

Risk-Based Solvency Capital Requirements

Damir Filipovic, University of Munich

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Overview

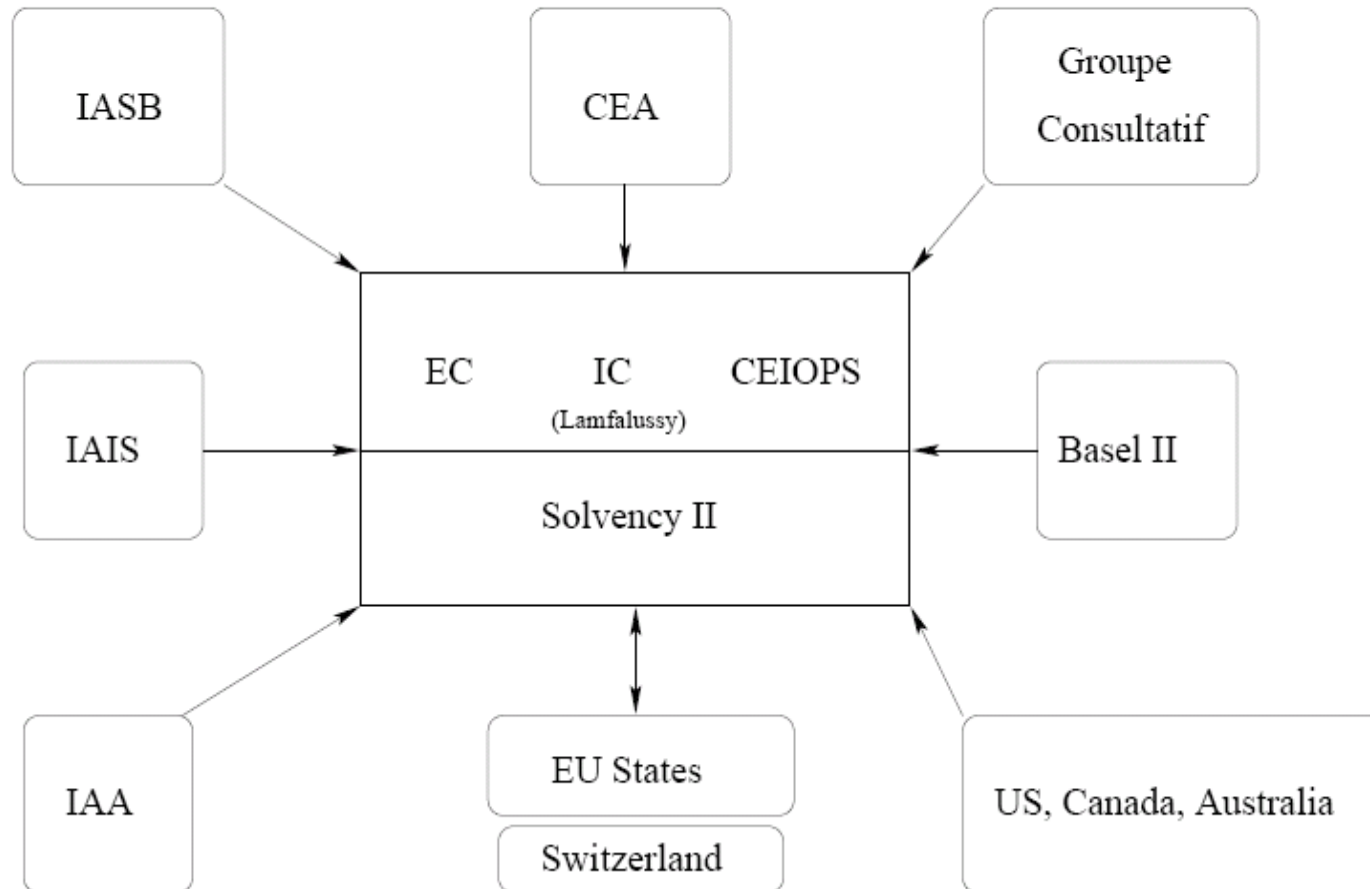
- Solvency II
- Available and Required Capital
- Diversification
- Industry Practice
- Equilibrium approach to value and measure capital and risk transfers

Solvency II

- New EU-system to assess overall solvency based on prospective risk-oriented approach (initiated 2001)
- 3 pillars: quantitative requirements, supervisory activities, public disclosure
- Pillar 1: Solvency Capital Requirement (SCR) and Minimum Capital Requirement (MCR), valuation of assets and liabilities, group ***diversification***, etc.
- Committee of European Insurance and Occupational Pension Supervisors (CEIOPS) in charge to advise the Solvency II project through three specific calls for advice (Aug 2004-Mar 2006)

http://europa.eu.int/comm/internal_market/insurance/solvency_en.htm

Solvency II: organisation/compatibility



Available Risk Capital (C)

$$C = A - L$$

available capital value of assets value of liabilities

- Depends on choice of and valuation principles for assets and liabilities
- Market consistent valuation of assets:
 - marked to market if available
 - marked to model else (e.g. risk-neutral valuation)

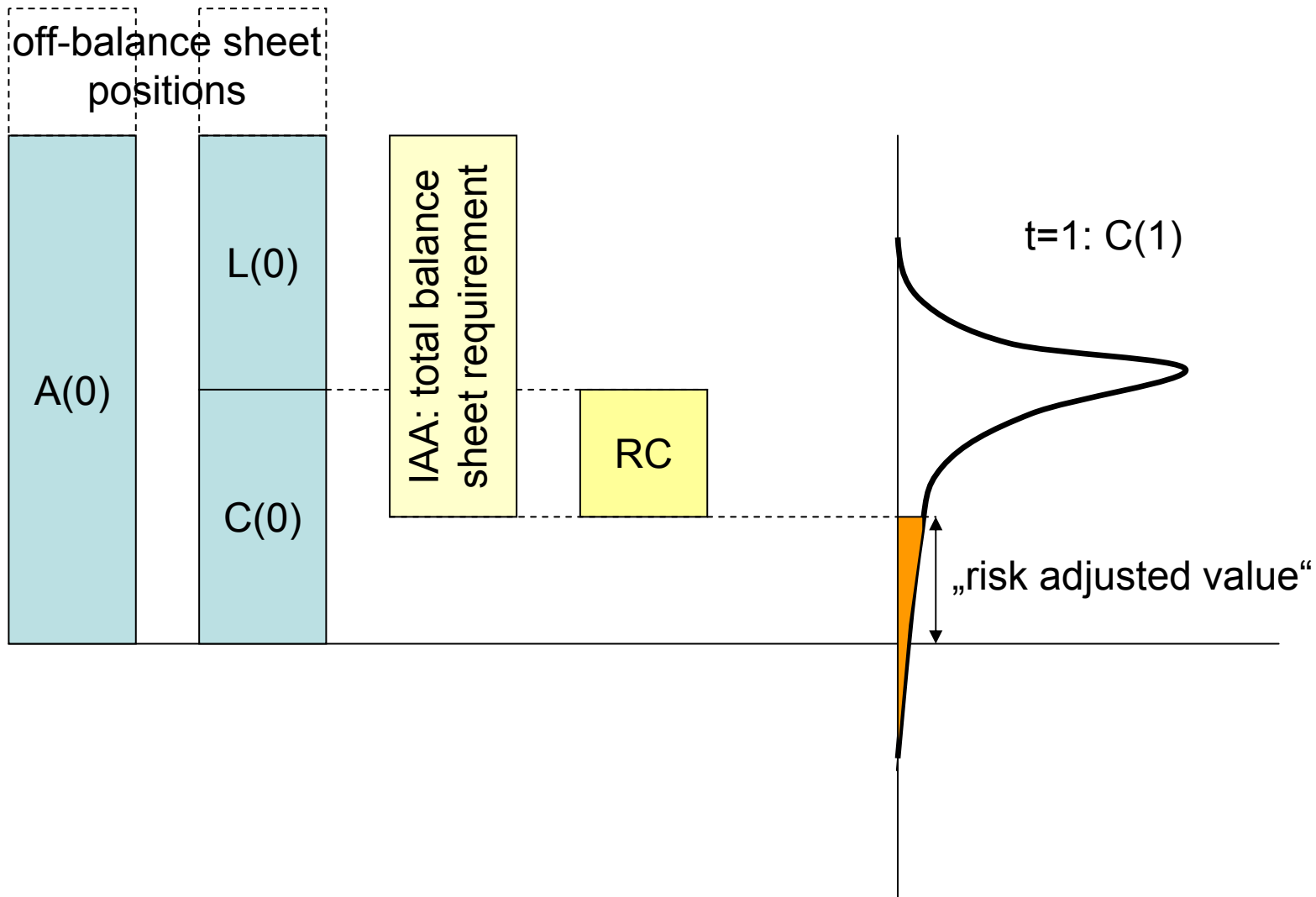
- Value of Liabilities:

Statutory reserves

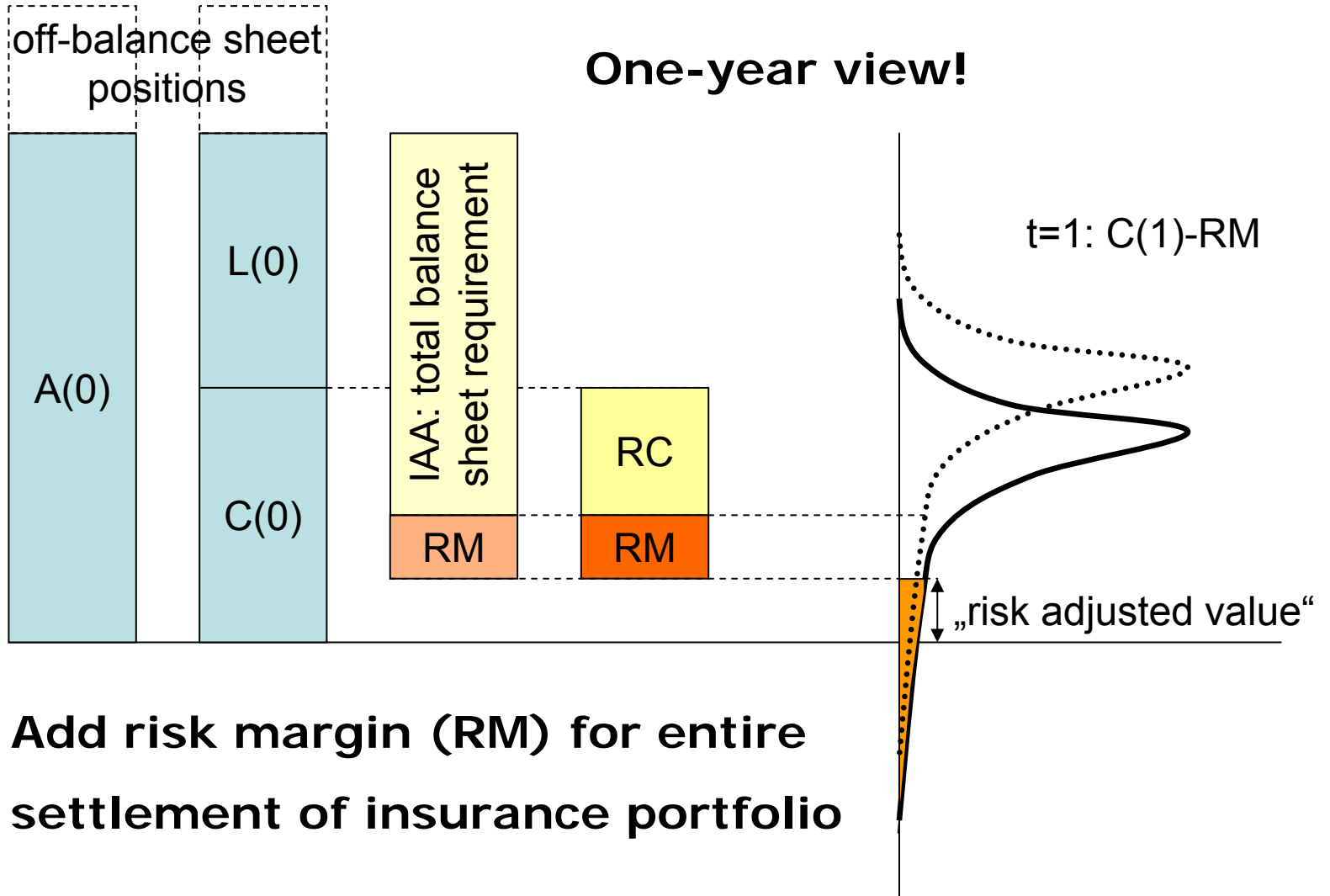
Risk margin
Best estimate

Best estimate

Required Risk Capital (RC)



Required Risk Capital (RC)



Add risk margin (RM) for entire settlement of insurance portfolio

Diversification

Diversification = spreading the portfolio over a variety of exposures (products, markets, legal entities, etc.), rather than only a few selected areas.

◇ The pooling of many independent individual risks results in a low coefficient of variation of the P&L.

◇ On a larger scale, independent risk types (such as market and technical insurance risks) have a statistically compensatory effect on the relative P&L variability.

◇ ALM, Hedging: compensation of opposite effects of risk factors by opposite portfolio sensitivities.

Stochastic Factor Model

$$C(1) = F(X)$$

1. Identification of relevant risk drivers $X = (X_1, \dots, X_n)$
2. Sensitivity analysis:
 - “delta”: $\delta_i F(x) = F(x + \epsilon_i) - F(x) \approx \partial_i F(x)$
 - “gamma” or scenario vector for non-linear instruments
3. Model joint distribution of $X = (X_1, \dots, X_n)$
4. $\Delta C = C(1) - C(0) = F(X) - F(x)$
5. Risk measurement $\rho(\Delta C)$

Covariance Model

$$\Delta C = \nabla F(x) \cdot (X - x) =: Z_1 + \dots + Z_n$$

- Z_1, \dots, Z_n multi-normal distributed
- Risk measure affine in $\sqrt{\text{Var}}$ (e.g. V@R or ES for normal d.)

$$\rho(Y) = \kappa \sqrt{\text{Var}(Y)} + \mathbb{E}[Y]$$

- Allocation to risk factors via marginal method

$$k_i = \frac{d}{d\epsilon} \rho(\Delta C + \epsilon Z_i)|_{\epsilon=0} = \kappa \frac{\sum_{j=1}^n \text{Cov}(Z_i, Z_j)}{\sqrt{\text{Var}(\Delta C)}} + \mathbb{E}[Z_i]$$

- Limited suitability for heavy tailed risk drivers and non-linear instruments

Industry Practice: Covariance Method (example)

	Geography A			Geography B			...
Risk type	BU1	BU2	...	BU1	BU2
Market	k						
Credit							
Non-life Insurance							
Life Insurance							

Level 1: within risk types

$$k = \kappa \sqrt{\text{Var}(Z)} + E[Z]$$

(CRO Forum: “A framework for incorporating diversification in the solvency assessment of insurers”)

Industry Practice: Covariance Method (example)

	Geography A			Geography B			...
Risk type	BU1	BU2	...	BU1	BU2
Market	k						
Credit							
Non-life Insurance							
Life Insurance							

Level 2: across risk types
within BUs

$$k = \kappa \frac{\sum_{BU1} Cov(Z, Z_j)}{\sqrt{Var\left(\sum_{BU1} Z_j\right)}} + E[Z]$$

(CRO Forum: “A framework for incorporating diversification in the solvency assessment of insurers”)

Industry Practice: Covariance Method (example)

	Geography A			Geography B			...
Risk type	BU1	BU2	...	BU1	BU2
Market	k						
Credit							
Non-life Insurance							
Life Insurance							

Level 3: across BUs
within geography

$$k = \kappa \frac{\sum_{GeographyA} Cov(Z, Z_j)}{\sqrt{Var\left(\sum_{GeographyA} Z_j\right)}} + E[Z]$$

(CRO Forum: “A framework for incorporating diversification in the solvency assessment of insurers”)

Industry Practice: Covariance Method (example)

	Geography A			Geography B			...
Risk type	BU1	BU2	...	BU1	BU2
Market	k						
Credit							
Non-life Insurance							
Life Insurance							

Level 4: across geographies
or regulatory jurisdictions

$$k = \kappa \frac{\sum_{Group} Cov(Z, Z_j)}{\sqrt{Var\left(\sum_{Group} Z_j\right)}} + E[Z]$$

(CRO Forum: “A framework for incorporating diversification in the solvency assessment of insurers”)

Problems

- Covariance method does not account for “tail dependence” (→ stress test, copulas!)
- Covariance capital allocation is not “fair”

Example:
3 BUs

A		B
BU1	BU2	BU3
k1	k2	k3

$$correlation = \begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

Level 1 (stand alone):

$$k1 = k2 = k3 = 100$$

Level 3 (within geography A):

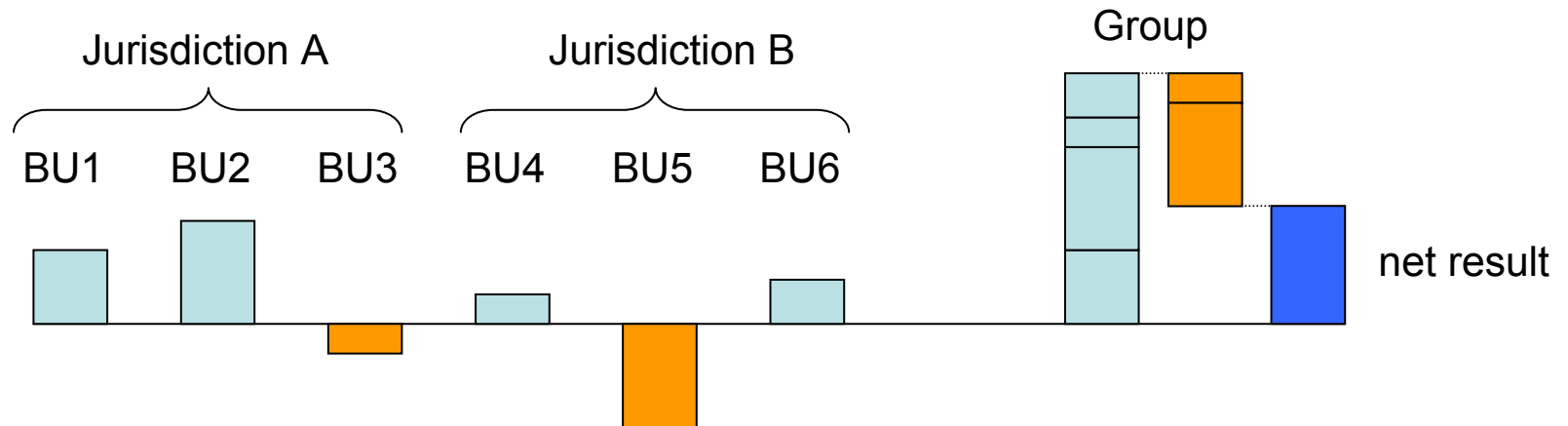
$$k1 = \frac{1+0}{\sqrt{1+1}} \times 100 = 70.71$$

Level 4 (full diversification):

$$k1 = \frac{1+1}{\sqrt{1+1+1+2 \times 1}} \times 100 = 89.44$$

Problems

- Diversification effects require full fungibility of capital:



Regulatory risk: regulators may prevent capital to be transferred between jurisdictions

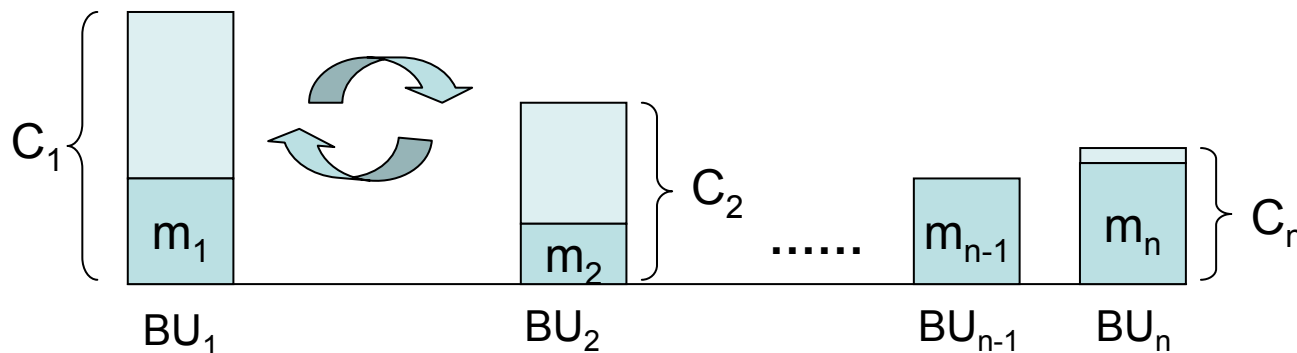
Management risk: companies' managements may refuse to provide necessary capital injections

→ **need of standardization for risk & capital transfers!**

Standardized Diversification. An Example

- n BUs with (random) future available capitals C_1, \dots, C_n
- Each BU faces a minimum capital requirement m_1, \dots, m_n
- All BUs can share excess gain $(C_i - m_i)^+$ of BU i at appropriate price
- New positions of BUs become

$$C_i + \sum_j \lambda_{ij} (C_j - m_j)^+ + \gamma_i$$



Optimization Problem

$$\inf_{\lambda_{ij}, \gamma_i} \sum_{i=1}^n \rho \left(C_i + \sum_{j=1}^n \lambda_{ij} (C_j - m_j)^+ + \gamma_i \right)$$

subject to the **(sub-)clearing condition**

$$\sum_{i=1}^n \left(C_i + \sum_{j=1}^n \lambda_{ij} (C_j - m_j)^+ + \gamma_i \right) \leq \sum_{i=1}^n C_i$$

Fact: “<” may happen for some $\omega \rightarrow$ dividends, group benefit

- Characterization of optimal allocation λ_{ij}, γ_i ?
- Fair value for surplus participation $(C_j - m_j)^+$?
- Extended formalization of this optimization problem?

Equilibrium Approach (Joint with Michael Kupper)

n agents with initial endowments $X_i \in L^\infty = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and convex risk measures $\rho_i : L^\infty \rightarrow \mathbb{R}$. Write $X := \sum_i X_i$.

1. monotone: $\rho_i(Z) \leq \rho_i(Y)$ if $Z \geq Y$
2. convex: $\rho_i(\lambda Z + (1 - \lambda)Y) \leq \lambda \rho_i(Z) + (1 - \lambda)\rho_i(Y)$, $\lambda \in [0, 1]$
3. cash invariant: $\rho_i(Z + \gamma) = \rho_i(Z) - \gamma$ for all $\gamma \in \mathbb{R}$
4. (coherent: $\rho_i(\lambda Y) = \lambda \rho_i(Y)$ for all $\lambda \geq 0$.)

Unconstrained optimization is trivial if $\rho_i \equiv \rho$ coherent:

$$\sum_{i=1}^n \rho(\lambda_i X + \gamma_i) = \rho(X) \leq \sum_{i=1}^n \rho(\xi_i) \quad \forall \sum_{i=1}^n \xi_i \leq X \text{ (sub-clearing)}$$

where $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$ (convex risk sharing), $\sum_i \gamma_i = 0$ (cash rebalancing)
s.t. $\rho(\lambda_i X + \gamma_i) \leq \rho(X_i)$ (individual rationality)

Capital and Risk Transfer Constraints

Agent i can only assume positions in $M_i \subset L^\infty$. $X_i \in M_i$.

Assumption: M_i closed, convex, cash invariant: $M_i + \mathbb{R} = M_i$ (cash is fungible)

The (M_i) -constrained convolution of ρ_1, \dots, ρ_n is defined by

$$\square_i^{(M_i)} \rho_i(X) := \inf_{\sum_i \xi_i \leq X, \xi_i \in M_i} \sum_{i=1}^n \rho_i(\xi_i).$$

Remark: the “sub-clearing condition” $\sum_i \xi_i \leq X$ is essential!

Assumption: $\square_i^{(M_i)} \rho_i(X) > -\infty$

Proposition 1. $\square_i^{(M_i)} \rho_i : L^\infty \rightarrow \mathbb{R}$ is a convex risk measure with “penalty fct”

$$\alpha^{(M_i)}(\mu) = \sum_{i=1}^n \sup_{\xi \in M_i} (\langle \mu, -\xi \rangle - \rho_i(\xi)), \quad \mu \in (L^\infty)^*.$$

Moreover,

$$\emptyset \neq \partial \square_i^{(M_i)} \rho_i(\xi) \subset (-\mathcal{P}), \quad \forall \xi \in L^\infty$$

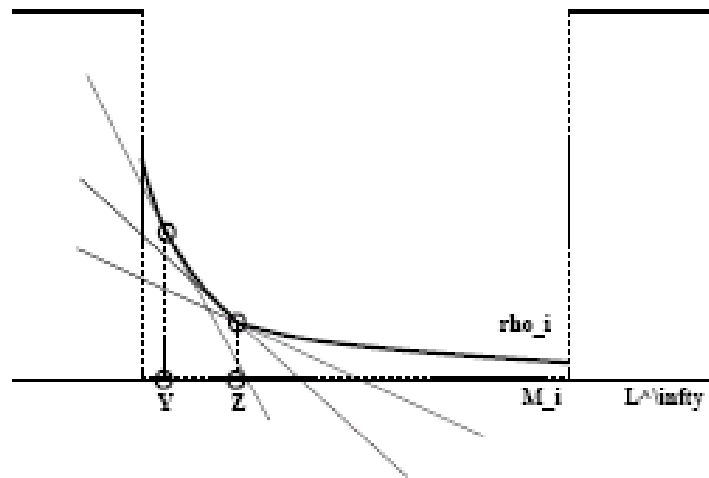
where $\mathcal{P} := \{\mu \in (L^\infty)^* \mid \langle \mu, 1 \rangle = 1, \mu \geq 0\}$ = set of pricing rules.

Sub-Gradient

Let $f : L^\infty \rightarrow \mathbb{R} \cup \{+\infty\}$ be convex. The **sub-gradient** is

$$\partial f(\xi) := \{v \in (L^\infty)^* \mid f(\eta) \geq f(\xi) + \langle v, \eta - \xi \rangle \forall \eta \in L^\infty\}$$

Example: $f(\xi) = \rho_i^{M_i}(\xi) := \begin{cases} \rho_i(\xi), & \xi \in M_i \\ +\infty, & \text{else,} \end{cases}$



Technical Results

Proposition 2. $\rho_t^{M_t} : L^\infty \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex, cash invariant, $\sigma(L^\infty, (L^\infty)^*)$ -lower semi-continuous and can be represented by

$$\rho_t^{M_t}(\xi) = \max_{\mu \in \mathcal{P}} (\langle \mu, -\xi \rangle - \alpha_t^{M_t}(\mu)), \quad \forall \xi \in M_t$$

with "penalty function" $\alpha_t^{M_t}(\mu) := \sup_{\xi \in M_t} (\langle \mu, -\xi \rangle - \rho_t(\xi))$.

Moreover,

$$\inf_{(\mu, \eta) \leq (\mu, \xi)} \rho_t^{M_t}(\eta) = \langle \mu, -\xi \rangle - \alpha_t^{M_t}(\mu), \quad \forall \mu \in (L^\infty)^*, \xi \in M_t.$$

And the following statements are equivalent:

1. $-\mu \in \partial \rho_t^{M_t}(\xi)$
2. $\rho_t^{M_t}(\xi) = \langle \mu, -\xi \rangle - \alpha_t^{M_t}(\mu)$
3. $\inf_{(\mu, \eta) \leq (\mu, \xi)} \rho_t^{M_t}(\eta) = \rho_t^{M_t}(\xi)$

Corollary 3. $\partial \rho_t^{M_t}(\xi) \cap (-\mathcal{P}) \neq \emptyset, \forall \xi \in M_t$.

Equilibrium and Optimality

An allocation ξ_1, \dots, ξ_n is **attainable** if $\xi_i \in M_i$ and $\sum_i \xi_i \leq X$.

An attainable allocation ξ_1, \dots, ξ_n is Pareto **optimal** if

$$\rho_i(\eta_i) \leq \rho_i(\xi_i) \quad \forall i \quad \Rightarrow \quad \rho_i(\eta_i) = \rho_i(\xi_i) \quad \forall i$$

for every attainable allocation η_1, \dots, η_n .

An attainable allocation ξ_1, \dots, ξ_n together with a pricing rule $\mu \in \mathcal{P}$ is an Arrow–Debreu **equilibrium** if

$$\langle \mu, \xi_i \rangle \leq \langle \mu, X_i \rangle \quad \text{and} \quad \rho_i(\xi_i) = \inf_{\langle \mu, \eta \rangle \leq \langle \mu, X_i \rangle, \eta \in M_i} \rho_i(\eta) \quad \forall i.$$

Characterization of Optimality

Let ξ_1, \dots, ξ_n be an attainable allocation, $\xi_0 := X - \sum_i \xi_i \geq 0$.

The following are equivalent:

1. ξ_1, \dots, ξ_n is optimal
2. $\square_i^{(M_i)} \rho_i(X) = \sum_{i=1}^n \rho_i(\xi_i)$
3. $\langle \mu, \xi_0 \rangle = 0$ (price clearing) and $-\mu \in \partial \rho_1^{M_1}(\xi_1) \cap \dots \cap \partial \rho_n^{M_n}(\xi_n)$ for some $\mu \in \mathcal{P}$.

Practical implementation: minimize 2. Or find allocations ξ_1, \dots, ξ_n such that $\bigcap_i \partial \rho_i^{M_i}(\xi_i)$ is not empty. Check attainability and $\langle \mu, \xi_0 \rangle = 0$.

Welfare Theorems

ξ_1, \dots, ξ_n, μ is an equilibrium if and only if

1. ξ_1, \dots, ξ_n is optimal and
2. $-\mu \in \partial \square_i^{(M_i)} \rho_i(X)$ (**optimal pricing rules**) and
3. $\langle \mu, \xi_i \rangle = \langle \mu, X_i \rangle$ for all i .

Moreover, if ξ_1, \dots, ξ_n is optimal and $-\mu \in \partial \square_i^{(M_i)} \rho_i(X)$ then

$$\xi_1 + \langle \mu, X_1 - \xi_1 \rangle, \dots, \xi_n + \langle \mu, X_n - \xi_n \rangle$$

is an equilibrium.

Approximate Equilibrium

Optimal allocations do not always exist! BUT:

Let $-\mu \in \partial \square_i^{(M_i)} \rho_i(X)$ ($\neq \emptyset, \subset (-\mathcal{P})$) and $\epsilon > 0$. Then there exists an attainable allocation ξ_1, \dots, ξ_n such that:

1. $\square_i^{(M_i)} \rho_i(X) \geq \sum_{i=1}^n \rho_i(\xi_i) - n\epsilon$ (“ ϵ -optimality”) and
2. $\langle \mu, -X_i \rangle - \alpha_i^{M_i}(\mu) \geq \rho_i(\xi_i) - \epsilon$ (“ ϵ -equilibrium”) and
3. $\langle \mu, X_i \rangle - \epsilon \leq \langle \mu, \xi_i \rangle \leq \langle \mu, X_i \rangle$ for all i .

Practical limitations: optimal pricing rules $\partial \square_i^{(M_i)} \rho_i(X)$ may be difficult to find

Special Case

Let $Z_0, \dots, Z_m \in L^\infty$ be linearly independent, with $Z_0 \equiv 1$ ("cash").

Assumptions:

- X_i and Z_0, \dots, Z_m linearly independent for all i
- $M_i = \left\{ X_i + \sum_{j=0}^m \lambda_j Z_j \mid \lambda_j \in \mathbb{R} \right\}$ finite-dimensional affine spaces
- $v_i(\lambda_0, \dots, \lambda_m) := \rho_i(X_i + \sum_{j=0}^m \lambda_j Z_j)$ differentiable in λ_j

Then

1. $v_i(\lambda_0, \dots, \lambda_m) = v_i(0, \lambda_1, \dots, \lambda_m) - \lambda_0 =: u_i(\lambda_1, \dots, \lambda_m) - \lambda_0$
2. $\nu \in \partial \rho_i^{M_i}(X_i + \sum_j \lambda_j Z_j) \Leftrightarrow \langle \nu, 1 \rangle = -1$ and $\langle \nu, Z_j \rangle = \partial_{\lambda_j} u_i(\lambda_1, \dots, \lambda_m) \forall j \geq 1$

→ Equilibrium pricing rule (if it exists) does not interfere with any a priori values of X_1, \dots, X_n !

The clearing condition

$$\sum_{i=1}^n \left(X_i + \sum_{j=0}^m \lambda_{ij} Z_j \right) \leq X \iff \sum_{j=0}^m \left(\sum_{i=1}^n \lambda_{ij} \right) Z_j \leq 0$$

Is always satisfied (“and vice versa”) for

$$\sum_{j=1}^m \left(\sum_{i=1}^n \lambda_{ij} \right) Z_j - \underbrace{\text{ess sup} \left(\sum_{j=1}^m \sum_{i=1}^n \lambda_{ij} Z_j \right)}_{\text{net cash}} Z_0$$

→ $(m \times n)$ -dimensional optimization problem

$$\square_i^{(M_i)} \rho_i(X) = \inf_{\lambda_y} \left(\sum_{j=1}^m \sum_{i=1}^n u_i(\lambda_{i1}, \dots, \lambda_{im}) + \text{ess sup} \left(\sum_{j=1}^m \sum_{i=1}^n \lambda_{ij} Z_j \right) \right)$$

Characterization of Optimum

Corollary. The optimum is attained:

$$\square_i^{(M_i)} \rho_i(X) = \sum_{j=1}^m \sum_{i=1}^n u_i(\lambda_{i1}^*, \dots, \lambda_{im}^*) + \text{ess sup} \left(\sum_{j=1}^m \sum_{i=1}^n \lambda_{ij}^* Z_j \right)$$

If and only if

1. $\partial_{\lambda_j} u_i(\lambda_{i1}^*, \dots, \lambda_{im}^*) = \partial_{\lambda_j} u_1(\lambda_{11}^*, \dots, \lambda_{1m}^*) =: \langle \mu, -Z_j \rangle \quad \forall i, j$ and
2. $\sum_{j=1}^m \sum_{i=1}^n \lambda_{ij}^* \langle \mu, Z_j \rangle = \text{ess sup} \left(\sum_{j=1}^m \sum_{i=1}^n \lambda_{ij}^* Z_j \right)$ (**price clearing**)

An equilibrium allocation is then obtained by

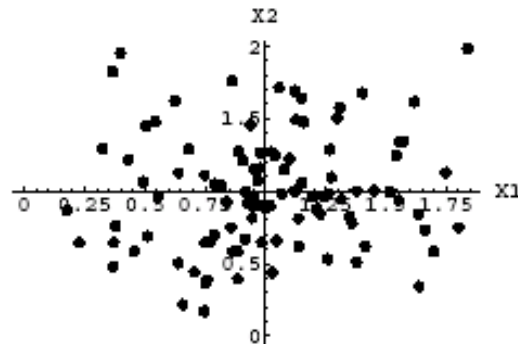
$$X_i + \sum_{j=1}^m \lambda_{ij}^* Z_j - \sum_{j=1}^m \lambda_{ij}^* \langle \mu, Z_j \rangle$$

and $\langle \mu, Z_j \rangle$ are the **fair prices** for the risk and capital transfers Z_j

Concrete Example

$m = n = 2$ insurance units. $Z_i = (C_i - m_i)^+$ surplus participation.

$(C_1, C_2) = \text{random (100pt) sample of } N \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \sigma^2 \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix} \right)$



Assumption: available ($= C_i(0) = 1$) = required capital:

$$0 = \text{ES}(C_i) = \sigma \text{ES}(N(0, 1)) - 1 \Leftrightarrow \sigma = \frac{1}{\text{ES}(N(0, 1))} = 0.375204,$$

for confidence level $\alpha = 0.01$.

$$ES(C_1) = -0.175, ES(C_2) = -0.171, ES(C_1 + C_2) = -0.850.$$

Minimum capital $m_i := 0.4 \times (1 + ES(C_i))$ (=SST risk margin)

$$m_1 = 0.330, m_2 = 0.332.$$

$$\text{Optimal allocation } (\lambda_{11}^*, \lambda_{12}^*) = (-0.58, 0.75), (\lambda_{21}^*, \lambda_{22}^*) = (0.50, -0.81).$$

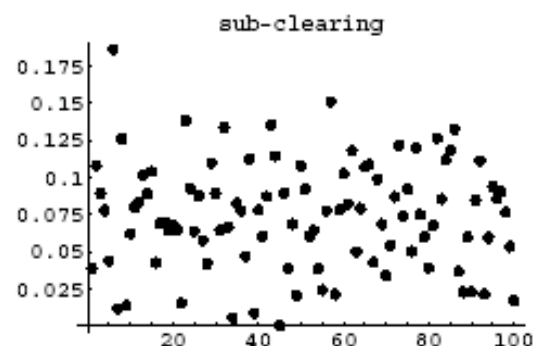
net cash flow = 0.01,

$$\text{fair values: } \langle \mu, (C_1 - m_1)^+ \rangle = 0.04, \langle \mu, (C_2 - m_2)^+ \rangle = 0.15.$$

$$ES(C_1 - 0.58((C_1 - m_1)^+ - 0.04) + 0.75((C_2 - m_2)^+ - 0.15)) = -0.37$$

$$ES(C_2 + 0.50((C_1 - m_1)^+ - 0.04) - 0.81((C_2 - m_2)^+ - 0.15)) = -0.48$$

sum = -0.85 \Rightarrow full diversification effect (In general not)



Summary

We propose an equilibrium approach to value and measure contingent capital and risk transfers between insurance units in order to account for diversification benefits.

The available capital does not change (equilibrium) while the maximal reduction of required capital is achieved (optimum).

The first example with surplus participations $Z_i = (C_i - m_i)^+$ can be generalized. Results are available for arbitrary convex and cash invariant sets $M_i \subset L^\infty$ of capital and risk transfers (D.F. and M.Kupper: "Equilibrium and optimality for monetary utility functions under constraints")

Also the framework L^∞ can be extended (L^∞ does not contain normal distributed rvs.). This is work in progress (joint with Michael Kupper and Gregor Svindland, University of Munich).