

Credit derivatives pricing using the Cox process with shot noise intensity

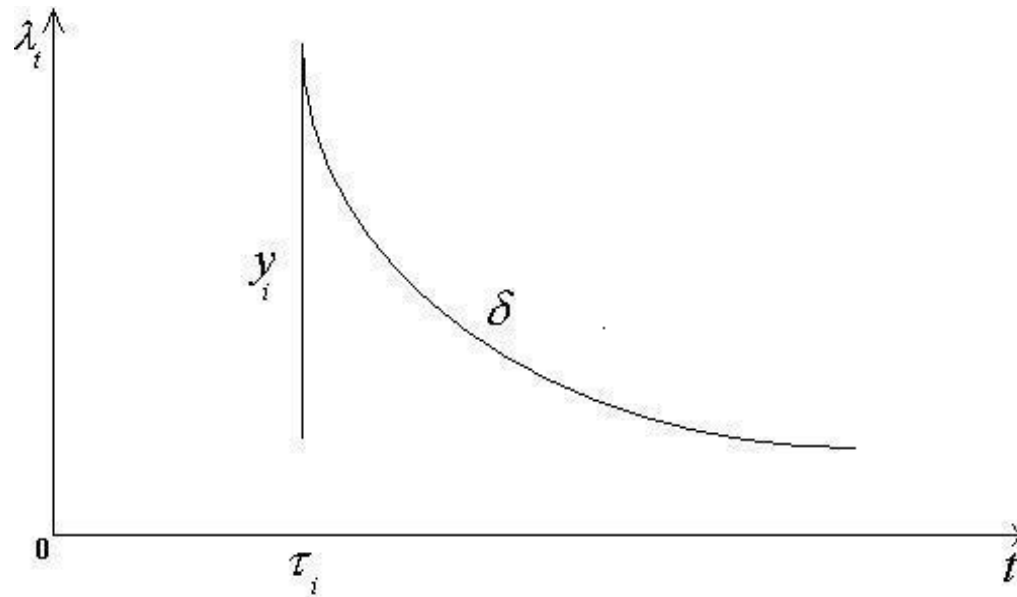
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Overview

- The primary events arrival process: Shot noise process.
- The survival probability using the Cox process with shot noise intensity.
- Three building blocks for credit derivatives pricing.
- The Esscher transform and an equivalent martingale probability measure.
- Defaultable fixed-coupon bond price and market credit default swaps (CDS) rate with their numerical illustrations.

Illustration of shot noise process with a primary event



- i is a primary event (i.e. the government's fiscal and monetary policies, the release of corporate financial reports, the political and social decisions, the rumours of mergers and acquisitions between firms, etc.)
- y_i is the jump size of primary event i (i.e. magnitude of contribution of primary event i to intensity)
- s_i is the time at which primary event i occurs
- δ is exponential decay.

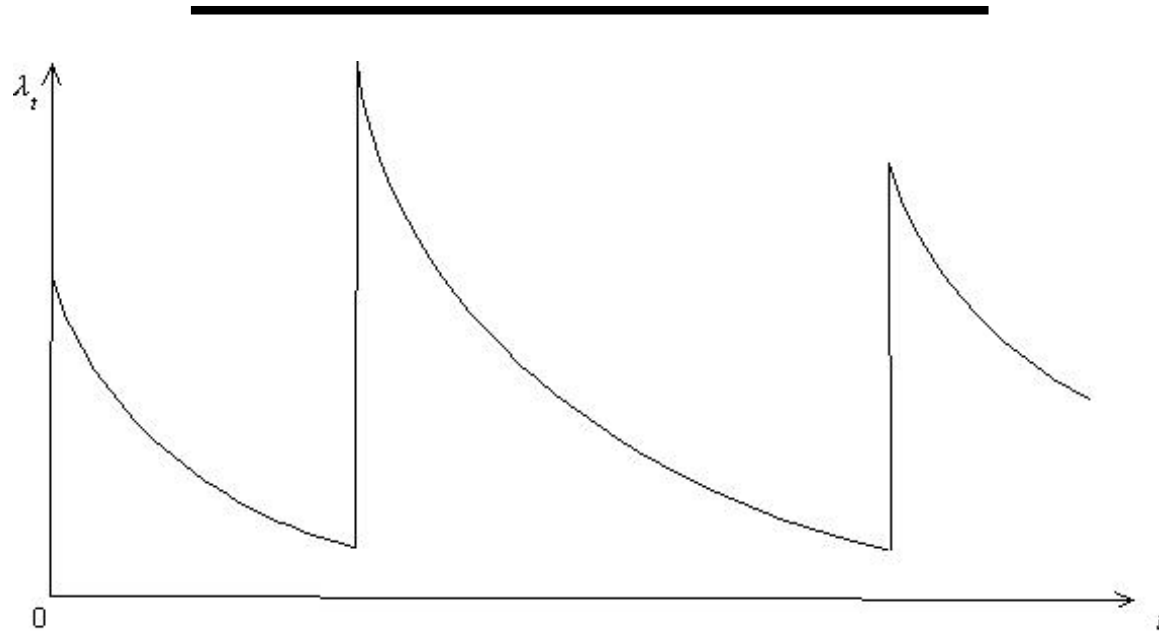
Number of defaults arising from a primary event i

- The number of defaults arising from a primary event i following the Poisson process is given by

$$N_t^{(i)} \sim \text{Poisson} \left(y_i e^{-\delta(t-s_i)} \right)$$

where $s_i < t < \infty$ and $E(y_i) < \infty$.

Illustration of shot noise process with a primary events over a period of time



where ρ is the number of primary events in time period t .

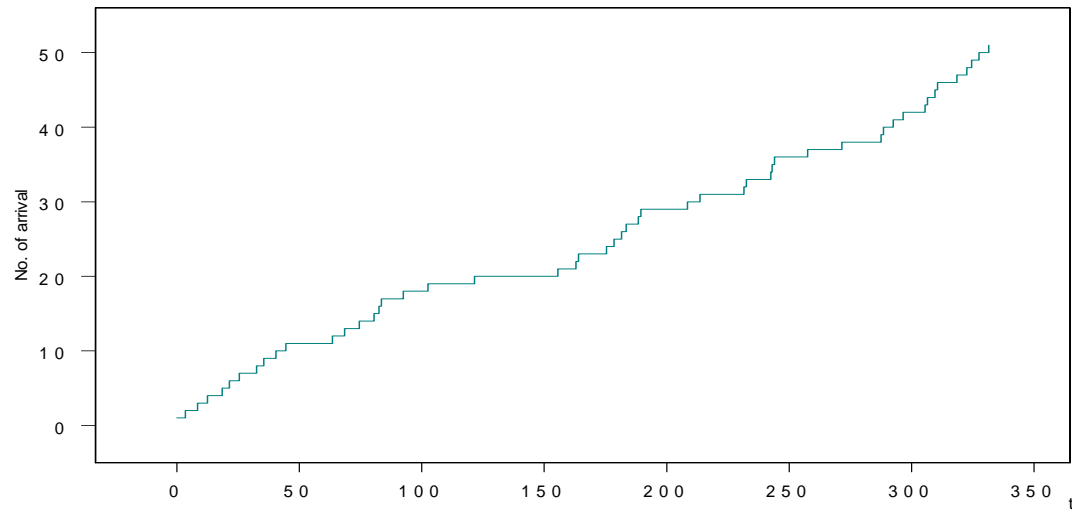
Number of defaults arising from primary events in time period t

- The number of defaults arising from primary events following the Poisson process is given by

$$N_t \sim \text{Poisson} \left(\lambda_0 e^{-\delta t} + \sum_{i=1}^{M_t} y_i e^{-\delta(t-s_i)} \right),$$

where λ_0 is the initial value of λ_t , $\{y_i\}_{i=1,2,\dots}$ is a sequence of *i.i.d.* random variables with distribution function $G(y)$ ($y > 0$), $\{s_i\}_{i=1,2,\dots}$ is the sequence representing the event times of a Poisson process M_t with constant intensity ρ and δ is the rate of exponential decay.

- Let $\lambda_t = \lambda_0 e^{-\delta t} + \sum_i^{M_t} y_i e^{-\delta(t-s_i)}$, which is a generalized Lévy process as it can be expressed as $d\lambda_t = -\delta\lambda_t dt + dC_t$ where $C_t = \sum_{i=1}^{M_t} y_i$ is a pure-jump process (Poisson arrivals of jumps of a given distribution).
- We have $N_t \sim \text{Poisson}(\lambda_t)$, the Cox process with shot noise intensity, that can be illustrated as below:



Graph illustrating the Cox process with shot noise intensity

Survival probability

- From the Cox process of N_t , the survival probability is given by

$$\Pr(\tau > t) = \mathbb{E} \left\{ \exp \left(- \int_0^t \lambda_s ds \right) \mid \lambda_0 \right\}$$

where $\tau \equiv \inf \{t : N_t = 1 \mid N_0 = 0\}$ is the default arrival time that is equivalent to the first jump time of the Cox process N_t .

- Define $\Lambda_t = \int_0^t \lambda_s ds$ then it becomes $\Pr(\tau > t) = \mathbb{E} \left(e^{-\Lambda_t} \mid \lambda_0 \right)$ and the expectation is calculated under an appropriate probability measure.

The price of default-free zero-coupon bond

- We assume that the default-free short rate follows a generalised CIR model, i.e. $dr_t = c(b - ar_t)dt + \sigma r_t^{1/2}dB_t$, where $a > 0$, $b > 0$, $c > 0$ and B_t is a standard Brownian motion.
- The price of zero-coupon default-free bond paying 1 at time t is given by

$$B(0, t) = \mathbb{E} \left\{ \exp \left(- \int_0^t r_s ds \right) \mid r_0 \right\} = \mathbb{E} \left\{ e^{-R_t} \mid r_0 \right\},$$

where $B(0, t)$ denotes the price of a default -free zero-coupon bond, $R_t = \int_0^t r_s ds$ and the expectation is calculated under an appropriate probability measure.

The price of zero-coupon defaultable bond

- Assuming that r_t and λ_t are independent, the price of zero-coupon defaultable bond paying $\mathbf{1}_{(\tau>t)}$ at time t is given by

$$\begin{aligned}\bar{B}(0, t) &= \mathbb{E} \left\{ \exp \left(- \int_0^t r_s ds \right) \mathbf{1}_{(\tau>t)} \mid r_0, \lambda_0 \right\} \\ &= \mathbb{E} \left\{ \exp \left(- \int_0^t (r_s + \lambda_s) ds \right) \mid r_0, \lambda_0 \right\} \\ &= \mathbb{E} \left(e^{-Rt} \mid r_0 \right) \mathbb{E} \left(e^{-\Lambda t} \mid \lambda_0 \right),\end{aligned}$$

where $\bar{B}(0, t)$ denotes the price of a defaultable zero-coupon bond and the expectation is calculated under an appropriate probability measure.

The deterministic recovery rate

- In reality, the lenders (i.e. the buyers of defaultable bonds) can receive the part (or whole) of coupon payments and principle after the liquidation of borrowers' assets. So we simply consider the recovery of par model introduced by Duffie (1998). For fractional recovery, we refer you Duffie and Singleton (2003).

The value of a deterministic payoff

- The value of a deterministic payoff, 1 that is paid at t_{k+1} if and only if a default happens in $]t_k, t_{k+1}]$, denoted by $e(0, t_k, t_{k+1})$, is given by $e(0, t_k, t_{k+1})$

$$= \mathbb{E} \left[\exp \left(- \int_0^{t_{k+1}} r_s ds \right) \left(\mathbf{1}_{\{N_{t_k}=0\}} - \mathbf{1}_{\{N_{t_{k+1}}=0\}} \right) \mid r_0, \lambda_0 \right]$$

$$= \mathbb{E} \left(e^{-Rt_{k+1}} \mid r_0 \right) \left\{ \mathbb{E} \left(e^{-\Lambda t_k} \mid \lambda_0 \right) - \mathbb{E} \left(e^{-\Lambda t_{k+1}} \mid \lambda_0 \right) \right\},$$

where $0 = t_0 < t_k < t_{k+1}$ and the expectation is calculated under an appropriate probability measure.

The price of defaultable fixed-coupon bond

- The price of defaultable fixed-coupon bond at time 0, denoted by $\bar{C}(0)$, is given by

$$\bar{C}(0) = \sum_{k=1}^N \bar{c}_n \bar{B}(0, t_{k_n}) + \bar{B}(0, t_{k_N}) + \pi \sum_{k=1}^{k_N} e(0, t_{k-1}, t_k)$$

where $\bar{c}_n = \bar{c} \times (t_{k_n} - t_{k_{n-1}})$ are coupon payments at t_{k_n} ($n = 1, 2, \dots, N$), $t_{k_1} < t_{k_2} < \dots < t_{k_N}$ and π is a deterministic recovery rate.

The market credit default swaps (CDS) rate

- The market credit default swaps (CDS) rate, denoted by \bar{s} , is given by

$$\bar{s} = (1 - \pi) \frac{\sum_{k=1}^{k_n} e(0, t_{k-1}, t_k)}{\sum_{k=1}^N (t_{k_{n+1}} - t_{k_n}) \bar{B}(0, t_{k_n})}.$$

The Esscher transform

- Let X_t be a stochastic process such that $e^{h^* X_t}$ a martingale with $h^* \in \mathfrak{R}$. For a measurable function f , the expectation of the random variable $f(X_t)$ with respect to the equivalent martingale probability measure is

$$E^* [f(X_t)] = E \left[f(X_t) \frac{e^{h^* X_t}}{E(e^{h^* X_t})} \right] = \frac{E[f(X_t) e^{h^* X_t}]}{E[e^{h^* X_t}]},$$

where $E(e^{h^* X_t}) < \infty$.

- $\frac{e^{h^* X_t}}{E(e^{h^* X_t})}$ can be used to define a change of probability measure, i.e. it can be used to define the Radon-Nikodym derivative $\frac{dP^*}{dP}$.

A suitable martingale

- Considering constants θ^* , ψ^* and γ^* such that $\theta^* \geq 1$, $\psi^* \geq 1$ and $\gamma^* \leq 0$,

$$\theta^{*N_t} \exp\{-(\theta^* - 1)\Lambda_t\} \psi^{*M_t} \\ \times \exp(-\gamma^* \lambda_t e^{\delta t}) \exp\left[\rho \int_0^t \{1 - \psi^* \hat{g}(\gamma^* e^{\delta s})\} ds\right]$$

is a martingale, where $\hat{g}(u) = \int_0^\infty e^{-uy} dG(y) < \infty$.

The generator of the process $(\Lambda_t, N_t, \lambda_t, M_t, t)$ w.r.t. the equivalent martingale probability measure

- The generator of the process $(\Lambda_t, N_t, \lambda_t, M_t, t)$ acting on a function $f(\Lambda, n, \lambda, m, t)$ with respect to the equivalent martingale probability measure is given by

$$\begin{aligned}
 & A^* f(\Lambda, n, \lambda, m, t) \\
 &= \frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial \Lambda} + \theta^* \lambda \{f(\Lambda, n+1, \lambda, m, t) - f(\Lambda, n, \lambda, m, t)\} - \delta \lambda \frac{\partial f}{\partial \lambda} \\
 &+ \rho^*(t) \left\{ \int_0^\infty f(\Lambda, n, \lambda+y, m+1, t) dG^*(y; t) - f(\Lambda, n, \lambda, m, t) \right\},
 \end{aligned}$$

$$\text{where } \rho^*(t) = \rho \psi^* \hat{g}(\gamma^* e^{\delta t}) \text{ and } dG^*(y; t) = \frac{\exp(-\gamma^* e^{\delta t} y) dG(y)}{\hat{g}(\gamma^* e^{\delta t})}.$$

Key results

- (i) The intensity function λ_t has changed to $\theta^* \lambda_t$;
- (ii) The rate of jump arrival ρ has changed to $\rho^*(t) = \rho \psi^* \hat{g}(\gamma^* e^{\delta t})$
(it now depends on time);
- (iii) The jump size measure $dG(y)$ has changed to $dG^*(y; t) = \frac{\exp(-\gamma^* e^{\delta t} y) dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$
(it now depends on time).

- In other words, the risk-neutral Esscher measure is the measure with respect to which N_t becomes the Cox process with parameter $\theta^* \lambda_t$ where three parameters of the shot noise process λ_t are δ , $\rho^*(t) = \rho \psi^* \hat{g}(\gamma^* e^{\delta t})$, $dG^*(y; t) = \frac{\exp(-\gamma^* e^{\delta t} y) dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$.

The L. T. of the distribution of Λ_t with respect to the Esscher measure

- Assuming that the jump size distribution is exponential, i.e. $g(y) = \alpha e^{-\alpha y}$, $y > 0$, $\alpha > 0$ and that λ_t is ' $-\infty$ ' asymptotic, we can derive the Laplace transform of the distribution of Λ_t with respect to an equivalent martingale probability measure, i.e.

$$\mathbb{E}^* \left\{ e^{-v(\Lambda_{t_2} - \Lambda_{t_1})} \right\} = \left(\frac{\gamma^* e^{\delta t_1} + \alpha e^{-\delta(t_2 - t_1)}}{\gamma^* e^{\delta t_1} + \alpha + \frac{\theta^* \nu}{\delta} (1 - e^{-\delta(t_2 - t_1)})} \right)^{\frac{\psi^* \rho}{\delta}} \cdot \left(\frac{\gamma^* e^{\delta t_1} + \alpha + \frac{\theta^* \nu}{\delta} (1 - e^{-\delta(t_2 - t_1)})}{\gamma^* e^{\delta t_1} + \alpha e^{-\delta(t_2 - t_1)}} \right)^{\frac{\alpha \psi^* \rho}{\delta \alpha + \theta^* \nu}}$$

- In practice, the banks and traders need to calculate credit derivatives prices using $\theta^* > 1$, $\psi^* > 1$ and $\gamma^* < 0$. This results in banks and traders assuming that there will be a higher value of intensity itself, more primary events occurring in a given period of time and a higher value of jump size of intensity. These assumptions are necessary, as the banks and traders have to consider the risks involved in incomplete market. If $\theta^* = 1$, $\psi^* = 1$, and $\gamma^* = 0$ then non-arbitrage free credit derivatives price is calculated without considering any risks involved in incomplete market. However, as expected, we have quite a flexible family of equivalent probability measures by the combination of θ^* , ψ^* and γ^* . It means that the banks and traders have various ways of obtaining an non-arbitrage credit derivatives price (i.e. by changing equivalent martingale probability measures using the combination of θ^* , ψ^* and γ^*).

- For simplicity, we will use a zero-coupon default-free bond price at time 0 with respect to the original measure. Hence from CIR (1985), assuming that $c = 1$, we have

$$\begin{aligned}
 B(0, t) &= \mathbb{E} \left\{ \exp \left(- \int_0^t r_s ds \right) \mid r_0 \right\} = \mathbb{E} \left(e^{-Rt} \mid r_0 \right) = \\
 &\exp \left[- \left\{ \frac{2 \left\{ 1 - \exp \left(- (a^2 + 2\sigma^2)^{1/2} t \right) \right\}}{\left\{ \left((a^2 + 2\sigma^2)^{1/2} + a \right) + \left((a^2 + 2\sigma^2)^{1/2} - a \right) \exp \left(- (a^2 + 2\sigma^2)^{1/2} t \right) \right\}} \right\} r_0 \right] \\
 &\times \left\{ \frac{2(a^2 + 2\sigma^2)^{1/2} \exp \left(- \frac{\left((a^2 + 2\sigma^2)^{1/2} - a \right)}{2} t \right)}{\left((a^2 + 2\sigma^2)^{1/2} + a \right) + \left((a^2 + 2\sigma^2)^{1/2} - a \right) \exp \left(- (a^2 + 2\sigma^2)^{1/2} t \right)} \right\}^{\frac{2b}{\sigma^2}} .
 \end{aligned}$$

An arbitrage-free pricing

- The price of a zero-coupon corporate (defaultable) bond paying $1_{(\tau_1 > t)}$ at time t is given by

$$\bar{B}^Q(0, t) = \mathbb{E} \left(e^{-Rt} \mid r_0 \right) \mathbb{E}^* \left(e^{-\Lambda t} \right)$$

and the value of a deterministic payoff 1 that is paid at t_{k+1} if and only if a default happens in $]t_k, t_{k+1}]$ is given by

$$e^Q(0, t_k, t_{k+1}) = \mathbb{E} \left(e^{-Rt_{k+1}} \mid r_0 \right) \left\{ \mathbb{E}^* \left(e^{-\Lambda t_k} \right) - \mathbb{E}^* \left(e^{-\Lambda t_{k+1}} \right) \right\},$$

where $Q = P \times P^*$ is an equivalent martingale probability measure.

- Hence the price a defaultable fixed-coupon bond is given by

$$\bar{C}^Q(0) = \sum_{k=1}^N \bar{c}_n \bar{B}^Q(0, t_{k_n}) + \bar{B}^Q(0, t_{k_N}) + \pi \sum_{k=1}^{k_N} e^Q(0, t_{k-1}, t_k)$$

and the market credit default swaps (CDS) rate is given by

$$\bar{s}^Q = (1 - \pi) \frac{\sum_{k=1}^{k_n} e^Q(0, t_{k-1}, t_k)}{\sum_{k=1}^N (t_{k_{n+1}} - t_{k_n}) \bar{B}^Q(0, t_{k_n})}.$$

- Up to now it is assumed that the frequency and magnitude of primary events and time period needed to go back to the previous (or better) level of intensity immediately after primary events occur are the same among all firms. However some of these primary events might not affect at all to a specific firm e.g. 'AAA' rating firm. Also even if primary events affect firms' default intensities, their magnitude should be different to each firms. Time period needed to go back to the previous (or better) level of intensity also need to be discriminated among firms.

An arbitrage-free default probability

- An arbitrage-free default probability for the firm i is given by

$$\mathbf{1} - \mathbb{E}^* \left(e^{-\Lambda_t^i} \right) = \mathbf{1} - \left(\frac{\gamma^* + \alpha^i e^{-\delta^i t}}{\gamma^* + \alpha^i + \frac{\theta^*}{\delta^i} (1 - e^{-\delta^i t})} \right)^{\frac{\psi^* \rho^i}{\delta^i}} \\ \times \left(\frac{\gamma^* + \alpha^i + \frac{\theta^*}{\delta^i} (1 - e^{-\delta^i t})}{\gamma^* + \alpha^i e^{-\delta^i t}} \right)^{\frac{\alpha^i \psi^* \rho^i}{\delta^i \alpha^i + \theta^*}}$$

Example 1

- The parameter values used to calculate the formula are $\theta^* = 1.1$, $\psi^* = 1.1$, $\gamma^* = -0.01$, $\alpha^i = 10$, $\delta^i = 0.5$, $\rho^i = 4$ and $t = 1$. Then an arbitrage-free default probability is given by

$$1 - \mathbb{E}^* \left(e^{-\Lambda_t^i} \right) = 1 - 0.396 = 0.604.$$

Using $\theta^* = 1$, $\psi^* = 1$ and $\gamma^* = 0$, the net default probability is given by

$$1 - \mathbb{E} \left(e^{-\Lambda_t^i} \right) = 1 - 0.46409 = 0.53591.$$

Example 2

- We will now examine the effect on arbitrage-free default probability caused by changes in the value of α^i , δ^i and ρ^i . The calculation of arbitrage-free default probability are shown in the Table 1, assuming other parameter values are the same as in Example 1.

Table 1.

	Default probability
$\alpha^i = 0.1$	1 (100%)
$\alpha^i = 20$	0.37705 (37.705%)
$\delta^i = 0.1$	0.98999 (98.999%)
$\delta^i = 5$	0.09349 (9.349%)
$\rho^i = 0$	0 (0%)
$\rho^i = 8$	0.84318 (84.318%)

Example 3

- The parameter values used to calculate an arbitrage-free defaultable fixed-coupon bond price are

$$r_0 = 0.05, a = 0.05, b = 0.025 \text{ and } \sigma = 0.8 \text{ for } r_t$$

and

$$\psi^* = 1.1, \gamma^* = -0.1, \theta^* = 1.1, \alpha^i = 10, \delta^i = 0.5 \text{ and } \rho^i = 4 \text{ for } \lambda_t$$

and

$$\bar{c} = 5\% \text{ and } \pi = 50\%$$

and

$$N = 2, t_{k_0} = 0, t_{k_1} = 0.5, t_{k_2} = 1.$$

- Then an arbitrage-free defaultable fixed-coupon bond price is given by

$${}^{\overline{Q}}C_i(0) = 0.024357 + 0.37052 + 0.28753 = 0.68241$$

and an arbitrage-free credit default swaps (CDS) rate is given by

$${}^{\overline{Q}}s_i = (1 - 0.5) \left(\frac{0.57506}{0.48715} \right) = 0.59023 = 5,902.3\text{bp.}$$

Example 4

- We will now examine the effect on arbitrage-free defaultable fixed-coupon bond price and credit default swaps (CDS) rate caused by changes in the value of α^i , δ^i and ρ^i . The calculation of arbitrage-free defaultable fixed-coupon bond prices and credit default swaps (CDS) rates are shown in Table 2 and Table 3 respectively, assuming other parameter values are the same as in Example 3.

Table 2.

	$\frac{-Q}{C_i}$
$\alpha^i = 1$	0.47337
$\alpha^i = 20$	0.80033
$\delta^i = 0.1$	0.47981
$\delta^i = 4$	0.92659
$\rho^i = 0$	0.99354
$\rho^i = 8$	0.55836

Table 3.

	$\frac{-Q}{s_i}$
$\alpha^i = 1$	704,280bp
$\alpha^i = 20$	2,647.4bp
$\delta^i = 0.1$	94,499bp
$\delta^i = 4$	718.74bp
$\rho^i = 0$	0bp
$\rho^i = 8$	15,399bp