VALUATION OF LIFE INSURANCE USING DIFFUSION INTEREST RATE MODEL

Tereza Jarolímková

Faculty of Mathematics and Physics, Charles University Prague
Department of Probability and Mathematical Statistics

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Introduction

- Components of risks – volatility, uncertainty
- Principle of parsimony
- Types of risks
  - ✓Technical risk – can be diversified,
    – deterministic approach
  - ✓Investment risk – stochastic approach

- Fair value principles
  - ✓Risk free rate of interest for discounting
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Risk free rate of interest

- Risk free rate of interest in the k^th policy year:

$$1 + I_k = e^{\int_{k-1}^{k} r_t \, dt} = e^{(R_k - R_{k-1})}$$

Random variable $1 + I_k \sim logN(\mu_k, \sigma_k{}^2)$

- Diffusion process of r_t:

Hull-White model $dr_t = (\rho(t) - \kappa \cdot r_t)dt + \delta_1 \, dW_t^1$

- Markovian character: on condition that $r_s = y$

$\{r_t, t \geq s\}$ and $\{r_u, u \leq s\}$ are independent

Parameters of the model

- Parameters of the model in compliance with the value of zero-coupon bond:

$$P(r,t,T) = E\left\{ e^{-\int_t^T r_s\, ds} \,\middle|\, r_t = r \right\}$$

- Market yield curve $f^M(0,t) = -\frac{\partial}{\partial t} \ln P(r,0,t)$

- Parameter of the Hull-White model

$$\rho(t) = f^M(0,t)' + \kappa \cdot f^M(0,t) + \frac{\delta_1^2}{2 \cdot \kappa^2} \cdot \left(1 - e^{-2 \cdot \kappa \cdot t}\right)$$

and $r_0 = f^M(0,0)$

Characteristics of $\mathbf{r_k}$

- On condition $r_0 = s$: $E_s\, r_k = e^{-\kappa \cdot k} \cdot r_0 + \int_0^k e^{-\kappa\cdot(k-u)} \rho(u)du$

$$Var\ r_k = \delta_1^2 \cdot \int_0^k e^{-2\cdot\kappa\cdot u} du\ , \quad Cov(r_k, R_k) = \frac{\delta_1^2}{\kappa} \cdot \int_0^k \left(1 - e^{-\kappa\cdot u}\right) \cdot e^{-\kappa\cdot u} du$$

- On condition $r_{k-1} = y$:

$$E_y(R_k - R_{k-1}) = \frac{1}{\kappa} \cdot \left(1 - e^{-\kappa}\right) \cdot y + \frac{1}{\kappa} \cdot \int_{k-1}^{k} \left(1 - e^{-\kappa\cdot(k-z)}\right) \cdot \rho(z)dz$$

$$Var_y(R_k - R_{k-1}) = \frac{\delta_1^2}{\kappa^2} \cdot \int_{k-1}^{k} \left(1 - e^{-\kappa\cdot(k-z)}\right)^2 dz$$

$$Cov_y(r_k, R_k - R_{k-1}) = E\left(\delta_1 \cdot \int_{k-1}^{k} e^{-\kappa\cdot(k-z)} dW_z^1\right) \cdot \left(\frac{\delta_1}{\kappa} \int_{k-1}^{k} (1 - e^{-\kappa\cdot(k-z)}) dW_z^1\right)$$

Rate of return achieved on investment

- Actually achieved rate of return in the kth policy year:

$$1+J_k = e^{\,q\cdot\int_{k-1}^{k} r_u\,du \;+(1-q)\cdot\left[\nu+\delta_2\cdot\left(W_k^2-W_{k-1}^2\right)\right]}$$

- Consist of
  - ✓ Risk free rate of return
  - ✓ Risk premium
  - ✓ Noise component

- Parameter q: mixed portfolio

Value of the liability

- Endowment contract with the guaranteed interest rate i'
- Profit sharing $Q_k = 0{,}8 \cdot (J'_k - i')_+ \cdot {}_{k-1}V_x + \left(1 + i' + 0{,}8 \cdot (J'_k - i')_+\right) \cdot Q_{k-1}$
- Value of the liability

$$
\begin{aligned}
C = & -K \cdot {}_n p_x \cdot E v^{(n)} \left(1 + Q_n\right) - \\
& - K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot s_{k+1,n} \cdot E\, v^{(k+1)} \cdot \left({}_{k+1}V_x^{surrender} + 0{,}9 \cdot Q_k\right) - \\
& - K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot q_{x+k} \cdot E v^{(k+1)} \cdot \left(1 + Q_k\right).
\end{aligned}
$$

Valuation using diffusion process (1)

- Recursive formula

$$E_s^z v^{(k)} Q_k = 0{,}8 \cdot {}_{k-1}V_x \cdot E_s^z (J_k' - i')_+ \cdot (1 + I_k)^{-1} \cdot v^{(k-1)} +$$
$$+ (1 + i') \cdot E_s^z v^{(k)} \cdot Q_{k-1} + 0{,}8 \cdot E_s^z (J_k' - i')_+ \cdot (1 + I_k)^{-1} \cdot v^{(k-1)} \cdot Q_{k-1}.$$

- Markovian character of diffusion process

  on condition $r_{k-1} = y$:

$$E_s^z (J_k' - i')_+ (1 + I_k)^{-1} v^{(k-1)} = \int_{-\infty}^{\infty} E_y^z (J_k' - i')_+ (1 + I_k)^{-1} \cdot E_s^y v^{(k-1)} \cdot \frac{1}{\sqrt{2\pi \cdot Var\, r_{k-1}}} \cdot e^{-\frac{(y - E_s r_{k-1})^2}{2 \cdot Var\, r_{k-1}}} dy,$$

$$E_s^z v^{(k)} \cdot Q_{k-1} = \int_{-\infty}^{\infty} e^{-E_y^z \Delta R_k + \frac{1}{2} Var^z \Delta R_k} \cdot E_s^y v^{(k-1)} \cdot Q_{k-1} \cdot \frac{1}{\sqrt{2\pi \cdot Var\, r_{k-1}}} \cdot e^{-\frac{(y - E_s r_{k-1})^2}{2 \cdot Var\, r_{k-1}}} dy.$$

Valuation using diffusion process (2)

- $$E_y^z(J'_k - i')_+ \cdot (1+I_k)^{-1} = e^{u+\frac{(q-1)\cdot\left(2\cdot\mu+\sigma^2\cdot(q-1)\right)}{2}} \cdot \Phi\left(-\frac{\frac{\ln(1+i')-u}{q} - \left(\mu_k + \sigma^2\cdot(1-q)\right)}{\sigma_k}\right) -$$
$$-(1+i')\cdot e^{-\mu_k+\frac{\sigma_k^2}{2}} \cdot \Phi\left(-\frac{\frac{\ln(1+i')-u}{q} - \left(\mu_k - \sigma_k^2\right)}{\sigma_k}\right)$$

- The contingent distribution of $R_k - R_{k-1}$, while $r_{k-1} = y, r_k = z$ is:

$$N\left(\mu_k, \sigma_k^2\right) = N\left(E_y \Delta R_k + \frac{Cov_y(r_k, \Delta R_k)}{Var_y\ r_k}\cdot(z - E_y r_k), \left(1 - \frac{Cov_y(r_k, \Delta R_k)^2}{Var_y\ r_k \cdot Var_y\ \Delta R_k}\right)\cdot Var_y\ \Delta R_k\right)$$

Example – value of the liability

k	Discounted death benefit	Discounted surrender benefit	Discounted maturity benefit	Total value of the liability C
1	126	632	0	758
2	119	1 073	0	1 192
3	121	1 364	0	1 485
4	120	1 536	0	1 656
5	119	1 616	0	1 735
6	120	1 627	0	1 747
7	124	1 585	0	1 709
8	126	1 503	0	1 629
9	127	1 392	0	1 519
10	135	1 259	35 758	37 151
Total	1 238	13 586	35 758	50 582

Sum assured 100 000, duration 10 years,

Parameters:

$$\kappa = 0{,}95\ , \delta_1 = 0{,}015,\ r_0 = 0{,}028,\ \nu = \ln(1 + 0{,}08),\ \delta_2 = 0{,}12,\ q = 85\%$$

Example – value of the guarantee

The value of techical interest rate guarantee					
k	q=10%	q=20%	q=50%	q=80%	q=90%
1	0	0	0	0	0
2	204	189	144	124	138
3	380	349	256	200	211
4	533	486	345	246	247
5	666	602	416	274	260
6	783	704	475	291	263
7	887	793	523	302	261
8	982	873	565	309	257
9	1 070	946	601	315	253
10	1 153	1 015	634	320	250
	6 659	5 956	3 958	2 381	2 142

Value of the interest rate guarantee:

$$K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot E\, v^{(k+1)} \cdot \left(i' - J'_{k+1}\right)_+ \cdot \left({}_k V_x + Q_k\right)$$

Portion of the risk-free assets in portfolio: parameter q

Profit and release from risk

The method can be also used for valuation of:

- Expected value of surpluses $\sum_{k=0}^{n-1} E\, v^{(k+1)} S_{k+1}$
- Reserve for risk and uncertainty

$$\overline{E}_{k+1} = S_{k+1} - \frac{{}_k p_x \cdot \alpha \cdot K}{\sum_{k=0}^{n-1} E\ {}_k p_x \cdot v^{(k)}} \cdot (1 + I_{k+1})$$

➡ Zero profit at issue

$$P_0 - \alpha \cdot K = \sum_{k=0}^{n-1} E\,\left(S_{k+1} - \overline{E}_{k+1}\right) \cdot v^{(k+1)} - \alpha \cdot K = 0$$

- Questions? Discussion…

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