the Merton ratio, which reduces from one-third to one-tenth. As we would have expected, the level of consumption and desired bequest show moderate reduction, while risky investment reduced inline with the Merton ratio. This result will serve as a starting point for our comparison with the introduction of the market crash component.

Table 6 Dynamics of the control variables for two investors with different risk aversion setting at various ages in a complete market. The values are taken from the wealth grid at 10.0.

Age	$r(x) = 1.5$			$r(x) = 5.0$		
	Consumption	Risky investment	Bequest	Consumption	Risky investment	Bequest
30	0.1781	33.1%	7.5132	0.1744	9.80%	7.4421
40	0.2023	33.1%	7.6810	0.1984	9.80%	7.6109
50	0.2358	33.1%	7.8957	0.2317	9.81%	7.8265
60	0.2842	33.1%	8.1627	0.2797	9.82%	8.0940
70	0.3584	33.2%	8.4783	0.3534	9.83%	8.4098
80	0.4837	33.2%	8.8394	0.4778	9.83%	8.7706
90	0.7255	33.2%	9.1052	0.7178	9.84%	9.0358
100	1.4023	33.2%	9.0687	1.3893	9.84%	8.9983

Table 7 Dynamics of the control variables for two investors with different relative risk aversions. The values are taken from the wealth grid at 10.0, at age 30.

Market crash frequency	Relative risk aversion	Optimal Controls		
		Consumption	Risky Investment	Desired bequest
1%	1.5	0.1781	31.3%	7.5126
	5.0	0.1744	9.28%	7.4417
5%	1.5	0.1778	24.5%	7.4969
	5.0	0.1742	7.26%	7.4308
10%	1.5	0.1766	16.1%	7.4503
	5.0	0.1734	4.78%	7.3983

Impact with jump process

Table 7 provides the dynamics of the control variables, at age 30 and wealth level of 10.0, for two investors with different risk aversion parameters, under the other three jump frequencies.

From the table, we see a similar pattern as in a complete market. First we notice, the level of consumption and desired bequest have reduced moderately under all three market crash frequency settings. Here, the size of the reduction in consumption and desired bequest are similar to the results of table 4, where market crashes are introduced to an investor with relative risk aversion of 1.5. This suggests that investors reacts to increases in market crash frequencies similarly regardless of their level of risk aversion.

And we see that the level of risky investment was significantly affected by the increase in risk aversion, with reductions of similar patterns as in a complete market. Here, in the second row of each market crash frequency, the optimal controls are impacted by both increase in risk aversion, and

increase in market crash frequency compared to the complete market setting. Therefore, it is not surprising to see the level of risky investment was further reduced. And this in turn explains the further reduction in consumption and desired bequest.

Another interesting result noticed is the relative size of the reduction in risky investment and the increase in risk aversion. In a complete market setting, the risky investment reduced from one-third to one-tenth of the total wealth, as described by the Merton ratio. In our incomplete market setting, the reduction appeared to have similar size. If we compare the relative size of the reduction for the case of 10% market crash frequency to that of a complete market, we can see the reductions are both approximately 70%. This suggests that the relative reduction in risky investment should not be affected by market crash frequencies, and an investor should reduce his/her exposure to the risky asset at the same proportion as in a complete market.

4.3 Jump impact in high return setting

Here we change our return settings to mimic the Australian securities market. We have chosen the risk free rate to be 2%. This was chosen to represent the real rate of return, calculated by the cash rate of return¹¹ of 5% minus the higher range of the inflation target 3%.¹² The investor's rate of time preference is adjusted to be the same level as the risk free rate. For the risky investment return, we have taken 17 years of the All Ordinaries daily return, from 1987 to 2003,¹³ to calculate annual return and variance. We approximate the result by setting the annual risky return and standard deviation of the return at 4.8% and 16% respectively. In our data, we have included the effect of the 1987 crash.

Other parameters are the same as in the low return setting, and here, we again would like to compare the results from a complete market to that of an incomplete market with market crashes. The market crash frequency is set at 10%, averaging once per decade, and the size of the market crash is set at 10% of the then market value.

Results

The resulting dynamics of the control variables under the high return setting are shown in table 8. Again, as we expected, the control variables are

¹¹This is the cash rate of return at 5th November 2003, taken from the Reserve Bank of Australia.

¹²This reflects the Reserve Bank of Australia's stance on monetary policy.

¹³Obtained from Bloomberg database.

Table 8 Dynamics of the control variables under our high return settings at various ages. The values are taken from the wealth grid at 10.0.

Age	Complete market			Market crash frequency at 10%		
	Consumption	Risky investment	Bequest	Consumption	Risky investment	Bequest
30	0.2973	44.5%	7.1663	0.2967	43.0%	7.1531
40	0.3190	45.0%	7.3224	0.3186	43.5%	7.3113
50	0.3510	45.5%	7.5442	0.3506	44.0%	7.5353
60	0.3992	46.0%	7.8420	0.3988	44.3%	7.8353
70	0.4744	46.2%	8.2143	0.4741	44.6%	8.2097
80	0.6024	46.4%	8.6564	0.6022	44.8%	8.6536
90	0.8453	46.4%	9.0003	0.8451	44.8%	8.9988
100	1.5171	46.4%	9.0174	1.5169	44.8%	9.0166

negatively affected by the market crash component. However, compared to table 4, we see that the size of the reduction in all control variables are limited in this high return setting. Intuitively, given the high returns offered by the risky asset, the investors are more able to recover the losses if a market crashes. Therefore, their decisions regarding the optimal consumption and investment are less affected by the market crash component.

4.4 Expected lifetime dynamics

Thus far we have been primarily concerned with isolated point comparisons of our model results. Another perspective for examining our results is by considering the expected paths the dynamic state and control variables take over time. Again, the introduction of event risk will impact these expected paths. From our results above, it is clear that the expected lifetime path of wealth will be lowered, but what of the control variables we are modelling?

Answering this question is neatly provided by the approximating Markov chain solution method we have adopted. This approach yields transition probabilities at every grid point. By simple recursion we can then use these transition probabilities to determine expected values. We do this by using these transition probabilities to move back from a grid at age x ($x > 30$) to the original grid at age 30. This gives us the expected values of state or control variables at age x conditional on wealth and salary at age 30.

Figure 1 gives the results for expected lifetime consumption: consumption here is convex and increasing. This is wholly in line with the results one obtains in the complete markets case. In this model, the introduction of event risk, although lowering the dynamic path of consumption, doesn't essentially change its convexity.

This is an interesting result, as it contrasts to other findings of optimal

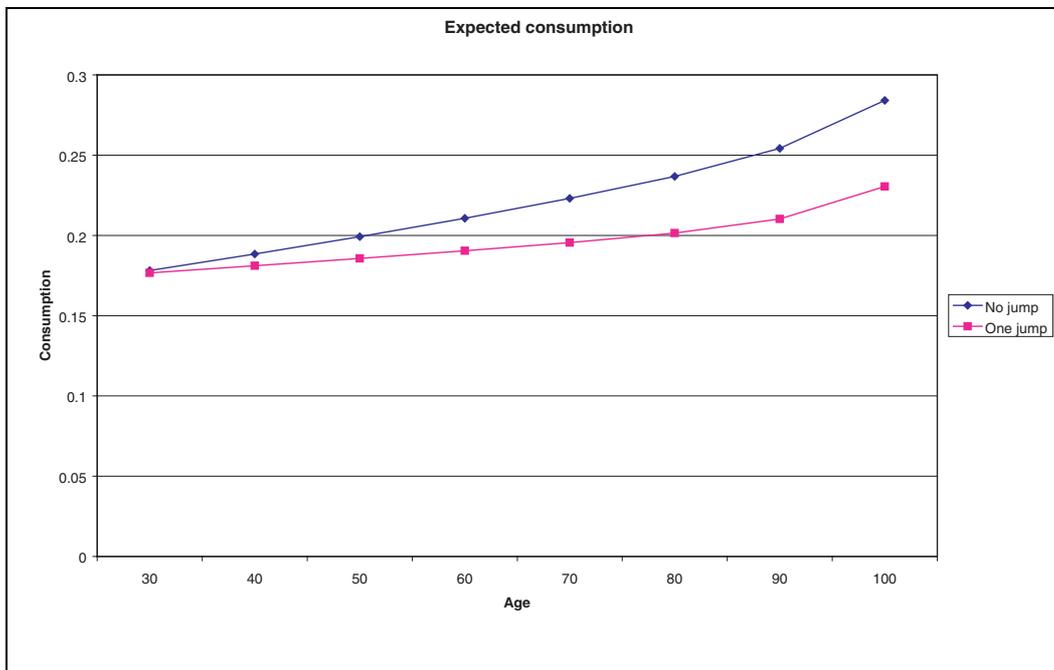


Figure 1: Expected optimal consumption paths for an individual aged 30 with initial total wealth of 10.0.

consumption and investment models which are built on incomplete markets. Some researchers, working with models which include stochastic labour income, found humped shaped optimal expected lifetime consumption paths (Gourinchas & Parker 2002, Campbell & Viciera 2002, Purcal 2003), which are, in fact, in accordance with empirical evidence. Our modelling of event risk doesn't appear to be of sufficient concern to investors to alter the general pattern of lifetime consumption we observe in the complete markets case.

5 Conclusion

In this paper we have extended the Richard (1975) model to a Lévy environment, with the introduction of a jump-diffusion process. Our jump component was asymmetric, aimed at modelling market crashes. The effect this jump process has on the state and control variables was examined. Comparisons were provided for different jump parameters, risk aversion parameters, and return parameters.

Our results reveal a negative impact on the state and control variables. We see that the level of consumption and desired bequest are moderately affected, while risky investment as a proportion to wealth was found to be sig-

nificantly reduced. This pattern of results was evident regardless of changes in jump frequency and risk aversion.

We have also found that the size of the negative impact is dependent on the economic environment. The effect is more striking in a low return environment, while less so in a economy experiencing higher real returns. Intuitively, the negative impact of the market crash component depends on the likelihood that the investors will be able to recover their losses.

Finally, examination of the expected paths of the state and control variables also showed lower values in our model which allows for event risk. In contrast to some other models of optimal consumption and investment in an incomplete market setting, our modelling of event risk doesn't seem to be of sufficient concern to investors to alter the general pattern of lifetime consumption we observe in the complete markets case.

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