Validation of Investment Models for Actuarial Applications

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AFIR Symposium

September 2005

University of Waterloo

Acknowledgements

- Keith Freeland
- Matthew Till
- Institute for Quantitative Finance and Insurance
- CIBC
- NSERC

Outline

- Some history
- The models
- Does it matter?
- Traditional model selection
- Bootstrap evidence
- Abusing the bootstrap

History

- Single premium equity linked insurance in North America
  - Segregated Funds in Canada
  - Variable Annuities in USA
- Carry guarantees on death and maturity
- Guarantee may be fixed or increasing

History

- 25 years ago, UK faced the same issue
- MGWP published paper in 1980
  - Stochastic simulation of liabilities (and underlying assets)
  - Quantile (VaR) reserve.
  - Early application of early Wilkie Model

Canadian Method

- Stochastic simulation of liabilities
- CTE (Tail-VaR) reserve
- Not much hedging
  - If hedged, simulate and reserve for unhedged risk
- Equity model : ‘freedom with calibration’

Canadian Calibration Method

- Use any model
- Check the left-tail accumulation factor probabilities, using standard data set
- Adjust parameters to meet calibration fatness requirement
- Table calculated using ‘Regime-Switching Lognormal –2’ model

Accumulation Factors

- Let Y_t represent log return in t^{th} month
- 1-year accumulation factors are $\exp(Y_t + Y_{t+1} + \ldots Y_{t+11})$
- Similarly for 5-year and 10-year
- 40 years data $\Rightarrow$ 4 non-overlapping observations of 10-Year accumulation factor

Canadian Calibration Table

Accumulation Factor	2.5 %ile	5 %ile	10%ile
1-year	0.76	0.82	0.90
5-year	0.75	0.85	1.05
10-year	0.85	1.05	1.35

US approach

- C3P2
- Similar to Canadian approach
- Calibration Table applied to left and right tails
- US table derived from

  ‘Stochastic Log-Volatility’ model

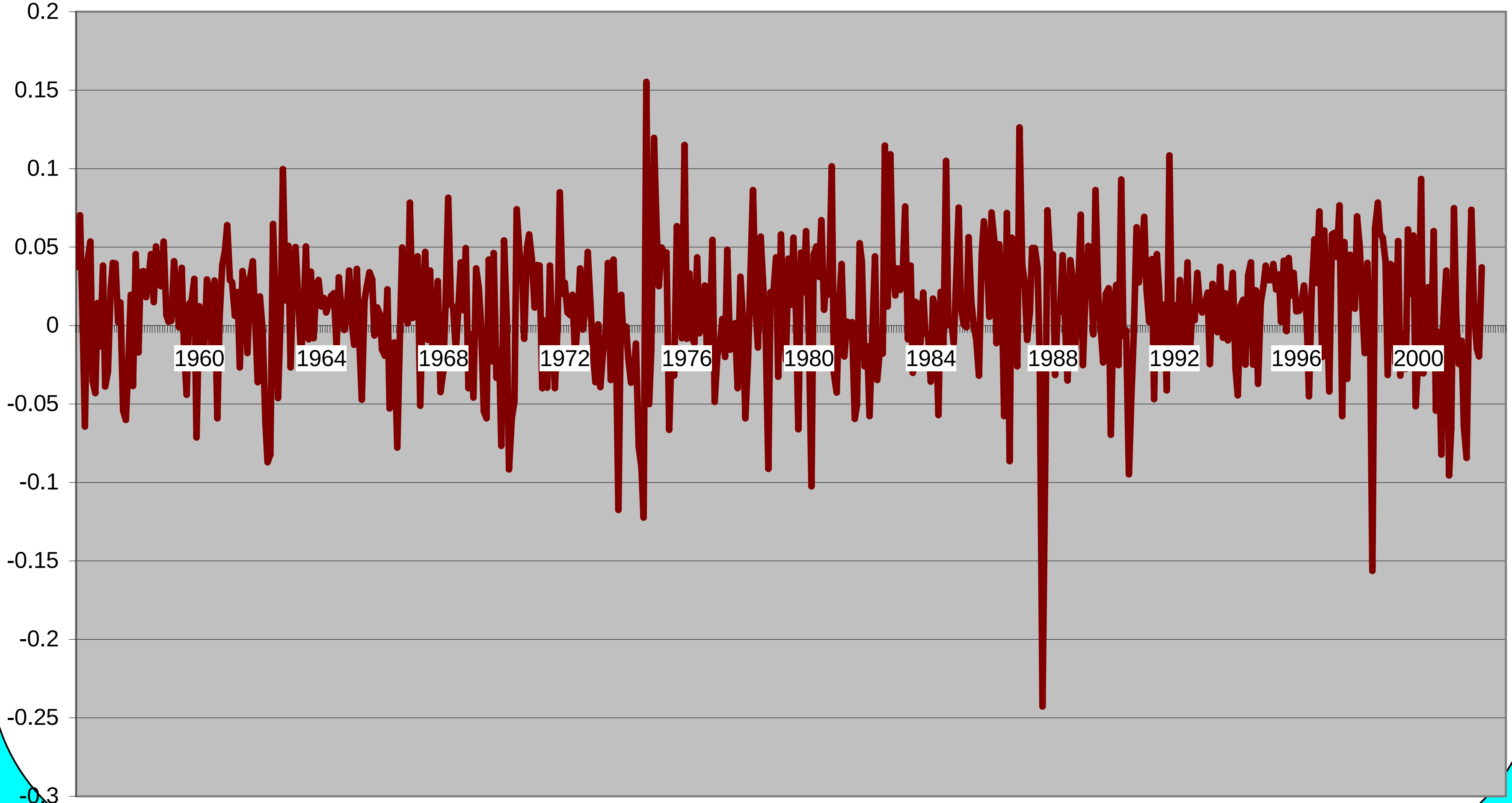
Some outcomes…

- UK
  - no more maturity guarantees
- Canada
  - cut back on generous guarantees
  - Plethora of equity models proposed
  - Still little hedging
- USA
  - Hedging common

Some equity models ….

- Regime Switching Log Normal (Hardy, 2001 and CIA Seg Fund Report, 2001)
- GARCH(1,1)
- MARCH (Chan and Wong, 2005)
- ‘Stochastic Log Volatility’ (AAA C3-Phase 2)
- Regime Switching Draw Down (Panneton, 2003)

S&P 500 Total Return
Log returns

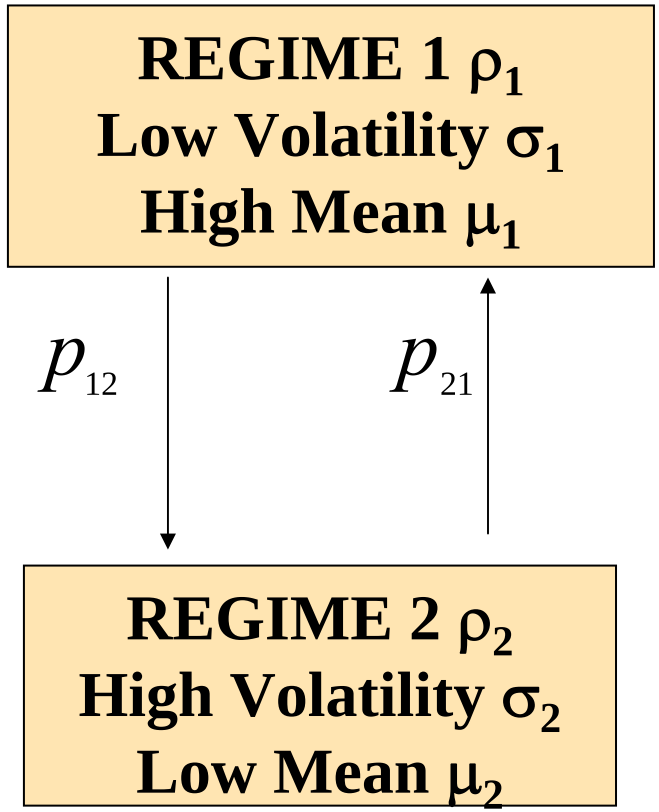


S&P data

- not much auto-correlation
- but correlation is not always a good measure of independence
- notice bunching of poor returns (eg last 2 years)
- and association of high volatility with crashes
  - ie high down more than high up

RSLN-2

$$Y_t \mid \rho_t = \mu_{\rho_t} + \sigma_{\rho_t} \varepsilon_t$$



$$Y_t = \mu_1 + \sigma_1 \varepsilon_t$$

$$Y_t = \mu_2 + \sigma_2 \varepsilon_t$$

The RSLN-2 Model

- The regime process is a hidden Markov process
- 2 Regimes are usually enough for monthly data.
- 2 Regime model has 6 parameters:
  - $\Theta=\{\mu_1, \mu_2, \sigma_1, \sigma_2, p_{12}, p_{21}\}$
- Regime 1: Low Vol, High Mean, High Persistance (small p_{12})
- Regime 2: High Vol, Low Mean, Low Persistance (large p_{21})

GARCH(1,1)

$$Y_t = \mu + \sqrt{h_t}\,\varepsilon_t$$

$$h_t = \alpha_0 + \alpha_1 \left(Y_{t-1} - \mu\right)^2 + \beta h_{t-1}$$

- Where $\varepsilon_t \sim N(0,1)$, iid
- Given F_{t-1}, ε_t is the only stochastic element
- We generally require $\alpha_1 + \beta < 1.0$

MARCH (2;0,0;2,0)

$$Y_t \mid F_{t-1} \sim \begin{cases} Q_1 & \text{w.p. } \alpha_1 \\ Q_2 & \text{w.p. } (1-\alpha_1) \end{cases}$$

$$Q_1 \sim ARCH(2);\ Q_2 \sim ARCH(0)$$

$$h_{1,t} = \beta_{10} + \beta_{11}\left(Y_{t-1} - \phi_1\right)^2 + \beta_{12}\left(Y_{t-2} - \phi_1\right)^2$$

$$h_{2,t} = \beta_{20}$$

MARCH(2;0,0;2,0)

- MARCH(K; $p_1, \ldots, p_K; q_1, \ldots q_K$) is a mixture of K AR-ARCH models,
- p_j and q_j are the AR-order and ARCH-order of the j^{th} mixture RV
- According to Chan and Wong, provides superior fit to 3rd and 4th moments of monthly log-return disn cf RSLN

SLV

$$v_t = \log\sigma_t = (1-\varphi)v_{t-1} + \varphi\log\tau + \sigma_v Z_{v,t}$$

$$\mu_t = A + B\sigma_t + C\sigma^2{}_t$$

$$Y_t = \frac{\mu_t}{12} + \frac{\sigma_t}{\sqrt{12}} Z_{y,t}$$

- $Z_{v,t}$ and $Z_{y,t}$ are standard normal RVs, with correlation ρ
- The v_t process is constrained by upper and lower bounds

SLV

- According to C3P2, SLV
  - *"Captures the full benefits of stochastic volatility in an intuitive model suitable for real world projections"*
  - *Stoch vol models are widely used in capital markets to price derivatives…*
  - *Produces very "realistic" volatility paths*

Regime Switching Draw Down (RSDD)

$$Y_t \mid (\rho_t = s) = \kappa_s + \phi_s D_{t-1} + \sigma_s \varepsilon_t$$

$$D_{t-1} = \min(0, D_{t-2} + Y_{t-1})$$

$$\varepsilon_t \sim N(0,1), \text{iid}$$

ρ_t is a Markov regime switching process

RSDD

- 2 Regimes proposed by Panneton
- D_t is the draw-down factor
- RSLN-2 is recovered when $\phi_\rho=0$, for $\rho=1,2$
- Captures 'tendency to recover'

Does it matter?

- 5 models, each being championed by someone.
- 2 RS, 2 conditional heteroscedatic, 1 ‘stochastic volatility’.
- Each fitted by MLE (-ish) to S&P500 data
- Does it make any difference to the results for Equity-Linked Capital Requirements?

Two methods for Equity Linked Life Insurance

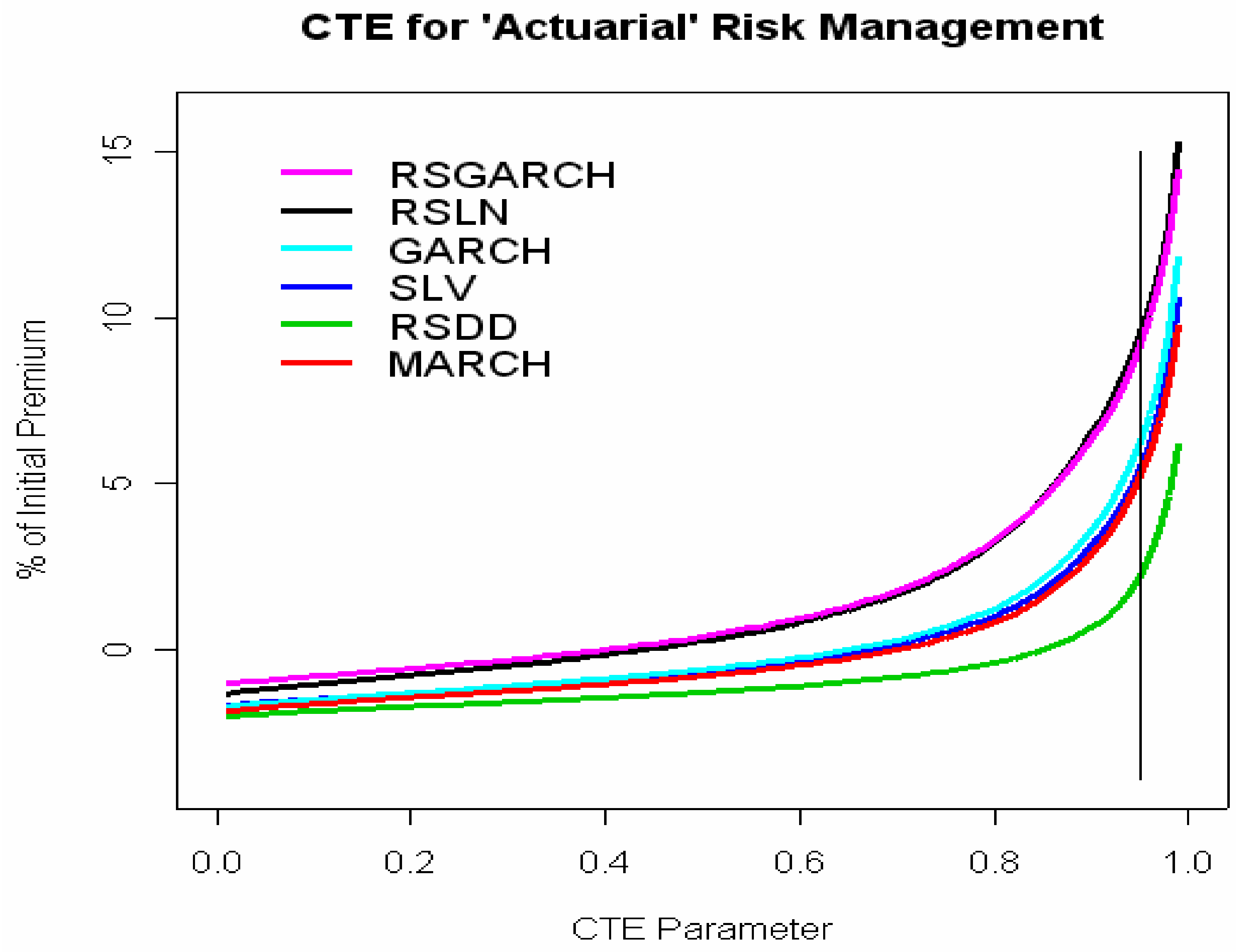
- Actuarial Approach:
  - Simulate liabilities,
  - apply risk measure,
  - discount at r-f rate
- Determines the economic capital requirement to write the contract for a given solvency standard.

Two methods for Equity Linked Life Insurance

- Dynamic Hedging Approach
  - Simulate Hedge under real world measure
  - Estimate distribution of unhedged liabilibility
  - Apply risk measure and discount at r-f rate
  - Add to cost of initial hedge

Actuarial Approach

- Estimate 95% CTE (= mean of worst 5% of simulated outcomes) for 20-year GMAB.
- Single Premium
- Guarantee payable on death or maturity
- Guarantee 'ratchet' at t=10
- Issue age 50, rf rate 5%, MER=3% p.y.
- Guarantee risk premium = 0.2% p.y.
- Deterministic mortality and lapses


CTE for 'Actuarial' Risk Management
RSGARCH
RSLN
GARCH
SLV
RSDD
MARCH
15
10
5
0
% of Initial Premium
0.0
0.2
0.4
0.6
0.8
1.0
CTE Parameter

Risk Measure, % of P; AA

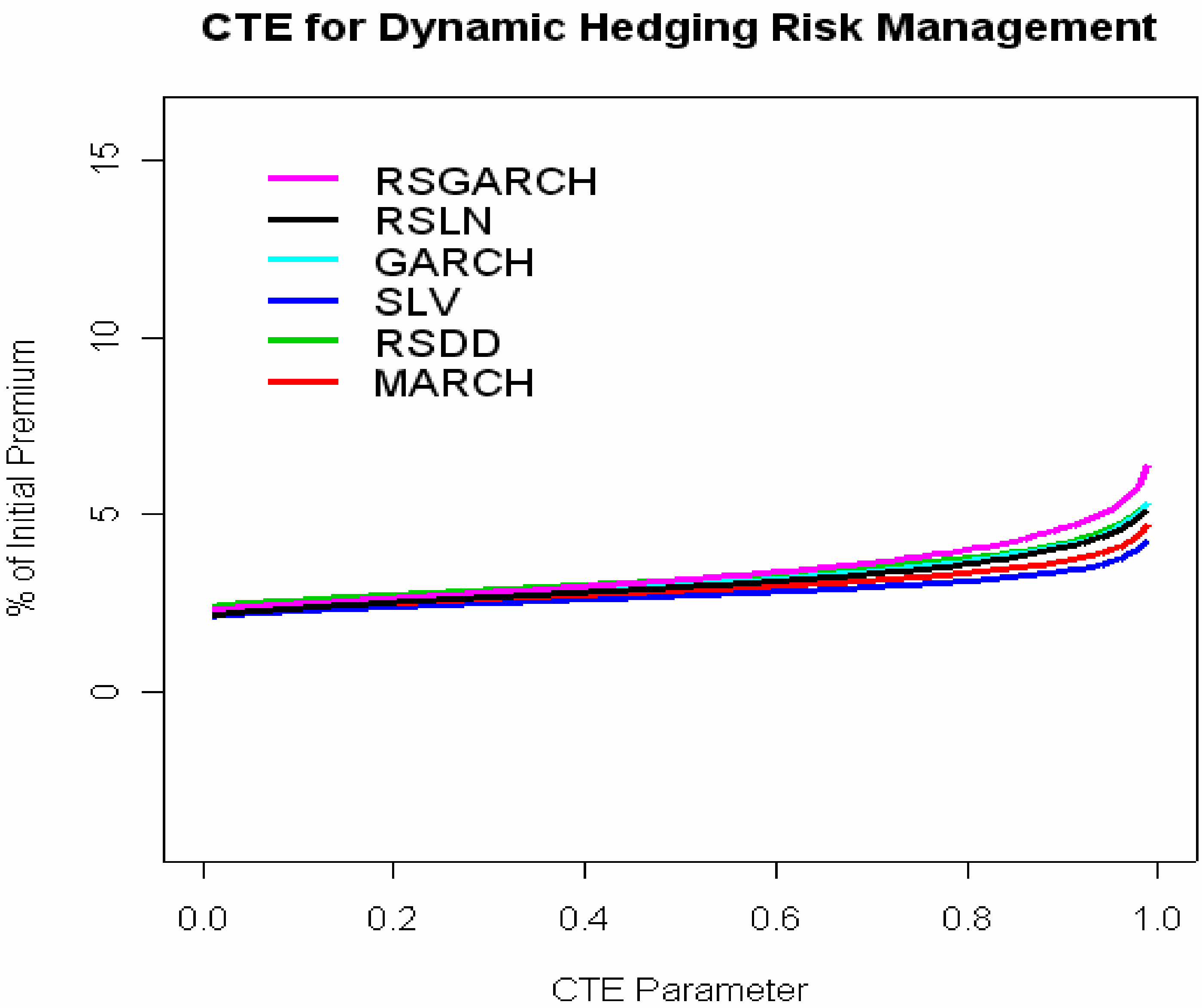
Model	90% CTE	**95% CTE**
RSDD	0.64 (0.09)	**2.25 (0.16)**
MARCH	2.85 (0.14)	**5.22 (0.19)**
SLV	3.12 (0.15)	**5.47 (0.20)**
GARCH	3.60 (0.19)	**6.27 (0.22)**
RSLN	6.50 (0.19)	**9.53 (0.23)**
RSGARCH	6.33 (0.17)	**9.18 (0.23)**

Does the model matter using the actuarial approach?

Oh Yes !!!

Using hedging?

- Straight Black-Scholes Hedge
- Simulate additional cost arising from
  - Discrete hedge
  - Model Error (ie P-measure is GARCH/RSDD etc)
  - Transactions costs


CTE for Dynamic Hedging Risk Management
RSGARCH
RSLN
GARCH
SLV
RSDD
MARCH
% of Initial Premium
0
5
10
15
0.0
0.2
0.4
0.6
0.8
1.0
CTE Parameter

Risk Measure, % of single premium

Model	90% CTE	**95% CTE**
RSDD	4.20 (0.08)	**4.62 (0.10)**
MARCH	3.67 (0.06)	**4.00 (0.09)**
SLV	3.39 (0.05)	**3.67 (0.09)**
GARCH	4.12 (0.08)	**4.52 (0.11)**
RSLN	4.06 (0.08)	**4.45 (0.09)**
RSGARCH	4.62 (0.12)	**5.15 (0.17)**

Does the model matter using the hedging approach?

Not so much….

But

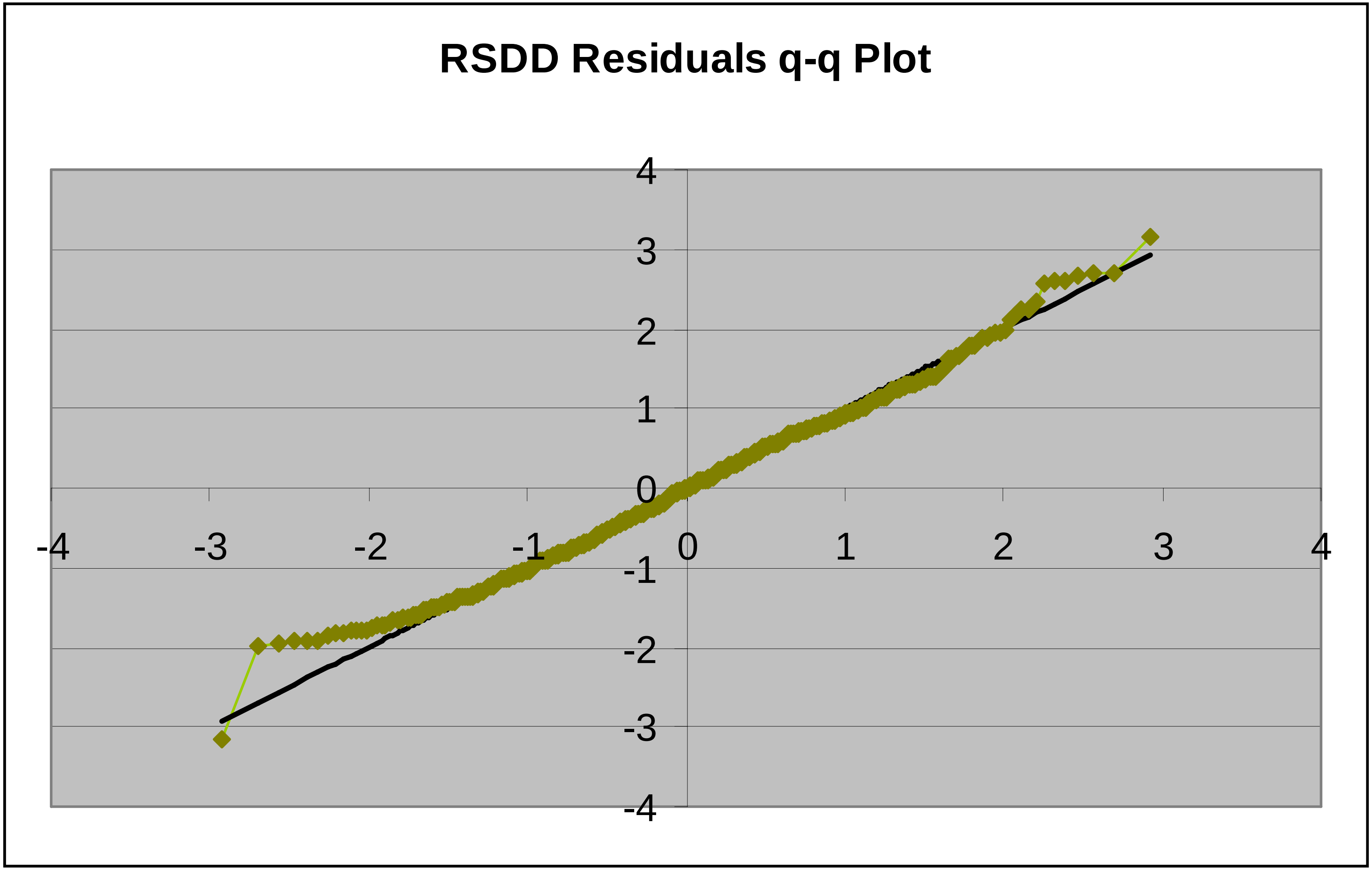
- Many companies are not hedging
- Pressure to adopt models giving lower capital requirements
- Can we use traditional methods to eliminate any of the models?

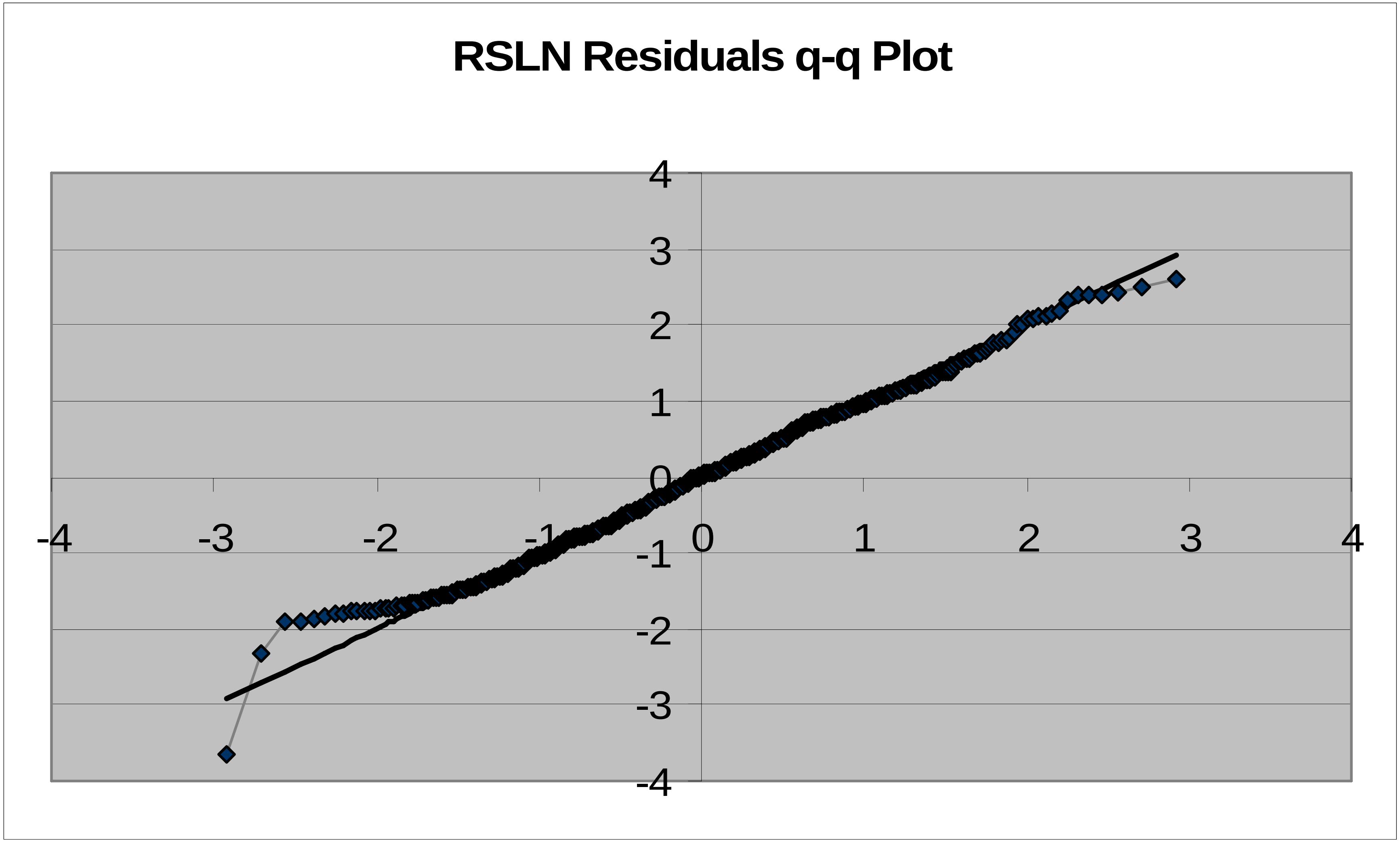
Likelihood Comparison

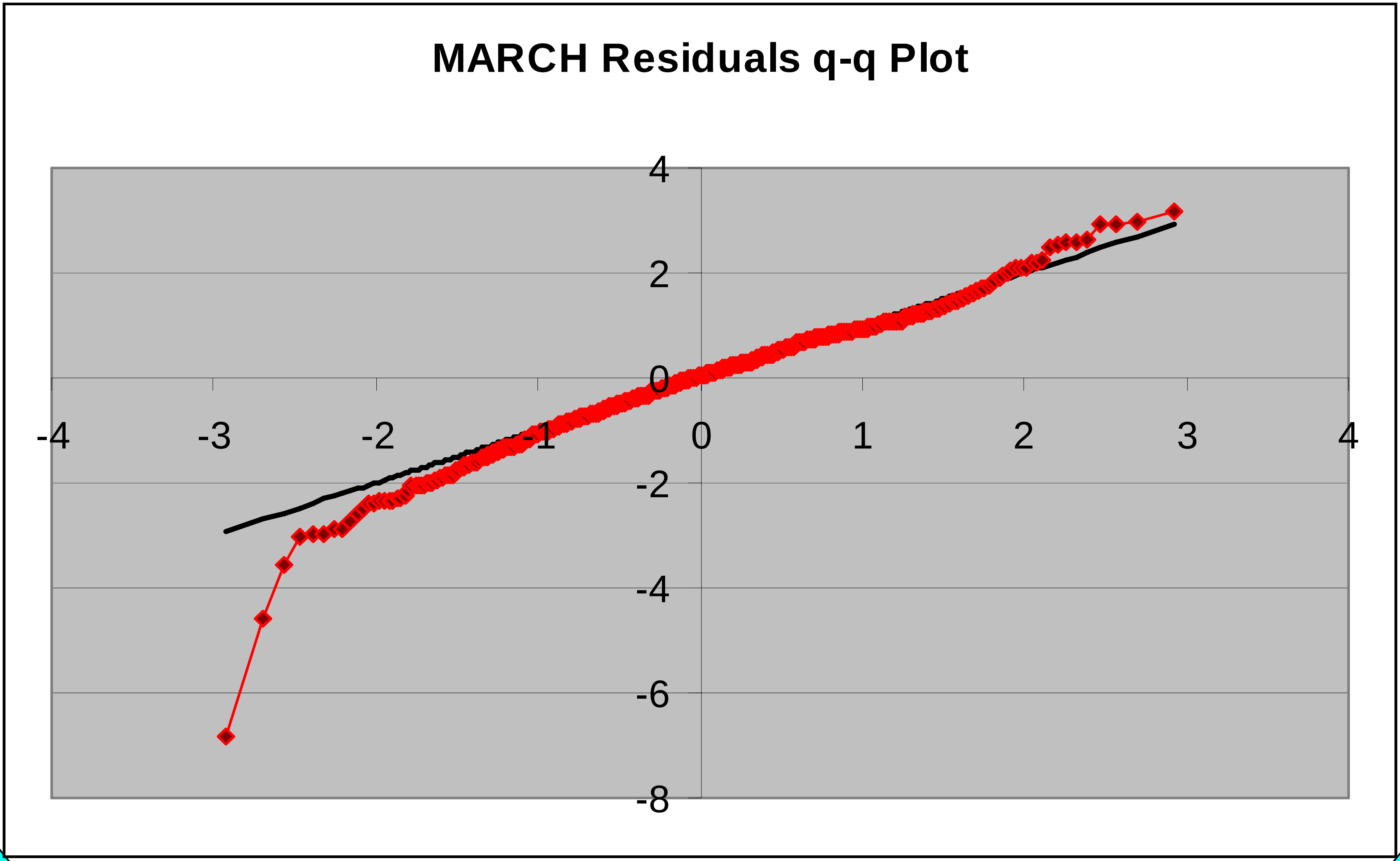
Model	# parameters	**Max LL**
RSDD	8	**1047.1**
MARCH	7	**1039.8**
SLV	7	**1032.9***
GARCH	4	**1030.1**
RSLN	6	**1042.0**
RSGARCH	8	**1054.9**

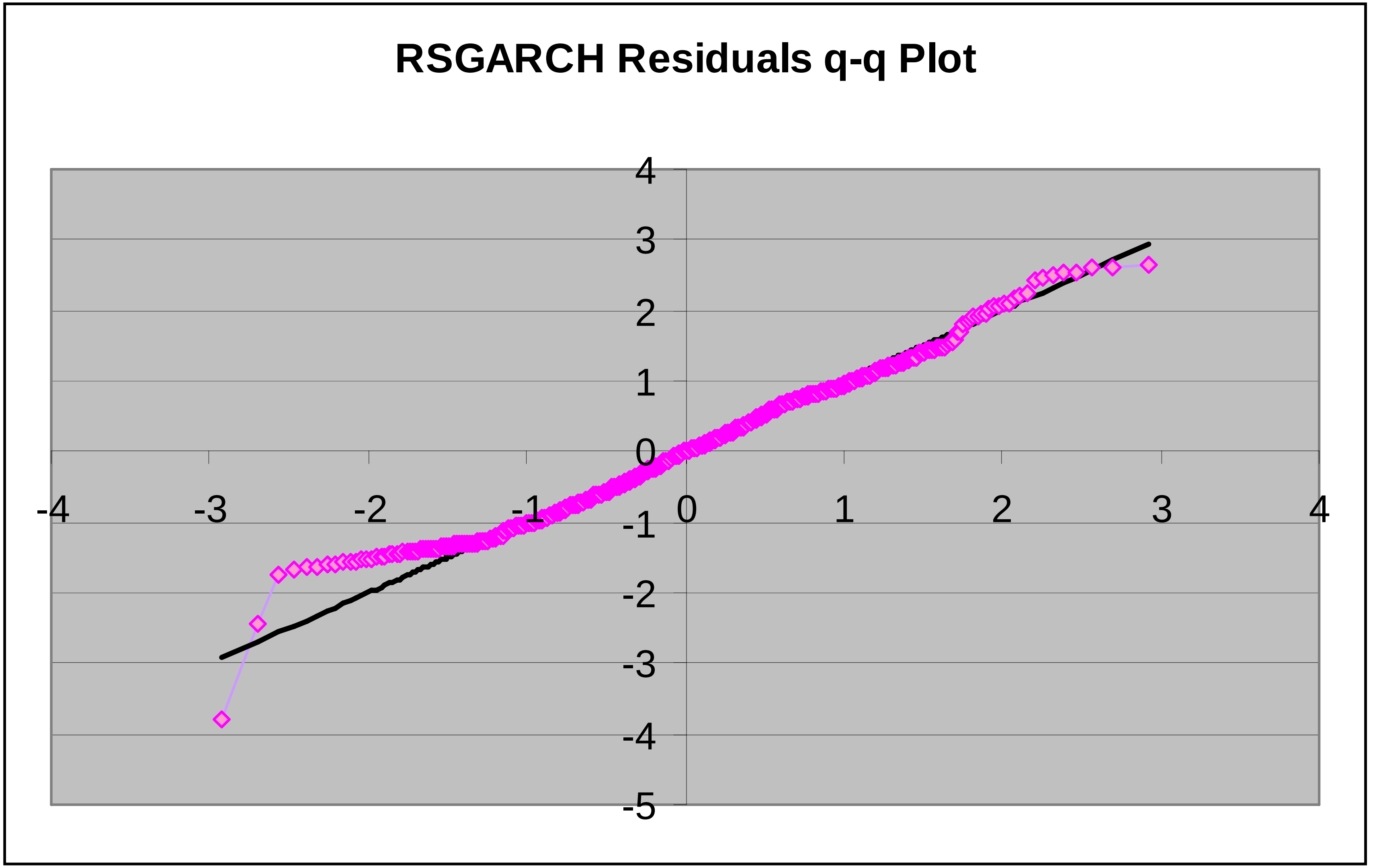
Residual analysis

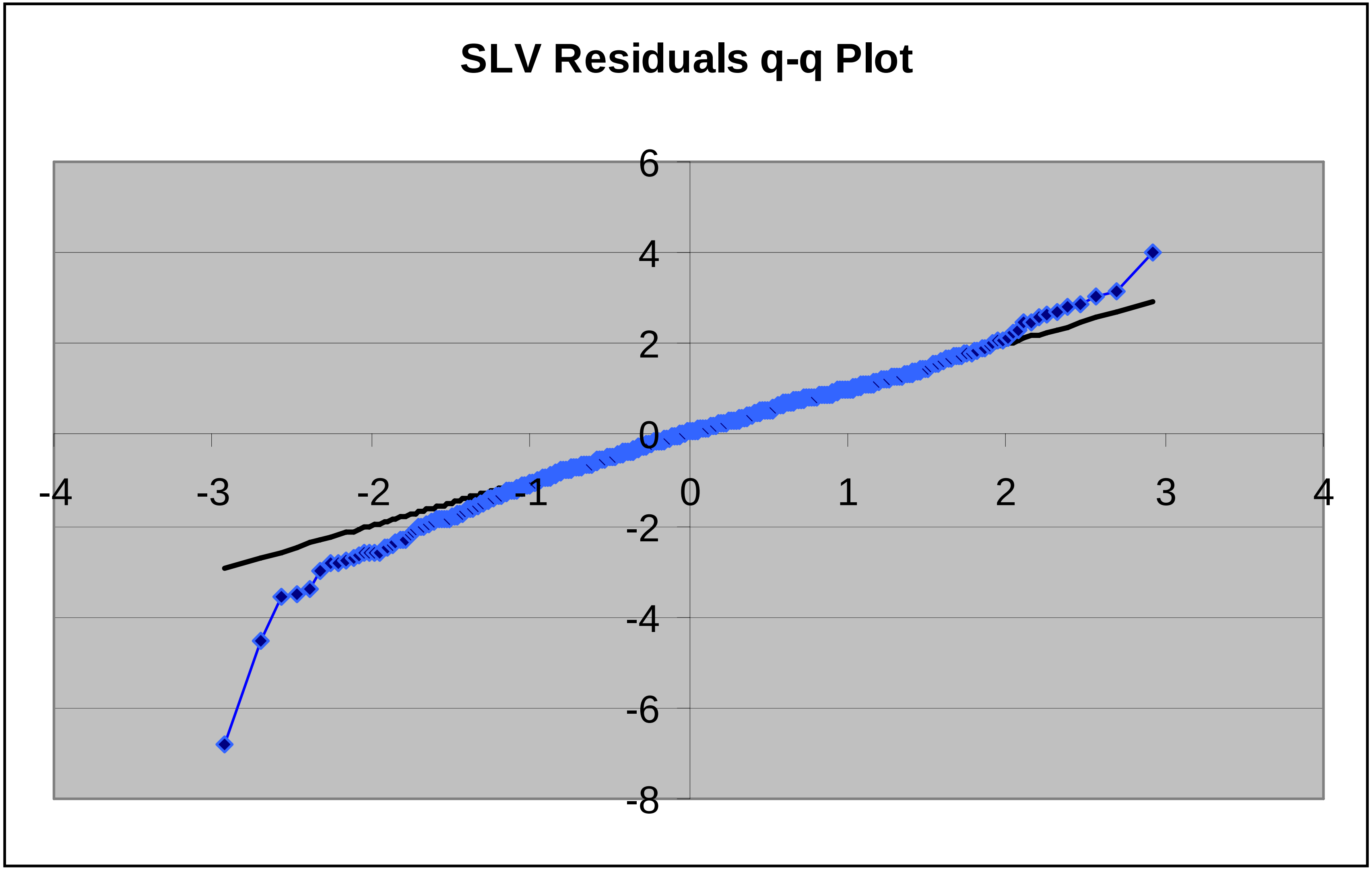
- Residuals for RS models – weighted from individual regimes
- Residuals for SLV – using simulated volatility paths


RSDD Residuals q-q Plot
4
3
2
1
0
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
3
4

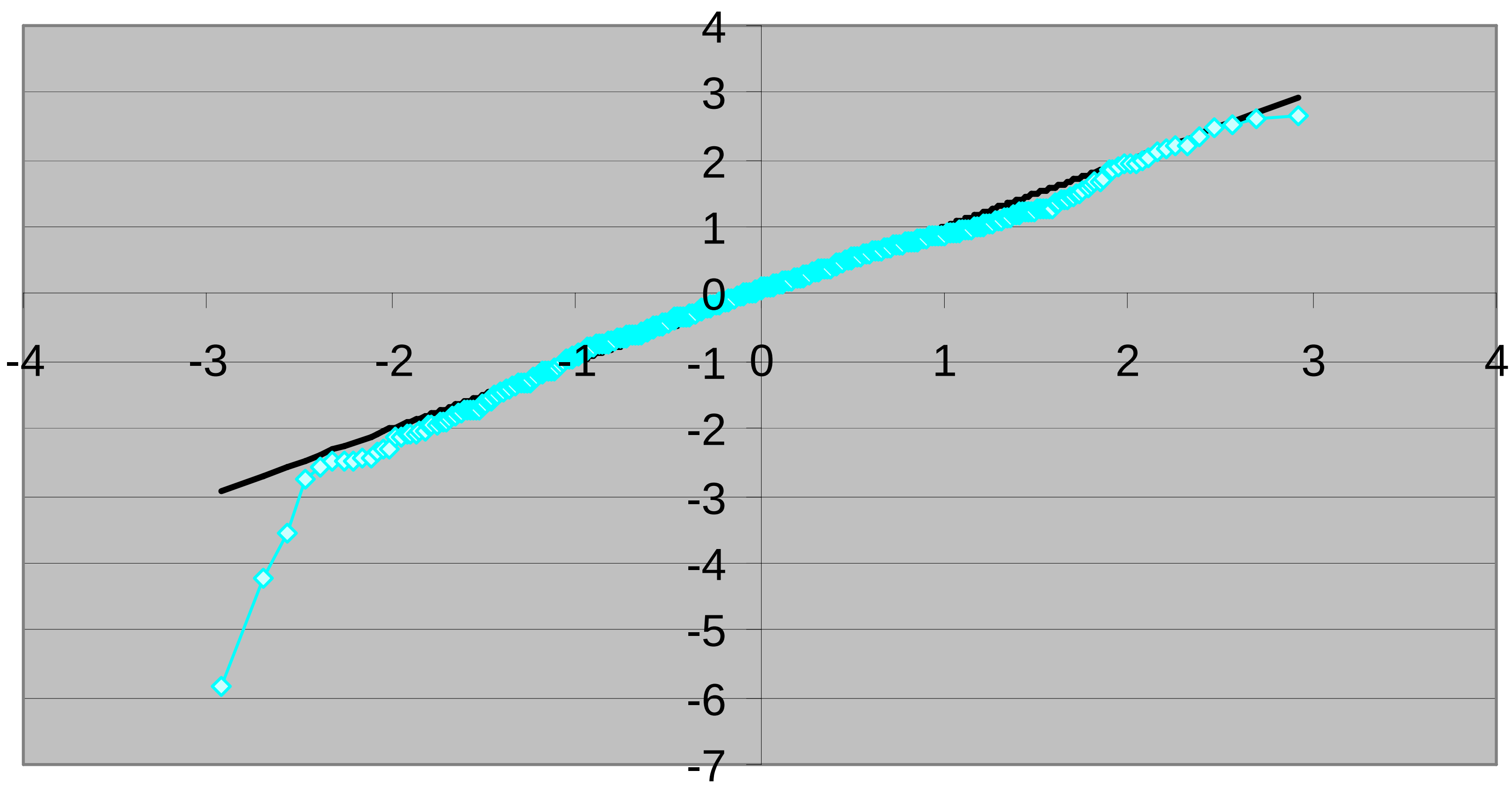

RSLN Residuals q-q Plot
4
3
2
1
0
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
3
4


MARCH Residuals q-q Plot
4
2
0
-2
-4
-6
-8
-4
-3
-2
-1
0
1
2
3
4


RSGARCH Residuals q-q Plot
4
3
2
1
0
-1
-2
-3
-4
-5
-4
-3
-2
-1
0
1
2
3
4


SLV Residuals q-q Plot
6
4
2
0
-2
-4
-6
-8
-4
-3
-2
-1
0
1
2
3
4

GARCH Residuals q-q Plot

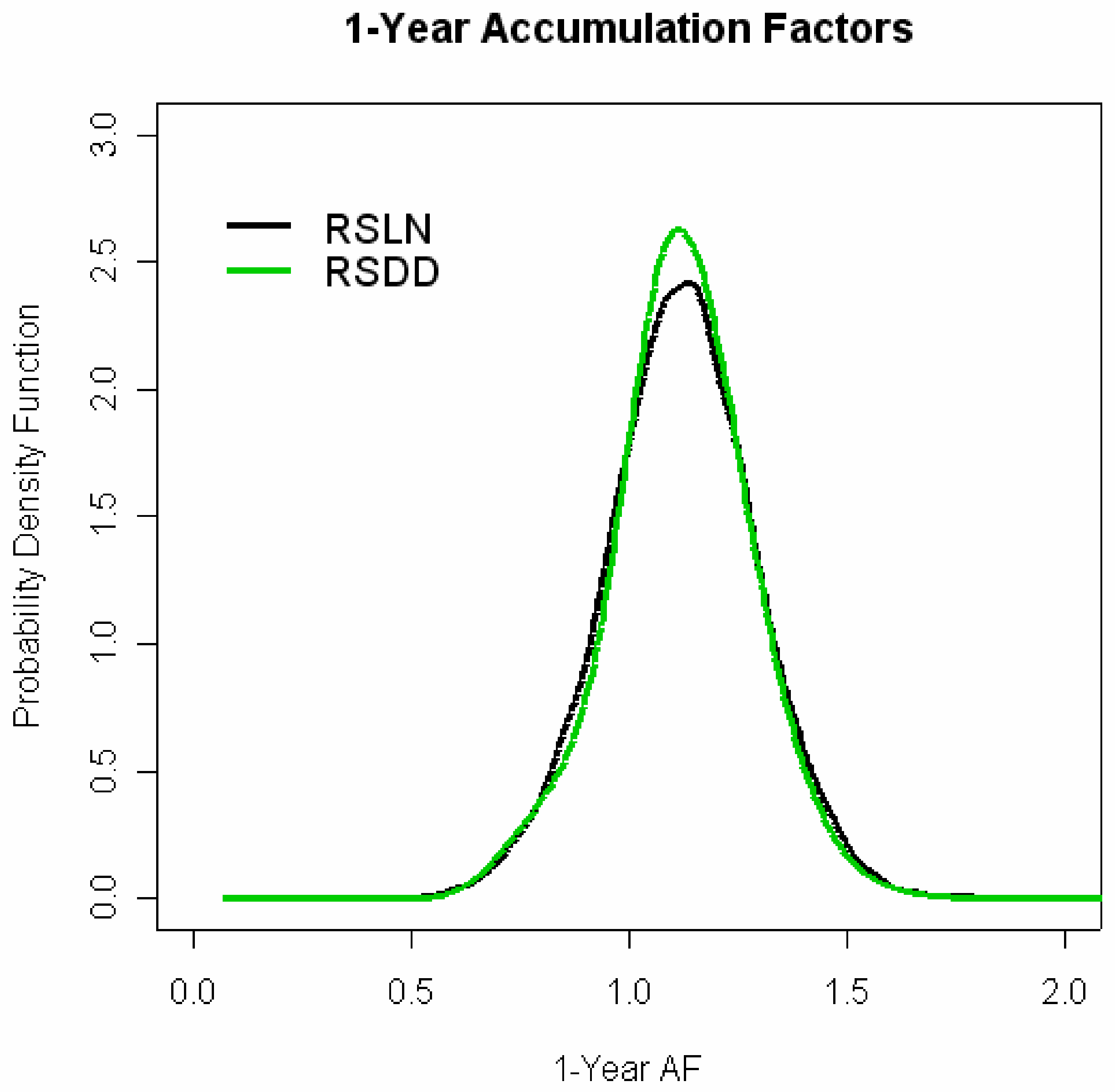


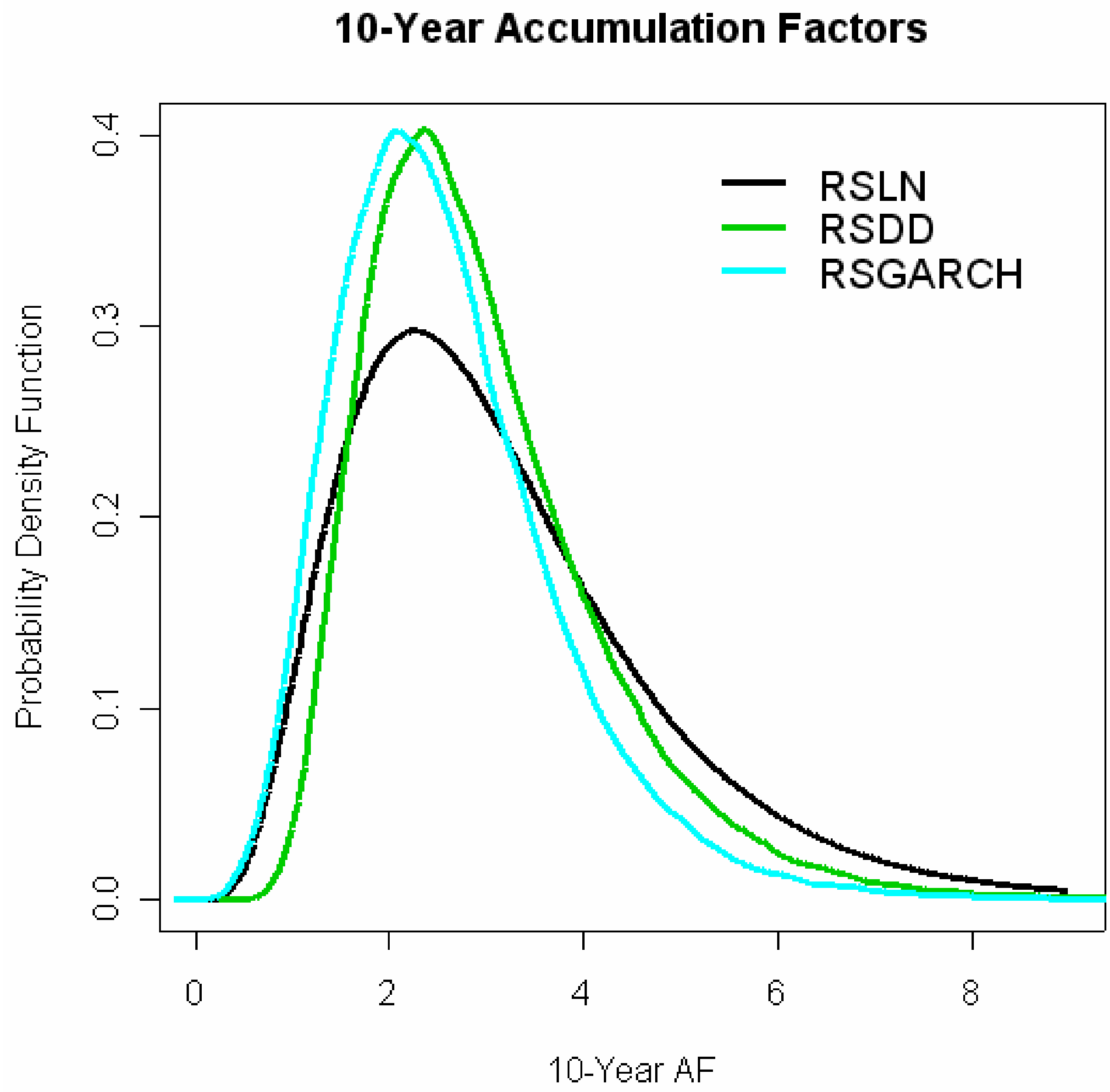
So far …

- Likelihood based selection doesn’t help much
- AIC is too simple, BIC depends on sample size, LRT has technical limitations
- Residuals can be useful, but are tricky in multifactor cases

so good…

- The regime switching models look good on likelihood and on residuals
- But there is a vast difference in application between rsdd and rsln or rsgarch
- What causes the big difference?
- Which rs model should we believe?


1-Year Accumulation Factors
RSLN
RSDD
Probability Density Function
0.0
0.5
1.0
1.5
2.0
2.5
3.0
0.0
0.5
1.0
1.5
2.0
1-Year AF


10-Year Accumulation Factors
Probability Density Function
0.0
0.1
0.2
0.3
0.4
0
2
4
6
8
10-Year AF
RSLN
RSDD
RSGARCH

Bootstrapping time series

- The traditional bootstrap is applied to independent observations.
- Dependent time series require different treatment.
- Order matters.

S&P 1-year Acc Factors

- If we take 1-year factors starting in January, empirical percentiles are (from 48 observations):
  - 2.5%ile – 0.84
  - 5%ile – 0.85
  - 10%ile – 0.94

S&P 1-year Acc Factors

- If we take 1-year factors starting in September, empirical percentiles are (47 observations):
  - 2.5%ile – 0.75
  - 5%ile – 0.87
  - 10%ile – 0.89
- Ranges are: 2.5%ile (0.74, 0.89)

  5%ile (0.83, 0.91)

  10%ile (0.89, 0.95)

1-year Acc factors

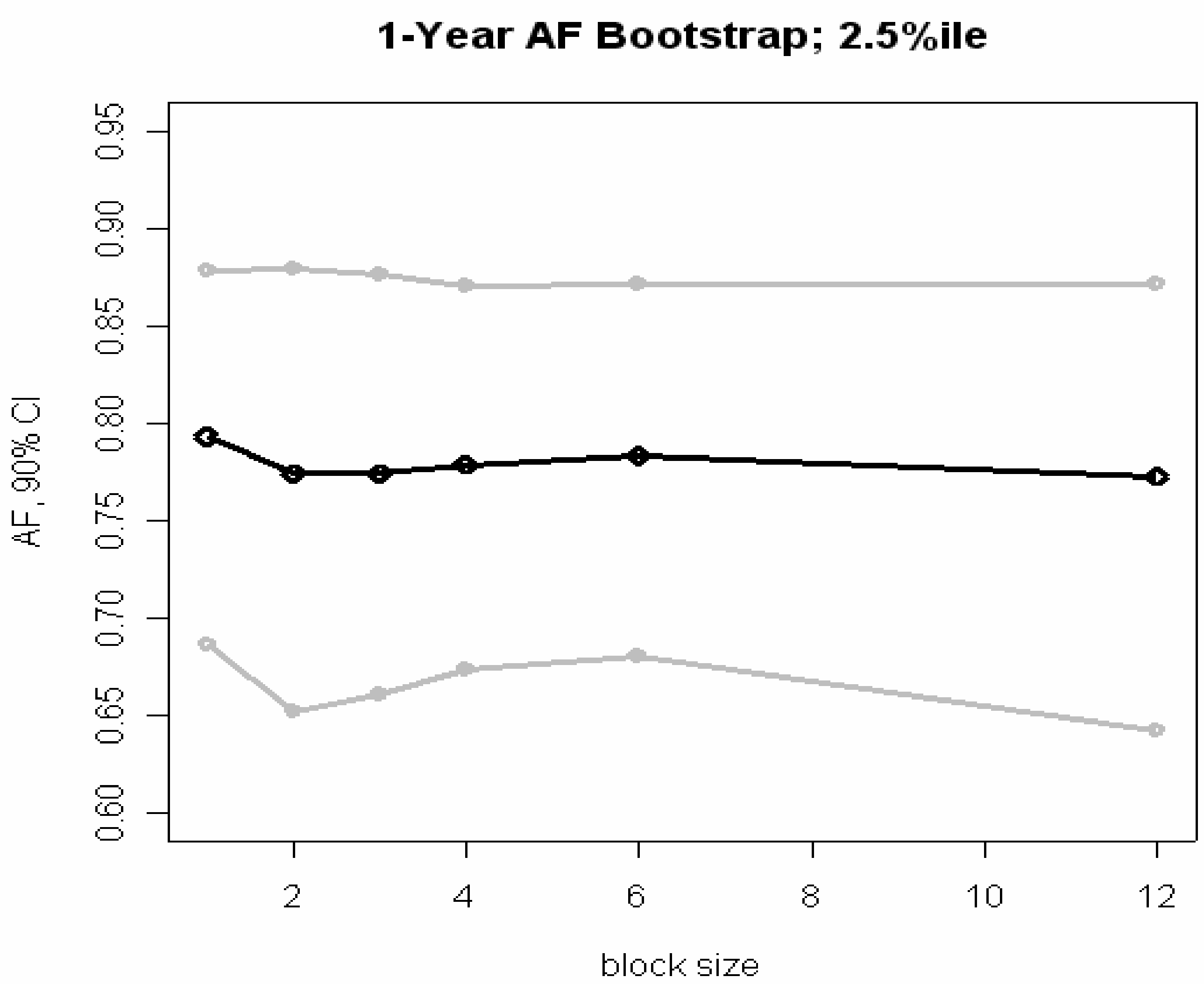
- Can't use all 1-year factors because of dependence
- If we only use (eg) January series, we are ignoring information
- Bootstrap the percentiles using time series bootstrap.

Time series bootstrap

- Bootstrap from original observations in blocks of b consecutive values.
- If the blocks are too small, lose dependence factor $\Rightarrow$ results too thin tailed (if +vely autocorrelated)
- If blocks are too large lose data, $\Rightarrow$ results too thin tailed (extreme results averaged out)

Block size

- So choose block size to maximize tail thickness.
- Other ways of selecting block size.
- No general agreement – see references.
- Randomized block length suggested.
- block resampling reduces exposure of end points → cycle from end to start.


1-Year AF Bootstrap; 2.5%ile
AF, 90% CI
0.60
0.65
0.70
0.75
0.80
0.85
0.90
0.95
2
4
6
8
10
12
block size

Bootstrap Quantile Estimates
1-Year Accumulation

Model	2.5%ile	5%ile	10%ile
Bootstrap 90% CI	0.67→0.87	0.76 →0.91	0.84 →0.97
RSDD	0.768	0.831	0.901
RSLN	0.764	0.829	0.908
RSGARCH	0.792	0.847	0.910

This doesn’t help us much.

10-year accumulation factor

- We can do the same thing
- But the original data only has 4 non-overlapping observations
- minimum 10-year observed AF is estimate of 1/5=20%ile
- So we bootstrap B samples of 4 observations

Bootstrap Quantile Estimates
10-Year Accumulation

Model	20 %ile
Bootstrap 90% CI	0.95→2.83 1.706
RSDD	1.953 *(1.92,1.97)*
RSLN	1.773
RSGARCH	1.660 *(1.63,1.68)*

And this doesn't help us much either.

Oversampling

- Bootstrapping re-samples from original data
- $\Rightarrow$ Four 10-year accumulation factors from 584 observations
- What happens if we break the rules and keep sampling?

Oversampling

- If data are independent then oversampling

  $\rightarrow$ thin tails

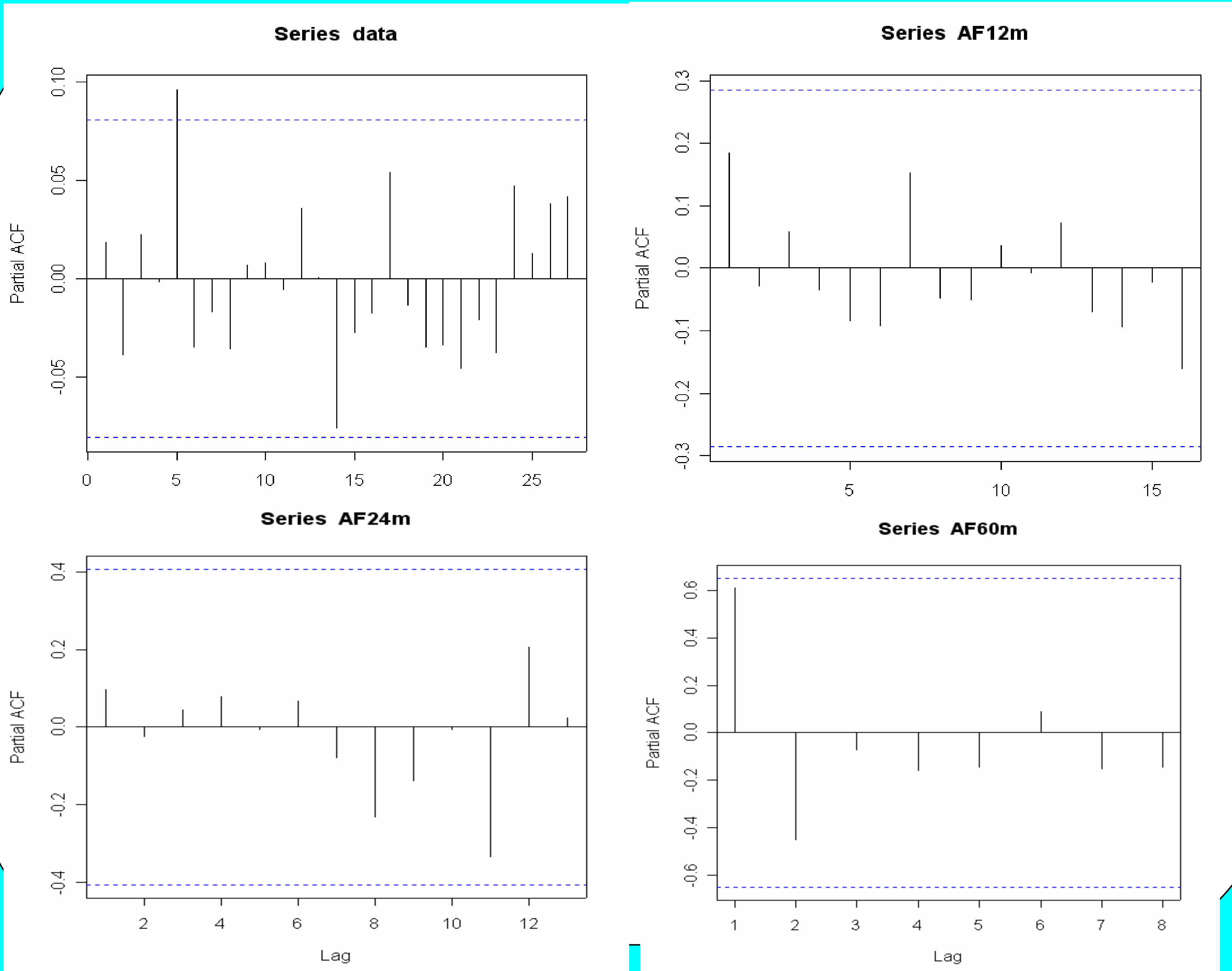
  - I.e. positive bias for low quantiles
  - Bias should be small for large original sample
- If data are +vely auto-correlated and block size is not large enough

  $\rightarrow$ thin tails

  - I.e. positive bias for low quantiles

Oversampling

- If data are –vely auto-correlated
  - Oversampling with small block size will fatten tails
  - Overall effect depends on correlation
- But we are estimating AFs so we also look at these correlations.



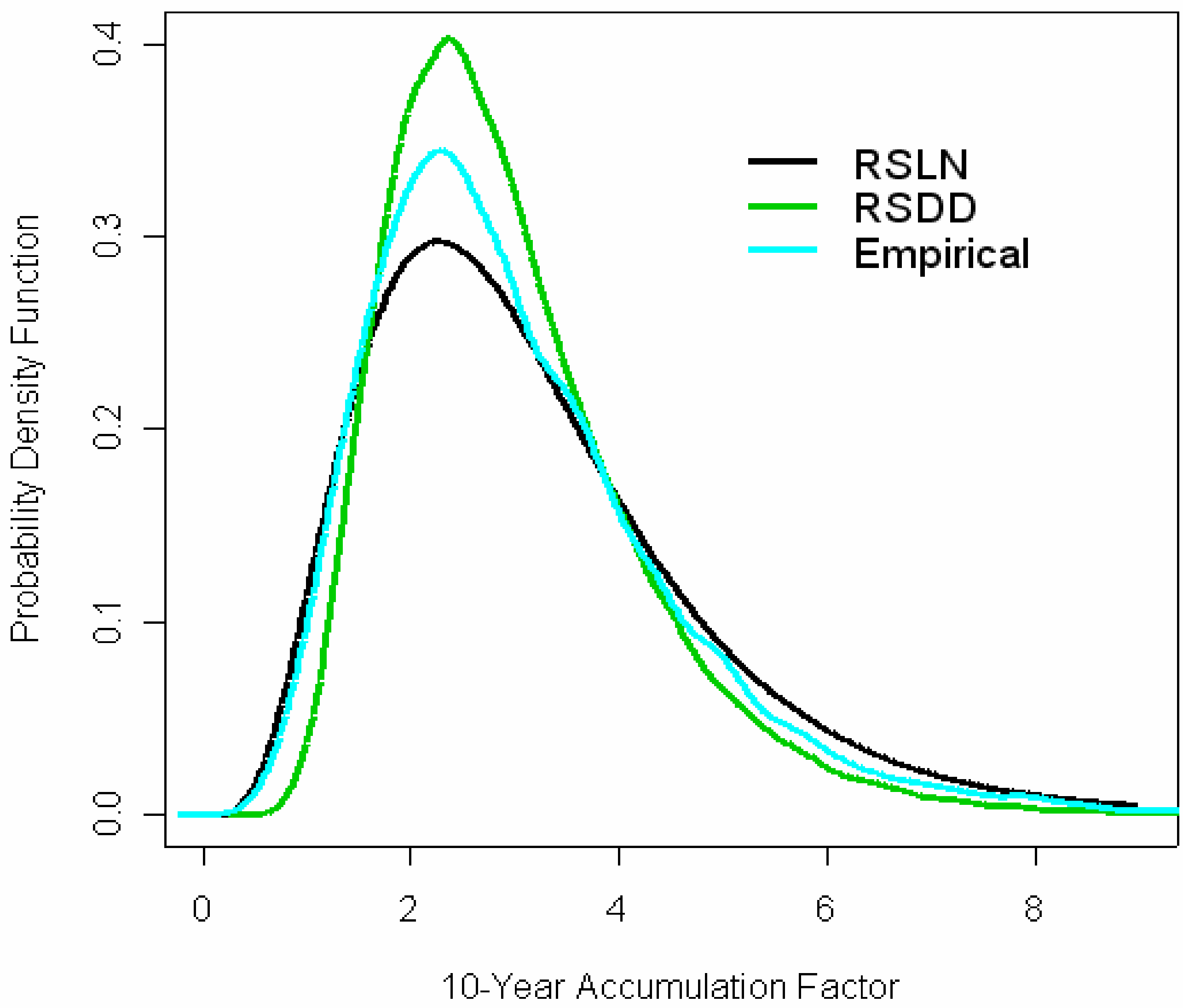
Series data
Partial ACF
0.10
0.05
0.00
-0.05
0
5
10
15
20
25
Series AF12m
Partial ACF
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
5
10
15
Series AF24m
Partial ACF
0.4
0.2
0.0
-0.2
-0.4
2
4
6
8
10
12
Lag
Series AF60m
Partial ACF
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
1
2
3
4
5
6
7
8
Lag

Back to the data

- No significant negative autocorrelations…
- So oversampling should over-estimate left tail quantiles (on average)

Left tail, 10-Year AFs

Model	2.5%	5%	10%
Bootstrap (sort of…)	1.041 *(1.03, 1.06)*	1.228 *(1.20,1.25)*	1.478 *(1.47,1.49)*
RSDD	1.277	1.439	1.653
SLV	1.082	1.254	1.468
RSGARCH	0.905	1.086	1.315
RSLN	0.914	1.105	1.378


RSLN
RSDD
Empirical
Probability Density Function
0.0
0.1
0.2
0.3
0.4
0
2
4
6
8
10-Year Accumulation Factor

Summing up

- We need to pay attention to model econometrics
- Huge financial implications – especially with traditional actuarial methods
- Abusing the bootstrap offers some info
- Multiple state models for equity returns.

References

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