VALUATION OF LIFE INSURANCE USING DIFFUSION INTEREST RATE MODEL

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Abstract:

This paper presents the methodology how to use diffusion interest rate models, e.g. Hull – White or Vasicek model for discounting cash flows. The methodology is based on Markovian character of diffusion processes and is generally usable in actuarial computations. Described method is applied on endowment contract with profit sharing and interest rate guarantee. The paper is concerned mostly with liability valuation, guarantee valuation and also with profit, release from risk and income statement.

Keywords:

Diffusion process, Markovian character, Interest rate model, Valuation of life insurance liability, Interest rate guarantee, Income statement.

1 Introduction

The stochastic models become an important tool in practical actuarial modelling, focusing especially on guarantees pricing and liability valuation. The aim of the paper is to propose a methodology based on the Markovian character of diffusion processes, which can be used in practical actuarial computations for discounting cash flows. We will apply the method on endowment contract with profit sharing, concerning on liability valuation. The method will be illustrated by numerical examples which were calculated by a stochastic model created in MS Excel.

The current development on the field of international accounting and reporting standards with its open questions was also the motivation for the paper. We are concerned especially with activating acquisition expense, profit at issue, accounting about adjustments for risk and uncertainty and its changes and release from risk on insurance liabilities.

1.1 Risk approach

There are two main components of risk which are important for our approach. These are volatility and uncertainty. Volatility can be diversified, while risks are independent. Choice of a model and setting parameters constitute the uncertainty risk.

The main principles used for modelling risks are recommended by International Association of Actuaries (IAA) in [10]. The insurer is exposed to technical and investment risk. We will assume that the technical risk can be diversified. It is proper to use a stochastic model to allow for the volatility of investment risk.

International Association of Actuaries also declares the principle of parsimony, which consist of using rather simple models. These models can by easily interpreted and used for practical calculations.

1.2 Fair value principles

Our model will be based on fair value concept, which is also the part of IASB a FASB framework. The key elements are the time value of money and allowing for market value margins. We will use three main assumptions

- Risk free rate of interest is used for discounting cash flows.

- The expected cash flows should be adjusted by market value margins in order to include the market value of risk connected with an adverse deviation of assumptions (referred also as adjustment for risk and uncertainty).
- The adjustment should include imperfections of the market. This approach can be found in Towers Perrin's study referred as [8] .

2 Instantaneous rate of interest

Suppose the space $\Omega = C^1_{[0,T]} \times C^2_{[0,T]}$ created by instantaneous trajectories of two Wiener processes with the real probability measure $\varpi^1 \times \varpi^2$.

For modelling the instantaneous rate of interest we will use a diffusion process $\{r_t\}$, which can be expressed by the equation $dr_t = \theta(t, r_t)dt + \delta(t, r_t)dW_t$, $t \in [0,T]$, where θ and δ are generally functions and W_t is Wiener process regarding to a filtration $F = \{F_t, t \in [0,T]\}$.

The Markovian character of diffusion processes is very important for our method. It says that on condition that $r_s = y$ there are $\{r_t, t \geq s\}$ and $\{r_u, u \leq s\}$ independent, as is shown in [6].

2.1 Risk free rate of interest

Denote I_k a random variable representing the risk free rate of interest in k^{th} policy year. We assume that

$$(1)\quad 1 + I_k = e^{\int_{k-1}^{k} r_t\, dt} = e^{(R_k - R_{k-1})}$$

and that $1 + I_k \sim \log N(\mu_k, \sigma_k^2)$, where parameters μ_k and σ_k^2 depend on a model used for $\{r_t\}$ and will be determined later.

We will use the Hull-White model $dr_t = (\rho(t) + \delta_1 \cdot \lambda_1 - \kappa \cdot r_t)dt + \delta_1\, dW_t^1$, $t \in [0,T]$, for modelling the instantaneous risk free rate of interest r_t.

Let's assume $d\widetilde{W}_t^1 - \lambda_1 dt = dW_t^1$, where $\{\widetilde{W}_t^1\}$ is $\widetilde{\varpi}^1$-Wiener process ($\widetilde{\varpi}^1$ is risk neutral probability measure). Thus Radon-Nikodym derivative is $\dfrac{d\widetilde{\varpi}^1}{d\varpi^1} = e^{-\lambda_1 \cdot W_T^1 - \frac{1}{2}\cdot\lambda_1^2 T}$.

2.2 Discounting

According to principles of the fair value, we will use a deflator (stochastic discounting factor) $D_k = e^{-\int_0^k r_u du - \lambda_1 \cdot W_k^1 - \frac{1}{2}\cdot\lambda_1^2 \cdot k}$ for discounting cash flows from the end of the period k to time 0. This stochastic discounting factor is the discounted value of the Radon-Nikodym derivative.

Using this deflator is the same as using $v^{(k)} = e^{-\int_0^k r_u du}$ with a probability measure $\tilde{\varpi}^1 \times \varpi^2$, while $\{r_t\}$ is the solution of the equation

$$(2)\quad dr_t = (\rho(t) - \kappa \cdot r_t)dt + \delta_1 \, d\tilde{W}_t^1 .$$

2.3 Parameters of the model

We will determine parameters of the model in compliance with the value of zero-coupon bond with maturity T expressed as (see [1])

$$(3)\quad P(r,t,T) = E_{\tilde{\varpi}^1 \times \varpi^2}\left\{ e^{-\int_t^T r_s \, ds} \,\middle|\, r_t = r \right\} .$$

Let $f^M(0,t)$ denote a market yield curve. Assuming $f^M(0,t) = -\frac{\partial}{\partial t} \ln P(r,0,t)$, we can express the parameter $\rho(t)$ given by the equation $dr_t = (\rho(t) - \kappa \cdot r_t)dt + \delta_1 \, d\tilde{W}_t^1$ as $\rho(t) = f^M(0,t)' + \kappa \cdot f^M(0,t) + \frac{\delta_1^2}{2\cdot\kappa^2}\cdot\left(1 - e^{-2\cdot\kappa\cdot t}\right)$ and $r_0 = f^M(0,0)$. This can be also found in [3].

It is useful to consider a simple function $f^M(0,t)$ for further analytical calculation, for example $f^M(0,t) = \alpha + \beta \cdot \left(1 - e^{-\frac{t}{\gamma}}\right)$.

Thus $\rho(t) = \left[\kappa \cdot (\alpha + \beta) + \frac{\delta_1^2}{2\cdot\kappa}\right] + \left(\frac{\beta}{\gamma} - \kappa \cdot \beta\right)\cdot e^{-\frac{t}{\gamma}} - \frac{\delta_1^2}{2\cdot\kappa}\cdot e^{-2\cdot\kappa\cdot t}$.

2.4 Characteristics of r_k a R_k

Solving the differential equation (2), the instantaneous rate of interest r_k can be expressed as

$$(4)\quad r_k = e^{-\kappa\cdot k}\cdot r_0 + \int_0^k e^{-\kappa\cdot(k-u)}\rho(u)du + \delta_1\cdot\int_0^k e^{-\kappa\cdot(k-u)}d\widetilde{W}_u^1\ , r_0 = s\ .$$

The variable R_k equals to

$$R_k = \int_0^k r_u du = \frac{1}{\kappa}\cdot\left(1-e^{-\kappa\cdot k}\right)\cdot r_0 + \frac{1}{\kappa}\cdot\int_0^k\left(1-e^{-\kappa\cdot(k-u)}\right)\rho(u)du + \frac{\delta_1}{\kappa}\cdot\int_0^k\left(1-e^{-\kappa\cdot(k-u)}\right)d\widetilde{W}_u^1\ .$$

We need to determine

$$E_s\ r_k = e^{-\kappa\cdot k}\cdot r_0 + \int_0^k e^{-\kappa\cdot(k-u)}\rho(u)du\ ,\ Var\ r_k = \delta_1^2\cdot\int_0^k e^{-2\cdot\kappa\cdot u}du = \delta_1^2\cdot\frac{1-e^{-2\cdot\kappa\cdot k}}{2\kappa}\ \text{ and}$$

$$Cov(r_k, R_k) = \frac{\delta_1^2}{\kappa}\cdot\int_0^k\left(1-e^{-\kappa\cdot u}\right)\cdot e^{-\kappa\cdot u}du = \frac{\delta_1^2}{\kappa}\cdot\left(\frac{1}{2\kappa} - \frac{e^{-2\cdot\kappa\cdot k}\cdot(-1+2e^{\kappa\cdot k})}{2\kappa}\right),$$

where an index s denotes the condition $r_0 = s$.

Finally we denote $\Delta R_k = R_k - R_{k-1}$. Assuming that $r_{k-1} = y$, we determine

$$E_y(R_k - R_{k-1}) = E\int_{k-1}^k r_u du = \frac{1}{\kappa}\cdot\left(1-e^{-\kappa}\right)\cdot y + \frac{1}{\kappa}\cdot\int_{k-1}^k\left(1-e^{-\kappa\cdot(k-z)}\right)\cdot\rho(z)dz\,,$$

$$Var_y(R_k - R_{k-1}) = Var\int_{k-1}^k r_u du = \frac{\delta_1^2}{\kappa^2}\cdot\int_{k-1}^k\left(1-e^{-\kappa\cdot(k-z)}\right)^2 dz = \frac{\delta_1^2}{\kappa^2}\cdot\left(-\frac{3-4e^{-\kappa}+e^{-2\kappa}-2\kappa}{2\kappa}\right),$$

$$Cov_y(r_k, R_k - R_{k-1}) = E\left(\delta_1\cdot\int_{k-1}^k e^{-\kappa\cdot(k-z)}d\widetilde{W}_z^1\right)\cdot\left(\frac{\delta_1}{\kappa}\int_{k-1}^k(1-e^{-\kappa\cdot(k-z)})d\widetilde{W}_z^1\right) =$$

$$= \frac{\delta_1^2}{\kappa}\cdot\left(\frac{1}{2\kappa} - \frac{e^{-2\cdot\kappa}\cdot(-1+2e^{\kappa})}{2\kappa}\right).$$

2.5 Actually achieved rate of return

Denote J_k a random variable representing the actually achieved rate of return in k^{th} policy year. We assume that

$$(5)\quad 1+J_k = e^{q\cdot\int_{k-1}^k r_u du + (1-q)\cdot\left[\nu + \delta_2\cdot\left(W_k^2 - W_{k-1}^2\right)\right]}\ .$$

$1+J_k$ consists of the risk free yield, a risk premium ν (the value of risk) and a noise component, which can be modelled by increases of second Wiener process $\{W_t^2, t \geq 0\}$, independent of $\{W_t^1, t \geq 0\}$. We can use this component for modelling investment into shares or also imperfections of the market. We use a parameter q for modelling of a mixed portfolio consisting of $q \cdot 100\%$ risk free instruments and $(1-q)\cdot 100\%$ risky instruments. We assume that $q \neq 0$.

When we need to consider an adjustment by market value margin according to fair value principle, we replace ν with $\nu' = \nu - \tau$. Thus

$$1+J_k' = e^{q\cdot \int_{k-1}^{k} r_u du + (1-q)\cdot\left[\nu' + \delta_2 \cdot \left(W_k^2 - W_{k-1}^2\right)\right]} .$$

The distribution of $(1+J_k')$ is $\log N(q\cdot\mu_k + (1-q)\cdot\nu', q^2\cdot{\sigma_k}^2 + (1-q)^2\cdot\delta_2^2)$, where

$\mu_k = E_y \Delta R_k$, $\sigma_k^2 = Var\ \Delta R_k$.

3 Value of the liability from the policyholder's point of view

We will apply the above-mentioned model to endowment insurance with profit sharing. Let K denotes sum assured, ${}_kV_x$ reserve of premium for 1 unit at the end of k^{th} policy year , n duration of insurance, i' technical interest rate, $s_{k,n}$ probability of surrender during k^{th} policy year, ${}_k p_x = {}_{k-1}p_x \cdot (1 - q_{x+k-1} - s_{k-1,n})$ probability, that the policy is in force at the and of k^{th} policy year.

3.1 Valuation of the liability using a diffusion process

For a bonus reserve at the end of period k we assume the formula

$Q_k = 0{,}8\cdot(J_k' - i')_+ \cdot {}_{k-1}V_x + \left(1 + i' + 0{,}8\cdot(J_k' - i')_+\right)\cdot Q_{k-1}$, for $k = 2,...,n$ and $Q_1 = 0$.

The present value of expected benefits is

$$C^{Policyholder} = -K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot q_{x+k} \cdot \left(Ev^{(k+1)} + Ev^{(k+1)} \cdot Q_k \right) -$$

$$(6) \qquad - K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot s_{k+1,n} \cdot \left({}_{k+1}V_x^{storno} Ev^{(k+1)} + 0{,}9 \cdot Ev^{(k+1)} Q_k \right) -$$

$$- K \cdot {}_n p_x \cdot \left(Ev^{(n)} + Ev^{(n)} \cdot Q_n \right).$$

It's clear that $Ev^{(k+1)} = e^{-E_s R_{k+1} + \frac{1}{2} Var R_{k+1}}$ and to determine $Ev^{(k+1)} \cdot Q_k$ we apply a recurrent formula

$$(7) \qquad \begin{aligned} E_s^z v^{(k)} Q_k &= 0{,}8 \cdot {}_{k-1}V_x \cdot E_s^z (J_k' - i')_+ \cdot (1 + I_k)^{-1} \cdot v^{(k-1)} + \\ &+ (1 + i') \cdot E_s^z v^{(k)} \cdot Q_{k-1} + 0{,}8 \cdot E_s^z (J_k' - i')_+ \cdot (1 + I_k)^{-1} \cdot v^{(k-1)} \cdot Q_{k-1}, \end{aligned}$$

where indices s and z denote conditions $r_0 = s, r_k = z$.

Utilizing the Markovian character of the diffusion process $\{r_t\}$ and assuming $r_{k-1} = y$ we can write

$$(8) \quad E_s^z (J_k' - i')_+ (1 + I_k)^{-1} v^{(k-1)} = \int_{-\infty}^{\infty} E_y^z (J_k' - i')_+ (1 + I_k)^{-1} \cdot E_s^y v^{(k-1)} \cdot \frac{1}{\sqrt{2\pi \cdot Var\, r_{k-1}}} \cdot e^{\frac{(y - E_s r_{k-1})^2}{2 \cdot Var\, r_{k-1}}} dy,$$

$$(9) \qquad \begin{aligned} &E_s^z (J_k' - i')_+ (1 + I_k)^{-1} v^{(k-1)} \cdot Q_{k-1} = \\ &= \int_{-\infty}^{\infty} E_y^z (J_k' - i')_+ (1 + I_k)^{-1} \cdot E_s^y v^{(k-1)} \cdot Q_{k-1} \cdot \frac{1}{\sqrt{2\pi \cdot Var\, r_{k-1}}} \cdot e^{\frac{(y - E_s r_{k-1})^2}{2 \cdot Var\, r_{k-1}}} dy, \end{aligned}$$

$$(10) \qquad E_s^z v^{(k)} \cdot Q_{k-1} = \int_{-\infty}^{\infty} e^{-E_y^z \Delta R_k + \frac{1}{2} Var^z \Delta R_k} \cdot E_s^y v^{(k-1)} \cdot Q_{k-1} \cdot \frac{1}{\sqrt{2\pi \cdot Var\, r_{k-1}}} \cdot e^{\frac{(y - E_s r_{k-1})^2}{2 \cdot Var\, r_{k-1}}} dy.$$

For a calculation of integrals (8), (9) we need to determine

$$E_y^z (J_k' - i')_+ \cdot (1 + I_k)^{-1} = E_y^z (1 + I_k)^{-1} \cdot \left((1 + J_k') - (1 + i') \right)_+ =$$

$$= E \int_{q \cdot x + U > \ln(1 + i')} e^{-x} \left(e^{q \cdot x + U} - (1 + i') \right) \cdot \frac{1}{\sqrt{2\pi} \cdot \sigma_k} \cdot e^{-\frac{(x - \mu_k)^2}{2 \cdot \sigma_k^2}} dx,$$

where $U = (1 - q) \cdot \left(\nu' + \delta_2 \cdot \left(W_k^2 - W_{k-1}^2 \right) \right)$ has distribution $N\left((1 - q) \cdot \nu', (1 - q)^2 \cdot \delta_2^2 \right)$. It is

$$\mu_k = E_y \Delta R_k + \frac{Cov(r_k, \Delta R_k)}{Var\, r_k} \cdot (z - E_y r_k) \quad \text{and} \quad \sigma_k^{\,2} = \left(1 - \frac{Cov(r_k, \Delta R_k)^2}{Var\, r_k \cdot Var\, \Delta R_k} \right) \cdot Var\, \Delta R_k,$$

because the contingent distribution of $R_k - R_{k-1}$ on conditions that $r_{k-1} = y, r_k = z$ is

$$N\left(E_y \Delta R_k + \frac{Cov_y(r_k, \Delta R_k)}{Var_y\ r_k} \cdot (z - E_y r_k), \left(1 - \frac{Cov_y(r_k, \Delta R_k)^2}{Var_y\ r_k \cdot Var_y\ \Delta R_k}\right) \cdot Var_y\ \Delta R_k \right).$$

Let's assume that $U = u$, then

$$E_y^z (J'_k - i')_+ \cdot (1 + I_k)^{-1} =$$

$$= e^{u + \frac{(q-1)\cdot(2\cdot\mu + \sigma^2 \cdot (q-1))}{2}} \cdot \Phi\left(- \frac{\frac{\ln(1+i') - u}{q} - (\mu_k + \sigma^2 \cdot (1-q))}{\sigma_k} \right) -$$

$$- (1+i') \cdot e^{-\mu_k + \frac{\sigma_k^2}{2}} \cdot \Phi\left(- \frac{\frac{\ln(1+i') - u}{q} - (\mu_k - \sigma_k^2)}{\sigma_k} \right) = g_k(u).$$

Finally we will integrate $\int_{-\infty}^{\infty} g_k(u) \cdot \frac{1}{\sqrt{2\pi} \cdot \delta_2} \cdot e^{\frac{-(u-v')^2}{2\cdot\delta_2^2}} du$.

Similarly $E_s^y v^{(k-1)} = e^{-E_s^y R_{k-1} + \frac{1}{2} Var^y R_{k-1}}$,

where $E_s^y R_{k-1} = E_s R_{k-1} + \frac{Cov(r_{k-1}, R_{k-1})}{Var\ r_{k-1}} \cdot (y - E_s r_{k-1})$

and $Var^y R_{k-1} = \left(1 - \frac{Cov(r_{k-1}, R_{k-1})^2}{Var\ r_{k-1} \cdot Var\ R_{k-1}}\right) \cdot Var\ R_{k-1}$.

To calculate integrals (9) and (10) we need the value of $E_s^y v^{(k-1)} \cdot Q_{k-1}$, which is given by a table of results from the previous step of the recursion formula (7). We will need some numerical method to calculate these integrals, which will be shown in chapter 3.2.

Finally to calculate (6) we have

$$E_s v^{(k+1)} \cdot Q_k = \int_{-\infty}^{\infty} e^{-E_z \Delta R_k + \frac{1}{2} Var \Delta R_k} E_s^z v^{(k)} \cdot Q_k \cdot \frac{1}{\sqrt{2\pi \cdot Var\ r_k}} \cdot e^{-\frac{(z - E_s r_k)^2}{2\cdot Var\ r_k}} dz.$$

The method described above is based on Markovian character of diffusion processes and therefore it can be used also with another diffusion interest rate model, for example Vašíček or Ho-Lee model.

Especially Vašíček model is suitable for easier and simpler computations, because its parameters are constants. Its mean reversion is typical for interest rates behaviour. It can be described by the equation

$$(11) \qquad d(r_t - \rho) = -\kappa(r_t - \rho)\cdot dt + \delta \cdot dW_t,\ t \in [0,T].$$

The method doesn't use any approximations as well as it can be easily used for practical computation, for example using MS Excel. These can be considered as an important advantage of the method.

3.1.1 Market value margin from the policyholder's point of view

To gain a conservative value of the liability from the policyholder's point of view, we have to adjust $(1+J_k)$ by replacing ν with $\nu' = \nu - \tau$. The value of the parameter τ can be set from an equation $C^{Policyholder} = K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot Ev^{(k)} \cdot \Pi_k$, where Π_k is market premium at the primary insurance market and $C^{Policyholder}$ is a function of the variable τ. This adjustment represents the market value of risk, because policyholders with just this assessment of risk will buy the contract at the primary insurance marker for premium Π_k.

3.2 Numerical valuation

We can use Gauss's formula for a numerical integration. The MS Excel model used for numerical examples applies 5 points. It is possible to find more details about this method in [5]. Firstly we have to transform integrals $\int_{-\infty}^{\infty} g(y)\cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$ to integrals $\int_0^1 g\left(\Phi^{-1}(u)\cdot\sigma + \mu\right)du$ to solve the problem of integration interval $(-\infty,\infty)$. It is possible to find values of $\Phi^{-1}(p) = -\Phi^{-1}(1-p)$ for particular parameters $p \in (0,1)$ in literature. For other values of parameters p we can use Bessel interpolation.

4 Profit and release from risk

In next chapters we will concentrate on insurer's point of view. Especially on value of interest rate guarantee, expected surplus and main items of income statement on insurance company.

4.1 Valuation and pricing assumptions

For deducing an expected surplus of life insurance we will follow up the method which is used particularly in Europe. The method is based on a difference between pricing (conservative) and valuation (best estimates) assumptions.

The provision for risk and uncertainty results from using pricing assumptions for determination of premium. This reserve is releasing during the insurance period, while the realization of assumptions is positive. We will designate pricing assumptions with comma as common.

4.2 Income statement

International Accounting Standards Board publicized in 2001 Insurance Contracts Draft [9] with a draft of income statement of insurance company. We will focus on main items for new business and for changes in estimates for previous year's business.

New business

A - Premium income,

B - Reinsurance premium expense,

C - Claims expense. For clarity we divide item C into following items C_1 - Maturity benefits, C_2 - Surrender claims, C_3 - Death benefits, C_4 - Value of an interest rate guarantee.

D - Reinsurance recoveries,

$\overline{E}$ - Provision for risk and uncertainty[1],

F - Acquisition costs,

G - Other operating costs.

Changes in estimates for previous year's business

H - Changes in estimates and assumptions,

I - Release of risk on insurance liabilities,

J - Changes to adjustments for risk and uncertainty.

Further items

$\overline{K}$ - Interest income[2],

R - Unwinding of discount rate,

S - Changes in discount rate.

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[1] We use a denotation with stripe to mark this item off from the denotation of mean value.

[2] We again use a denotation with stripe to mark this item off from the denotation of sum assured.

Let's assume issue date 1st January and a sequence $V_0, V_1, ..., V_n$ specifying a method of reserving of insurance company. With regard to technical interest guarantee, we assume

$$ {}_kV_x \cdot (1+i') \leq {}_{k+1}V_x \quad \text{for } k = 0, 1, ..., n-1 . $$

While α represents the percentage of sum assured standing for an acquisition cost at issue of a policy, we will account for F - acquisition costs as $F = -\alpha \cdot K$.

Furthermore we account G - other operating costs

$$ G = -\sum_{k=0}^{n-1} K \cdot E \cdot v^{(k+1)} \cdot {}_kp_x \cdot (\beta + \gamma \cdot \Pi_k) \cdot (1 + I_{k+1}) = $$
$$ = -\sum_{k=0}^{n-1} K \cdot E \cdot v^{(k)} \cdot {}_kp_x \cdot (\beta + \gamma \cdot \Pi_k). $$

According to DSOP [9] acquisition costs don't meet the definition of asset, consequently it is not recommended to defer and capitalize them as DAC (deferred acquisition costs). Regardless we will defer them respecting the matching principle, which says that revenues should be recognized in the same period in which the relevant costs are incurred. This approach can be found also in critical letters of three insurance associations to IASB.

4.2.1 Expected surplus

Principle of zero profit at issue can also be found in the mentioned critical letters. The profit should arise during the insurance period while releasing from risk. This can be managed by creating a provision for risk and uncertainty, as recommended in [7].

We assume that this provision arise from adjustments for risk and uncertainty to pricing assumptions, mentioned in paragraph 1.2. A profit is therefore given by releasing of these margins. We will denote S_{k+1} expected surplus at the end of year $k+1$. The present value of these surpluses is $\sum_{k=0}^{n-1} E\, v^{(k+1)} S_{k+1} =$

(a) $$ + \sum_{k=0}^{n-1} K\ E\, v^{(k+1)} \cdot {}_kp_x \cdot (1 + J_{k+1}) \cdot \Pi_k + $$

(b) $$ + \sum_{k=0}^{n-1} K\ E\, v^{(k+1)} \cdot {}_kp_x \cdot ({}_kV_x + Q_k) \cdot (1 + J_{k+1}) - $$

(c) $$ - \sum_{k=0}^{n-1} K\ E\, v^{(k+1)} \cdot {}_kp_x \cdot (\beta + \gamma \cdot \Pi_k) \cdot (1 + I_{k+1}) - $$

(d) $$ - \sum_{k=0}^{n-1} K\ E\, v^{(k+1)} \cdot {}_kp_x \cdot p_{x+k} \cdot ({}_{k+1}V_x + Q_{k+1}) - $$

(e) $$-\sum_{k=0}^{n-1} K\ E v^{(k+1)} \cdot {}_k p_x \cdot s_{k+1,n} \cdot \left({}_{k+1}V_x^{storno} + 0{,}9 \cdot Q_k \right) -$$

(f) $$-\sum_{k=0}^{n-1} K\ E v^{(k+1)} \cdot {}_k p_x \cdot q_{x+k} \cdot (1 + Q_k).$$

The first term (a) will be partly charge to account A - Premium income, amounting to $\sum_{k=0}^{n-1} K\ E v^{(k+1)} \cdot {}_k p_x \cdot \Pi_k$ and partly to account K - Interest income, amounting to $\sum_{k=0}^{n-1} K\ E v^{(k+1)} \cdot {}_k p_x \cdot J_{k+1} \cdot \Pi_k$. The term (c) will be charge to G - other operating costs, term (e) to C_2 - surrender claims and term (f) to C_3 - death benefit. We use the following formula for modification of lines (b) and (d)

$$\begin{aligned}(1 + J_{k+1}) \cdot \left({}_kV_x + Q_k \right) = {} & +(1 + i') \cdot \left({}_kV_x + Q_k \right) - \\ & - (i' - J_{k+1})_+ \cdot \left({}_kV_x + Q_k \right) + \\ & + 0{,}8 \cdot \left(J_{k+1} - i \right)_+ \cdot \left({}_kV_x + Q_k \right) + \\ & + 0{,}2 \cdot \left(J_{k+1} - i \right)_+ \cdot \left({}_kV_x + Q_k \right),\end{aligned}$$

where the first term represents guaranteed interest on reserves, the second one represents a supplement needed in case of $J_{k+1} < i'$, which can be interpreted as value of an interest rate guarantee. The third one is profit sharing belonging to a policyholder and the last term represents profit sharing retained by an insurer

Sum of terms (b) and (d) equals to

$$\begin{aligned} & -\sum_{k=0}^{n-1} K\ E v^{(k+1)} \cdot {}_k p_x \cdot \left[p_{x+k} \cdot \left({}_{k+1}V_x + Q_{k+1} \right) - \left({}_kV_x + Q_k \right) \cdot (1 + i') \right] - \\ & -\sum_{k=0}^{n-1} K\ E v^{(k+1)} \cdot {}_k p_x \cdot \left({}_kV_x + Q_k \right) \cdot (i' - J_{k+1})_+ + \\ (12) \quad & +\sum_{k=0}^{n-1} K\ E v^{(k+1)} \cdot {}_k p_x \cdot 0{,}8 \cdot \left({}_kV_x + Q_k \right) \cdot \left(J_{k+1} - i' \right)_+ \\ & +\sum_{k=0}^{n-1} K\ E v^{(k+1)} \cdot {}_k p_x \cdot 0{,}2 \cdot \left({}_kV_x + Q_k \right) \cdot \left(J_{k+1} - i' \right)_+ ,\end{aligned}$$

where the second term of the formula (12) represents C_4 - value of interest guarantee. The third plus fourth term of (12) will be charged to account $\overline{K}$ - interest income. We will convert the first term of (12) to

$$-\sum_{k=0}^{n-1} K\ E v^{(k+1)} \cdot {}_k p_x \cdot \left[p_{x+k} \cdot \left({}_{k+1}V_x + Q_{k+1} \right) - \left({}_k V_x + Q_k \right) \cdot (1+i') \right] =$$

$$= -\sum_{k=0}^{n-1} K\ E v^{(k+1)} \cdot {}_{k+1} p_x \cdot \left({}_{k+1}V_x + Q_{k+1} \right) + \sum_{k=0}^{n-1} K\ E v^{(k)} \cdot {}_k p_x \cdot \left({}_k V_x + Q_k \right) +$$

$$+ \sum_{k=0}^{n-1} K\ E v^{(k+1)} \cdot {}_k p_x \cdot \left({}_k V_x + Q_k \right) \cdot \left(i' - I_{k+1} \right) =$$

$$= -K\ E v^{(n)} \cdot {}_n p_x \cdot \left({}_n V_x + Q_n \right) + \sum_{k=0}^{n-1} K\ E v^{(k+1)} \cdot {}_k p_x \cdot \left({}_k V_x + Q_k \right) \cdot \left(i' - I_{k+1} \right).$$

The term $-K\ E v^{(n)} \cdot {}_n p_x \cdot \left({}_n V_x + Q_n \right)$ represents C_1 - maturity benefits and the last term $\sum_{k=0}^{n-1} K\ E v^{(k+1)} \cdot {}_k p_x \cdot \left({}_k V_x + Q_k \right) \cdot \left(i' - I_{k+1} \right)$ can be charged to account S - changes in discount rate. To sum up, we can write

$$(13) \qquad \sum_{k=0}^{n-1} E v^{(k+1)} S_{k+1} = A + C_1 + C_2 + C_3 + C_4 + G + \overline{K} + S \ .$$

4.3 Reserve for risk and uncertainty

The provision for risk and uncertainty $\overline{E}$ is created at issue amounting to $\overline{E} = \sum_{k=0}^{n-1} E\ \overline{E}_{k+1} \cdot v^{(k+1)}$. Lets denote $P_0 = \sum_{k=0}^{n-1} E\ \left(S_{k+1} - \overline{E}_{k+1} \right) \cdot v^{(k+1)}$, then the expected profit at issue including the provision for risk and uncertainty is

$$P_0 - \alpha \cdot K = \sum_{k=0}^{n-1} E\ \left(S_{k+1} - \overline{E}_{k+1} \right) \cdot v^{(k+1)} - \alpha \cdot K \ .$$

If we set $\overline{E}_{k+1} = S_{k+1} - \dfrac{{}_k p_x \cdot \alpha \cdot K}{\sum_{k=0}^{n-1} E\ {}_k p_x \cdot v^{(k)}} \cdot (1 + I_{k+1})$, the principle of zero profit at issue $P_0 + F = 0$ is fulfilled, we assume $\overline{E}_{k+1} \geq 0$.

However at the end of the first year, we have

$$(14) \qquad P_1 = \sum_{k=1}^{n-1} E \left(S_{k+1}^1 - \overline{E}_{k+1}^1 \right) \cdot \ {}^2 v_1^{(k)} \ ,$$

where the index 1 denotes best estimate assumptions based on experience from the first year. For discounting we use

$${}^2 v_1^{(k)} = \frac{1}{\left(1 + I_2\right) \cdot \left(1 + I_3\right) \cdots \left(1 + I_{k+1}\right)} \ .$$

The realized surplus in the first year is S_1^a and realized acquisition cost to end of the first year is F_1^a. This means that the asset arise by

$$P_1 + S_1^a + F_1^a = S_1^a + \left(\frac{K_1 \cdot \alpha \cdot \sum_{k=0}^{n-2} E\ {}_k p_{x+1} \cdot {}^2 v_1^{(k)}}{\sum_{k=0}^{n-1} E\ {}_k p_x \cdot v^{(k)}} \cdot (1 + I_1) + F_1^a \right),$$

where K_1 denotes the real sum assured in force at the end of the first year.

We should account $P_1 - P_0 + S_1^a - (F - F_1^a)$ at the end of the first year, which can be modified as

$$\sum_{k=2}^{n} E\left(S_k^1 - \overline{E}_k^1\right) \cdot {}^2 v_1^{(k-1)} + S_1^a - \sum_{k=2}^{n} E\left(S_k - \overline{E}_k\right) \cdot {}^2 v^{(k-1)} -$$
$$-\left(S_1 - \overline{E}_1\right) + I_1 \cdot P_0 - (F - F_1^a).$$

Particular elements of income statement will be:

Changes in estimates and assumptions

$$H = S_1^a - S_1 + \sum_{k=2}^{n} E\left(S_k^1 \cdot {}^2 v_1^{(k-1)} - S_k \cdot {}^2 v^{(k-1)}\right) + \left(F_1^a - F \cdot (1 + I_1)\right),$$

Release from risk on insurance liabilities $I = E_1$,

Changes to adjustment for risk and uncertainty $J = -\sum_{k=2}^{n} E\left(\overline{E}_k^1 \cdot {}^2 v_1^{(k-1)} - \overline{E}_k \cdot {}^2 v^{(k-1)}\right)$ and

Unwinding of discount rate $R = I_1 \cdot \left(P_0 + F\right)$.

So far we assumed that $\overline{E}_{k+1} \geq 0$. In case of $\overline{E} = \sum_{k=0}^{n-1} E\ \overline{E}_{k+1} \cdot v^{(k+1)} < 0$, the insufficiency of premium is indicated and the provision for risk and uncertainty should be accounted amounting to $\overline{E} = 0$. At the same time we establish a deficiency reserve $-\sum_{k=0}^{n-1} E\ \overline{E}_{k+1} \cdot v^{(k+1)}$ which will be a part of costs included in S_1^a .

There will be $\sum_{k=0}^{n-1} E\ \overline{E}_{k+1}^1 \cdot {}^2 v_1^{(k)}$ representing an expected change in the reserve in k^{th} year in the formula (14).

5 Value of the liability from the insurer's point of view

Valuating a liability from the insurer's point of view, we have to consider the value of a technical interest guarantee. We assume that the technical interest on a bonus reserve Q_k is guaranteed as well as on reserve of premium ${}_kV_x$. The expected present value of a liability can be expressed as

$$
\begin{aligned}
C^{Insurer} = & -K \cdot {}_n p_x \cdot Ev^{(n)}\left(1+Q_n\right)+ \\
& -K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot s_{k+1,n} \cdot E\, v^{(k+1)} \cdot \left({}_{k+1}V_x^{storno} + 0{,}9 \cdot Q_k\right) - \\
& -K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot q_{x+k} \cdot Ev^{(k+1)} \cdot \left(1+Q_k\right) - \\
& -K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot E\, v^{(k+1)} \cdot \left(i' - J'_{k+1}\right)_+ \cdot \left({}_kV_x + Q_k\right).
\end{aligned} \tag{15}
$$

5.1 Valuation of liability using diffusion process

The value of $Ev^{(k+1)}$ a $Ev^{(k+1)}Q_k$ will be calculated using the same method as in paragraph 3.1. Additionally we have to calculate

$$
\begin{aligned}
& E_s v^{(k+1)}(i' - J'_{k+1})_+ = \\
& = \int_{-\infty}^{\infty} E_s^z v^{(k)} \cdot E_z (1+I_{k+1})^{-1} \cdot (i' - J'_{k+1})_+ \frac{1}{\sqrt{2\pi\, Var\, r_k}} e^{-\frac{(z - E_s\, r_k)^2}{2 Var\, r_k}} dz,
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
& E_s v^{(k+1)}(i' - J'_{k+1})_+ \cdot Q_k = \\
& = \int_{-\infty}^{\infty} E_s^z v^{(k)} Q_k \cdot E_z (1+I_{k+1})^{-1} \cdot (i' - J'_{k+1})_+ \frac{1}{\sqrt{2\pi\, Var\, r_k}} e^{-\frac{(z - E_s\, r_k)^2}{2 Var\, r_k}} dz,
\end{aligned} \tag{17}
$$

using the Markovian character of diffusion process $\{r_t\}$. To calculate integrals (16) and (17) we need to determine

$$
\begin{aligned}
& E_z (1+I_{k+1})^{-1} \cdot \left((1+i') - (1+J'_k)\right)_+ = \\
& = E \int_{q\cdot x + U < \ln(1+i')} e^{-x}\left((1+i') - e^{q\cdot x + U}\right) \cdot \frac{1}{\sqrt{2\pi} \cdot \sigma_k} \cdot e^{-\frac{(x-\mu_k)^2}{2\cdot\sigma_k^2}} dx,
\end{aligned}
$$

where $U = (1-q)\cdot\left(\nu' + \delta_2 \cdot \left(W_k^2 - W_{k-1}^2\right)\right)$ has a distribution $N\left((1-q)\cdot\nu', (1-q)^2 \cdot \delta_2^2\right)$.

Assuming $U = u$, we can write

$$E_z(1+I_{k+1})^{-1}(i'-J'_{k+1})_+ =$$

$$= \left(1+i'\right)\cdot e^{-\mu_k+\frac{\sigma_k^2}{2}}\cdot\Phi\left(\frac{\frac{\ln\left(1+i'\right)-u}{q}-\left(\mu_k-\sigma_k^2\right)}{\sigma_k}\right)-$$

$$-e^{u+\frac{(q-1)\cdot\left(2\cdot\mu+\sigma^2\cdot(q-1)\right)}{2}}\cdot\Phi\left(\frac{\frac{\ln\left(1+i'\right)-u}{q}-\left(\mu_k+\sigma^2\cdot(1-q)\right)}{\sigma_k}\right)=m_k(u),$$

where $\mu_k = E_z(R_{k+1}-R_k)$ and $\sigma_k^2 = Var(R_{k+1}-R_k)$. Finally we integrate

$\int_{-\infty}^{\infty} m_k(u)\cdot\frac{1}{\sqrt{2\pi}\cdot(1-q)\cdot\delta_2}\cdot e^{\frac{-[u-(1-q)\nu']^2}{2\cdot(1-q)^2\cdot\delta_2^2}}\,du$ using the Gauss's formula.

5.1.1 Market value margin from the insurer's point of view

According to the principle from the paragraph 1.2 we will adjust the variable J_k in the expression $\left({}_kV_x+Q_k\right)\cdot\left(i'-J'_{k+1}\right)_+$ replacing ν by $\nu'=\nu-\tau$ to get more conservative result. It is more complicated to adjust J_k in the expression $\left(J_{k+1}-i'\right)_+\cdot\left({}_kV_x+Q_k\right)$ because it is not clear which adjustment ($\nu'=\nu-\tau$ or $\nu'=\nu-\tau$) will lead to the more conservative result, so we keep $\left(J_{k+1}-i'\right)_+\cdot\left({}_kV_x+Q_k\right)$ without any adjustment in our model. In practice this can be solved according to actuary's best estimate of future development of variable J_k. If J_k is above i' in a long term, the higher value of J_k is positive for the insurer. But in case of decreasing J_k below i', the insurer guarantees the technical interest on the higher bonus reserve and subsidizes this guarantee from other sources.

The value of the parameter τ can be deducted form the equation $C^{Insurer} = K\cdot\sum_{k=0}^{n-1} {}_kp_x\cdot Ev^{(k)}\cdot\Pi_k$, where Π_k is premium at the primary market.

5.2 Valuation of life insurance

Exploiting formulas derived in the paragraph 4.2.1 and using all interrelated cash flows, as premium, interest on reserves, changes in reserves, expense, benefits including surrenders and bonuses we can determine value of insurance as

$$
\begin{aligned}
H^{Insurer} = & -K \cdot {}_n p_x \cdot Ev^{(n)} \cdot \left(1+Q_n\right)+ \\
& -K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot E\,v^{(k+1)} \cdot \left(i' - J_{k+1}\right)_+ \cdot \left({}_k V_x + Q_k\right) - \\
& -K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot s_{k+1,n} \cdot E\,v^{(k+1)} \cdot \left({}_{k+1}V_x^{storno} + 0{,}9 \cdot Q_k\right) - \\
& -K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot q_{x+k} \cdot Ev^{(k+1)} \cdot \left(1+Q_k\right) - \\
& -K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot E\,v^{(k)} \cdot \left(\beta + \gamma \cdot \Pi_k\right) - \alpha \cdot K + \\
& +K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot E\,v^{(k+1)} \cdot \left(1 + J_{k+1}\right) \cdot \Pi_k + \\
& +K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot E\,v^{(k+1)} \cdot \left(J_{k+1} - i'\right)_+ \cdot \left({}_k V_x + Q_k\right) + \\
& +K \cdot \sum_{k=0}^{n-1} {}_k p_x \cdot E\,v^{(k+1)} \cdot \left(i' - I_{k+1}\right)_+ \cdot \left({}_k V_x + Q_k\right).
\end{aligned}
$$

We are able to calculate the value of $H^{Insurer}$ using the method and formulas from paragraphs 3.1 and 5.1.

5.3 Example

It is quite easy to construct a computing model in MS Excel, which is a significant advantage of the method described above. In the following tables, we can find results for endowment with profit sharing insurance, with sum assured 100 000 and duration 10 years. Parameters of Hull-White model in this example are $\kappa = 0{,}95$, $\delta_1 = 0{,}015$, $r_0 = 0{,}028$. Furthermore we assume $\nu = \ln(1+0{,}08)$ and $\delta_2 = 0{,}12$.

Firstly we will show how the value of the liability $C^{Insurer}$ is affected by the structure of underlying portfolio. We will change the value of parameter q, which represents the portion of risk free financial instruments in the portfolio. The technical interest rate is $i' = 4\%$

The value of the liability $C^{Insurer}$					
k	q = 10%	q =20%	q = 50%	q = 80%	q = 90%
1	758	758	758	758	758
2	1 396	1 381	1 336	1 317	1 331
3	1 886	1 852	1 749	1 685	1 695
4	2 244	2 188	2 023	1 904	1 898
5	2 495	2 417	2 190	2 011	1 987
6	2 665	2 565	2 277	2 040	1 998
7	2 770	2 648	2 302	2 013	1 954
8	2 820	2 678	2 278	1 942	1 868
9	2 827	2 665	2 215	1 838	1 751
10	47 709	46 033	41 498	37 652	36 589
	67 571	65 186	58 626	53 160	51 830

Table No. 1: Value of the liability from the insurer's point of view in dependence on portion q of risk free financial instruments in portfolio.

We can see that value of the liability decrease with higher portion of risk free financial instruments in the portfolio, especially because of decreasing value of profit sharing belonging to policyholders. Concurrently the higher portion of risky instruments with its volatility stands for higher value of the interest rate guarantee, as can be seen from the table below.

The value of the interest rate guarantee					
k	q = 10%	q =20%	q = 50%	q = 80%	q = 90%
1	0	0	0	0	0
2	204	189	144	124	138
3	380	349	256	200	211
4	533	486	345	246	247
5	666	602	416	274	260
6	783	704	475	291	263
7	887	793	523	302	261
8	982	873	565	309	257
9	1 070	946	601	315	253
10	1 153	1 015	634	320	250
	6 659	5 956	3 958	2 381	2 142

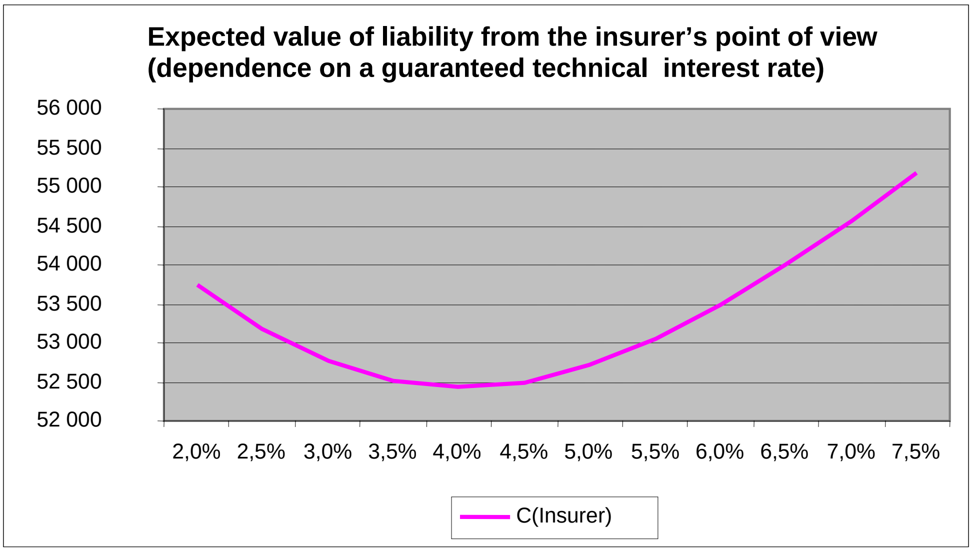
Table No. 2: Value of the interest rate guarantee in dependence on portion q of risk free financial instruments in portfolio.

In the next example we will assume $q = 85\%$, because the highest value of parameter q will be more realistic. Let's take a look, how can be the liability from the insurer's point of view affected by the volatility of risky assets.

Standard deviation of risky asset	Value of interest rate guarantee	Maturity benefits	Surrender claims	Death benefits	Value of the liability $C^{Insurer}$
4%	1 482	34 867	13 472	1 230	51 051
6%	1 630	34 986	13 487	1 231	51 334
8%	1 809	35 124	13 505	1 232	51 671
10%	2 007	35 274	13 524	1 234	52 039
12%	2 216	35 433	13 544	1 235	52 428
14%	2 433	35 597	13 564	1 237	52 831
16%	2 656	35 764	13 585	1 238	53 243

Table No. 3: Influence of volatility of risky asset.

Now we demonstrate how the value of the liability from insurer's point of view is affected by the guaranteed technical interest rate i'. Firstly the value of $C^{Insurer}$ decreases with increasing i'. The reason is decreasing value of profit sharing. Afterwards the value of $C^{Insurer}$ grows with increasing value of the guarantee.



Graph No.1: Value of liability (axis y) in dependence on technical interest rate (axis x).

i'	$C^{Insurer}$	i'	$C^{Insurer}$
2,0%	53 739	5,0%	52 711
2,5%	53 181	5,5%	53 052
3,0%	52 768	6,0%	53 490
3,5%	52 515	6,5%	54 002
4,0%	52 428	7,0%	54 569
4,5%	52 497	7,5%	55 173

Table No. 4: Table of values from graph No 1.

6 Conclusion

The paper describes the method, how to use a diffusion model of instantaneous interest rate for discounting cash flows, using the Markovian character of diffusion processes without any approximation. The described model is used for calculation of life insurance liability arising from endowment with profit sharing. The paper is also concerned with a numerical computation which could be practically exercisable. The method is universal and can be easily used for practical computation even in MS Excel.

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