Optimal Investment and Ruin Probability for Insurers

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ASTIN-AFIR Conference, september 2005

joint work with

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The market is composed by a bank account S^0 and a risky asset S_t, whose dynamics satisfy

$$S_t^0 = S_0^0 e^{\eta t}, \quad S_0^0 = 1,$$
$$dS_t = S_t(adt + \sigma dB_t), \quad S_0 = x,$$

where $\eta \geq 0$, a and $\sigma > 0$ are constants.

The risk process is based in the classical Lundberg model, using a compound Poisson process for the claims. Given the initial surplus z and the constant premium rate c, the risk process is defined as

$$R_t = z + ct - \sum_{i=1}^{N_t} Y_i,$$

where Y_i represents the claim amounts.

Several authors have been studied this problem in different settings, BETWEEN OTHERS

Paulsen and Gjessing [15], Paulsen[16], Kalashnikov and Norberg [13], Frovola, Pergamienschhikov, Kabanov [7]. In al these cases even if the claim size has exponential moments the ruin probability decrease only with some negative powers of the initial reserve.

Hipp and Plumb [11] minimize the the ruin probability. If the claims are exponential the ruin probability decreases exponentially.

Gaier, Grandits, Schachermayer [9] in the exponential case claims investigate whether are constants $\hat{r}$ and c such that the ruin probability $\Psi(x)$ satisfies

$$\Psi(x) \leq Ce^{-\hat{r}x}$$

Browne [2] investigate the problem where the risk process is also Brownian (not a compound Poisson) and obtain a minimal bound for the ruin by an exponential function.

(i) Finite horizon problem $T > 0$ fixed.

(ii) At each time $t \in [0, T]$, the insurer divides his wealth X_t between the risky and the riskless assets.

(iii) If a claim is received at time t, it is paid immediately.

(iv) Let π_t be the amount of wealth invested in the risky asset at time t.

(v) $X_t - \pi_t$ is invested in the bank account.

If at time $s < T$ the surplus of the company is x, the wealth process satisfies the dynamics

$$
\begin{aligned}
X_t^{s,x,\pi} &= x + c(t-s) - \sum_{j=N_s+1}^{N_t} Y_j + \int_s^t (a-\eta)\pi_r dr \\
&\quad + \int_s^t \eta X_r^{s,x,\pi} dr + \int_s^t \sigma \pi_r dB_r,
\end{aligned}
\tag{0.1}
$$

with the convention that $\sum_{j=1}^{0} = 0$. When $s = 0$ we write for simplicity X_t^{π}.

Definition 0.1. *We say that $\pi = \{\pi_t\}$ is an admissible strategy if it is a $\mathcal{F}_t$-progressively measurable process such that*

$$\mathbf{P}[|\pi_t| \leq A,\ 0 \leq t \leq T] = 1,$$

where the constant A may depend of the strategy, and the equation (0.3) has a unique solution. We denote the set of admissible strategies as $\mathcal{A}$.

A utility function $U : I\!R \to I\!R \in C^2$ strictly increasing and strictly concave.

THE OPTIMIZATION PROBLEM:

Maximize the expected utility of terminal wealth at time T, i.e. we are interested in the following value function

$$W(s,x) = \sup_{\pi \in \mathcal{A}} \mathbf{E}[U(X_T^{s,x,\pi})]. \tag{0.2}$$

We say that an admissible strategy π^* is optimal if

$$W(s,x) = \mathbf{E}[U(X_T^{s,x,\pi^*})]$$

.

$$\begin{aligned} X_t^{s,x,\pi} &= x + c(t-s) - \sum_{j=N_s+1}^{N_t} Y_j + \int_s^t (a-\eta)\pi_r dr \\ &\quad + \int_s^t \eta X_r^{s,x,\pi} dr + \int_s^t \sigma \pi_r dB_r, \end{aligned} \tag{0.3}$$

We prove:

1. A Verification Theorem associated to HJB-equation.

2. If $U(x) = -e^{-\gamma x}$

   (i) We obtain an explicit solution.

   (ii) Estimate Ruin Probability (Bounds of exponential type).

   (iii) When the claims are exponential we compare the results with that of Gaier, Grandits, Schachermayer [9].

   (iv) Numerical Examples.

Martingale Techniques

1 Verification Theorem

The Hamilton-Jacobi-Bellman (HJB) equation associated with the optimal stochastic control problem is given by

$$\begin{aligned} 0 &= \frac{\partial V}{\partial t}(t,x) + \max_{\pi \in I\!R} \left\{ \frac{\sigma^2}{2}\pi^2 \frac{\partial^2 V}{\partial x^2}(t,x)x + (\pi(a-\eta)+\eta x)\frac{\partial V}{\partial x}(t,x) \right\} \\ &\quad + c\frac{\partial V}{\partial x}(t,x) + \lambda \int_{I\!R} [V(t,x-y) - V(t,x)]\nu(dy), \end{aligned} \tag{1.4}$$

with terminal condition $V(T,x) = U(x)$.

Theorem 1.1. *Assume that there exists a classical solution $V(t,x) \in C^{1,2}([0,T] \times I\!R)$ to the HJB equation (1.4) with boundary conditions $V(T,x) = U(x)$. Assume also that for each $\pi \in \mathcal{A}$*

$$\int_0^T \int_{I\!R} \mathbf{E}|V(s, X^\pi_{s^-} - y) - V(s, X^\pi_{s^-})|\nu(dy)ds < \infty, \tag{1.5}$$

$$\int_0^T \mathbf{E}[\pi_{s^-} \frac{\partial V}{\partial x}(s, X^\pi_{s^-})]^2 ds < \infty. \tag{1.6}$$

Then, for each

$s \in [0,T],\ x \in I\!R,$

$$V(s,x) \geq W(s,x).$$

If, in addition, there exists a measurable function $\pi^* : [0,T] \times \mathbb{R} \to \mathbb{R}$ *such that*

$$\pi^*(t,x) \in \operatorname{argmax}_{\pi \in \mathbb{R}} \left\{ \frac{\sigma^2}{2}\pi^2 \frac{\partial^2 V}{\partial x^2}(t,x) + (\pi(a-\eta)+\eta x)\frac{\partial V}{\partial x}(t,x) \right\},$$

then π^* *defines an optimal investment strategy in feedback form if (0.3) admits a unique solution* $X_t^{\pi^*}$ *and*

$$V(s,x) = W(s,x) = \mathbf{E}U[X_T^{s,x,\pi^*}].$$

2 Explicit solutions for exponential utility function

The utility function is of exponential type, i.e.

$$U(x) = -e^{-\gamma x}.$$

Theorem 2.1. *Assume that*

$$\int_{I\!R} \exp\{2\gamma y e^{\eta T}\}\nu(dy) < \infty.$$

Then, the value function defined in (0.2) has the form

$$\begin{aligned} W(t,x) \;&=\; -\exp\left\{ -\frac{1}{2}\frac{(a-\eta)^2}{\sigma^2}(T-t) + \frac{c\gamma}{\eta}[1-e^{\eta(T-t)}] \right. \\ &+\; \left. \lambda \int_t^T \beta_s ds \right\} \cdot \exp\{-\gamma x e^{\eta(T-t)}\}, \end{aligned} \tag{2.7}$$

and

$$\pi^*(t,x) = \frac{a-\eta}{\gamma\sigma^2} e^{-\eta(T-t)}$$

is an optimal strategy.

In particular, when $\eta = 0$ we have that

$$W(t,x) = -\exp\left\{ -\frac{1}{2}\frac{a}{\sigma^2}(T-t) + c\gamma(T-t) + \lambda\beta(T-t) \right\} e^{-\gamma x} \tag{2.8}$$

and

$$\pi^*(t,x) = \frac{a}{\gamma\sigma^2}.$$

3 Ruin Probability

The wealth process associated with the optimal investment strategy π^* is given by

$$X_t^* = z + ct - \sum_{i=1}^{N_t} Y_j + \int_0^t \frac{(a-\eta)^2}{\gamma\sigma^2} e^{-\eta(T-r)} dr + \int_0^t \eta X_r^* dr + \int_0^t \frac{(a-\eta)}{\gamma\sigma} e^{-\eta(T-r)} dB_r, \quad \text{for } \eta > 0 \tag{3.9}$$

and

$$X_t^* = z + ct - \sum_{i=1}^{N_t} Y_i + \int_0^t \frac{a^2}{\gamma\sigma^2} dr + \int_0^t \frac{a}{\gamma\sigma} dB_r, \quad \text{for } \eta = 0. \tag{3.10}$$

Theorem 3.1. *Let us denote* $\theta = E[Y_1]$ *and assume that*

(a) The law of the random variables $Y_i,\ i \geq 1$ *admits a (finite) Laplace transform* $L(r)$ *for* $0 < r < K \leq \infty$,

(b) If $K < \infty$, *then* $\lim_{r \to K} L(r) = \infty$.

(c) The following safety loading conditions are satisfied

$$[e^{-\eta T}(c + \frac{(a-\eta)^2}{\gamma\sigma^2})] - \lambda\theta > 0, \qquad \text{if } \eta > 0,$$

and

$$c + \frac{a^2}{\gamma\sigma^2} - \lambda\theta > 0, \qquad \text{if } \eta > 0.$$

Then, the ruin probability satisfies

$$\mathbf{P}[\sup_{0 \leq s \leq T} -X_s^* \geq 0] \leq e^{-\delta^* z},$$

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where **(i)** *If $\eta > 0$, then, for each $\gamma > 0$, δ^* is the positive root of the equation:*

$$h(\delta,\gamma) = -\delta\{e^{-\eta T}(c+\frac{(a-\eta)^2}{\gamma\sigma^2})\}+\frac{\delta^2}{2}\frac{(a-\eta)^2}{\gamma^2\sigma^2}e^{-2\eta T}+\lambda(L(\delta)-1) = 0. \tag{3.11}$$

(ii) *If $\eta = 0$ then, for each $\gamma > 0$, δ^* is the positive root of the equation:*

$$h^0(\delta,\gamma) = -\delta\{c+\frac{a^2}{\gamma\sigma^2}\}+\frac{\delta^2}{2}\frac{a^2}{\gamma^2\sigma^2}+\lambda(L(\delta)-1) = 0. \tag{3.12}$$

In addition, let δ^1 be the root of the classic Cramér–Lundberg equation

$$h^1(\delta) = -\delta c + +\lambda(L(\delta)-1) = 0.$$

If $\eta = 0$ and $\frac{\delta^1}{2} < \gamma$, then

$$\delta^1 < \delta^*.$$

In the proof we shall use a simplified version of Lemma 3.1 in [3], which we state now:

Lemma 3.1. *Let*

$$L_t = z + \int_0^t b_s ds + \int_0^t d_s dB_s - \sum_{i=1}^{N_t} Y_i,$$

where everything is as stated in this paper, $(b_s)_{s\geq 0}$ *is an adapted integrable process and* $(d_s)_{s\geq 0}$ *is predictable with* $\mathbf{E}[\int_0^t d_s^2 ds] < \infty$*. Assume*

(i) The law of the random variables $(Y_i)_{i\geq 1}$ *has finite Laplace transform* $L(r)$ *for* $0 < r < K \leq \infty$*.*

(ii) There exist $\delta \in (0, K)$ *and a constant* $h_T(\delta) \geq 0$ *such that for all* $s \in [0, T]$*,*

$$\delta \int_0^s b_u du + \frac{\delta^2}{2} \int_0^s d_u^2 du + \lambda s(L(\delta) - 1) \leq h_T(\delta).$$

Then, for $m \geq z$

$$\mathbf{P}[\sup_{0<s\leq T} L_s > m] \leq \exp\{\delta(z - m) + h_T(\delta)\}.$$

The proof of this Lemma is based on maximal inequalities for martingales.

For estimate

$$\mathbf{P}[\sup_{0\le s\le t} -X_s^* \ge 0]$$

We can not apply directly Lemma 3.1 to $-X^*$, observe that

$$\mathbf{P}[\sup_{0\le s\le t} -X_s^* \ge 0] = \mathbf{P}[\sup_{0\le s\le t} -Z_s \ge 0].$$

where $Z_t = X_t^* e^{-\eta t}$

$$\begin{aligned} Z_t &= z + \int_0^t e^{-\eta r} c dr - \sum_{j=1}^{N_t} e^{-\eta \tau_j} Y_j + \int_0^t \frac{(a-\eta)^2}{\gamma\sigma^2} e^{-\eta T} dr \\ &+ \int_0^t \frac{a-\eta}{\gamma\sigma} e^{-\eta T} dB_r. \end{aligned}$$

Let $-Z^1$ be as follows

$$-Z_t^1 = -z - cte^{-\eta T} + \sum_{i=1}^{N_t} Y_j - t\frac{(a-\eta)^2}{\gamma\sigma^2} e^{-\eta T} - \int_0^t \frac{a-\eta}{\gamma\sigma} e^{-\eta T} dB_r,$$

Then it is clear that

$$-Z_s^1 \ge -Z_s = -X_s^* e^{-\eta s}.$$

and

$$\mathbf{P}[\sup_{0\le s\le t} -Z_s \ge 0] = \mathbf{P}[\sup_{0\le s\le t} -Z_s^1 \ge 0]$$

We apply the Lemma to $-Z_s^1$ with $h_T(\delta) = Th(\delta)$. The existence of the positive root is guaranteed by the safety loading condition.

Proposition 3.1. *We assume that the random variables* $Y_i,\ i \geq 1$ *are exponential with mean* θ *and*

$$0 < \gamma < \frac{e^{-\eta T}}{\theta}. \tag{3.13}$$

Then

$$\begin{aligned} W(t,x) \;=\; & -\exp\,\left\{ -\frac{1}{2}\frac{a-\eta}{\sigma^2}(T-t) + \frac{c\gamma}{\eta}[1-e^{\eta(T-t)}] \right. \\ & \left. -\frac{\lambda}{\eta}\log\,\left(\frac{1-\gamma\theta}{1-\gamma\theta e^{\eta(T-t)}}\right)\right\} \\ & \cdot\exp\,\{-\gamma x e^{\eta(T-t)}\}. \end{aligned}$$

In particular, if $\eta = 0$*,*

$$0 < \gamma < \frac{1}{\theta},$$

and

$$W(t,x) \;=\; -\exp\left\{-\frac{1}{2}\frac{a}{\sigma^2}(T-t) + c\gamma(T-t) - \lambda\frac{\gamma\theta}{1-\gamma\theta}(T-t)\right\} e^{-\gamma x}.$$

Compare with Gaier, Grandits, Schahermayer [9]:

For each γ we obtain a positive root δ_γ of h, using the implicit theorem, it can be shown that δ_γ is maximum when $\delta_\gamma = \gamma$. Gaier, Grandits and Schachermayer, see [9], obtain the strategy π_t that guarantees that the ruin probability is optimal: $\pi_t = \frac{a}{\widehat{r}\sigma}$, where $\widehat{r}$ is the solution of the following equation

$$\lambda(L[r] - 1) = \frac{a^2}{2\sigma^2} + cr. \tag{3.14}$$

It can be easily shown that $\delta_{\widehat{r}} = \widehat{r}$. So if we chose as $\gamma = \widehat{r}$ we get the strategy that is optimal for the exponential utility function and also that has the less ruin probability.

In the exponential case, for $\eta = 0$, $h(\delta, \gamma)$ becomes

$$h(\delta,\gamma) = \frac{a^2\theta}{2\gamma^2\sigma^2}\delta^2 - ((c+\frac{a^2}{\gamma\sigma^2})\theta + \frac{a^2}{2\gamma^2\sigma^2})\delta + (c+\frac{a^2}{\gamma^2\sigma^2} - \theta\lambda). \quad (3.15)$$

For each $\gamma \in (0, 1/\theta)$ we obtain a positive root δ_γ of h of the form

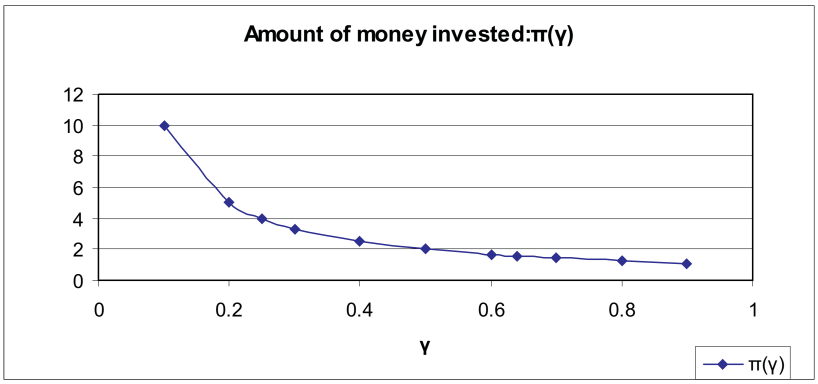
$$\delta_\gamma = (\frac{1}{2\theta} + \frac{c+Ka}{K^2\sigma^2}) + \sqrt{(\frac{c+Ka}{K^2\sigma^2})^2 + \frac{1}{4\theta^2} - \frac{c+Ka-\lambda\theta}{\theta K^2\sigma^2}},$$

In the exponential case equation(3.13) becomes

$$f(r) = c\theta r^2 + (\frac{a^2\theta}{2\sigma^2} + \lambda\theta - c)r - \frac{a^2}{2\sigma^2} \quad (3.16)$$

whose solution satisfies $\delta_{\widehat{r}} = \widehat{r}$.

In order to illustrate the behavior of the ruin probability for infinite horizon when the optimal strategy of investment $\pi_t = \frac{a}{\gamma\sigma}$ is applied, we present some numerical results for the exponential case, with data used by Hipp and Plum, see [11], for different values of $\gamma \in (0, 1)$. The parameters have the following values: $a = \sigma = \theta = \lambda = 1$, $c = 2$, and $\eta = 0$.



Graph 1

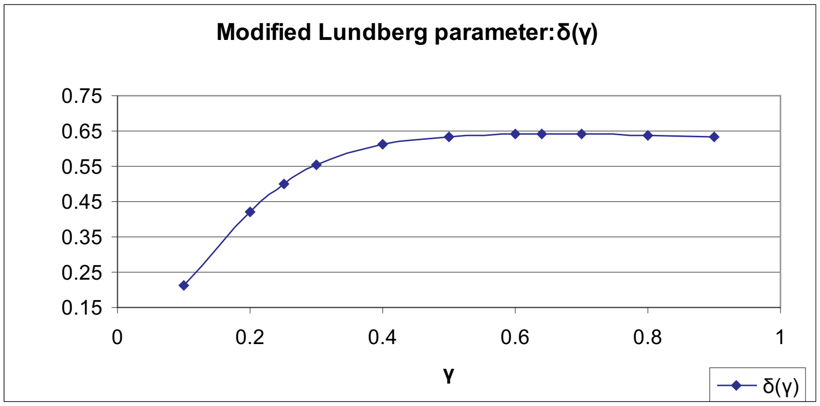
Graph 1 shows how the root $\delta(\gamma)$ of $h(\delta)$ varies for different values of γ. For our data the root of (3.16) is $\widehat{r} = 0.640388$ and the Lundberg parameter for the classical case is 0.5. As it was expected the maximum value of δ is obtained at 0.640388 and for $\gamma \in [.25, .9]$ the root is larger that 0.5.

Graph 2 shows how K decreases as γ increases. This has the advantage that the ruin probability is almost the same as in the optimal case without needing a large sum of money to invest in the risk asset.

Let

$$S_t = \sum_{i=1}^{N_t} Y_i - ct - \int_0^t \frac{a^2}{\gamma\sigma^2} dr - \int_0^t \frac{a}{\gamma\sigma} dB_r, \qquad (3.17)$$

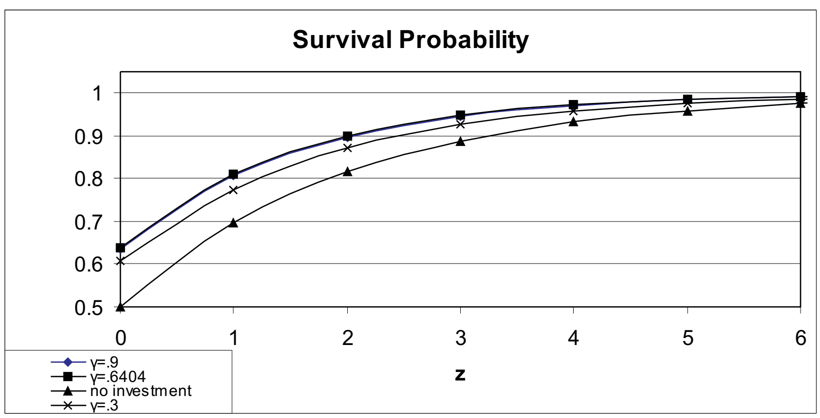
denote the surplus; observe that $S_t = z - X_t$. Let $\tau(z) = \inf_{0 \le t < \infty}\{t >$



Graph 2

$0|S_\tau > z\}$, we are interested on estimating

$$P[\tau(z) < \infty] = E(1_{\tau(z)<\infty}).$$



Graph 3

We use a Monte-Carlo method with importance sampling to estimate the ruin probability. Importance sampling is applied to overcome several difficulties:

1. Given that the horizon is infinite, a stopping time T must be defined for the simulation which introduce an error difficult to estimate.

2. When the probability is small, less than 10^{-3}, which is the case for our data when $z > 7$, we are simulating a rare event. In order to do it well we have to generate an impractical number of paths.

3. When a crude Monte-Carlo method is used the relative error increases as z becomes large.

These problems can be handle if we change the probability measure to one that increases the probability of occurrence of $\{\tau(z) < \infty\}$. Asmussen [1] propose to use an exponential change of measure. Let P^* be the equivalent probability of P given by the Radon-Nykodin derivative

$$\frac{dP^*}{dP} = e^{\delta S_{\tau(z)} - \tau(z) h(\delta)},$$

where $h(\delta)$ is given by 3.15. If we chose as δ the root δ^* of h we have that the calculation of the ruin probability reduces to

$$E(\mathbb{1}_{[\tau(z)<\infty]}) = E^*(e^{-\delta^* S_{\tau(z)}} \mathbb{1}_{[\tau(z)<\infty]}).$$

As $P^*(\tau(z) < \infty) = 1$, we don't have to worry about the stopping time. We also obtain a considerable reduction of the variance which implies a lesser number of paths for Monte-Carlo. When $\delta = \delta^*$ the estimation is optimal, in an asymptotic sense, for variance reduction; the variance is bounded by $e^{-2\delta^* z}$ which tends to zero when z goes to infinity.

Graph 3 compares the probability of survival, equal to 1 minus the ruin probability, for values of $z \in [0, 6]$ for $\gamma = .9$, $\gamma = 0.640388$ and when there is no investment. As it can be seen the ruin probability is almost the same for the first two cases even when we need to invest for $\gamma = .9$ a smaller amount of money.

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