Optimal Investment and Ruin Probability for Insurers

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In this paper we study an optimal investment problem of an insurer when the company has the opportunity to invest in a risky asset using stochastic control techniques. A closed form solution is given when the risk preferences are exponential as well as an estimate of the ruin probability when the optimal strategy is used.

Let $(\Omega, \mathcal{F}, P)$ be the underlying probability space, where a Brownian motion B_t, a Poisson process N_t with constant intensity λ, and a sequence of independent random variables Y_i with identical distribution ν are defined. It is assumed that $(B_t)_{t\geq 0}$, $(N_t)_{t\geq 0}$ and $(Y_i)_{i\geq 1}$ are independent, and for each $t > 0$ the filtration $\mathcal{F}_t$ containing the information up to time t is defined by

$$\mathcal{F}_t = \sigma\{B_s,\ N_s, Y_j 1\!\!1_{[j\leq N_s]},\ s \leq t,\ j \geq 1\}.$$

The market where the insurer can invest is composed by a bank account S^0 and a risky asset S_t, whose dynamics satisfy

$$S_t^0 = S_0^0 e^{\eta t}, \quad S_0^0 = 1,$$
$$dS_t = S_t(a dt + \sigma dB_t), \quad S_0 = x,$$

where η, a and σ are constants.

On the other hand, the risk process is based in the classical Lundberg model, using a compound Poisson process for the claims. Given the initial surplus z and the constant premium rate c, the risk process is defined as

$$R_t = z + ct - \sum_{i=1}^{N_t} Y_i,$$

where Y_i represents the claim amounts.

In this paper we are interested in the finite horizon problem. Then, at each time $t \in [0, T]$, with $T > 0$ fixed, the insurer divides his wealth X_t between the risky and the riskless assets and, if a claim is received at that time, it is paid immediately. Let π_t be the amount of wealth invested in the risky asset at time t, which takes values in $I\!\!R$, while the rest of his wealth $X_t - \pi_t$ is invested in the bank account. Then, if at time $s < T$ the surplus of the company is x, the wealth process satisfies the dynamics

$$X_t^{s,x,\pi} \quad = \quad x + c(t-s) - \sum_{j=N_s+1}^{N_t} Y_j + \int_s^t (a-\eta)\pi_r dr$$

$$+\int_s^t \eta X_r^{s,x,\pi} dr + \int_s^t \sigma \pi_r dB_r, \tag{0.1}$$

with the convention that $\sum_{j=1}^{0} = 0$. When $s = 0$ we write for simplicity X_t^π.

We say that $\pi = \{\pi_t\}$ is an admissible strategy if it is a $\mathcal{F}_t$-progressively measurable process such that

$$\mathbf{P}[|\pi_t| \leq A,\ 0 \leq t \leq T] = 1,$$

where the constant A may depend of the strategy, and the equation (0.1) has a unique solution. We denote the set of admissible strategies as $\mathcal{A}$.

A utility function $U : I\!R \to I\!R$ is defined as a twice continuously differentiable function, with the property that $U(\cdot)$ is strictly increasing and strictly concave. Now, we consider the following optimization problem, consisting in maximizing the expected utility of terminal wealth at time T, i.e. we are interested in the following value function

$$W(s,x) = \max_{\pi \in \mathcal{A}} \mathbf{E}[U(X_T^{s,x,\pi})]. \tag{0.2}$$

We say that an admissible strategy π^* is optimal if $W(s,x) = \mathbf{E}[U(X_T^{s,x,\pi^*})]$.

The main results of this paper can be summarized as follows: when the risk preferences of the insurer are exponential the optimal investment problem described above can be solved explicitly and, for the optimal investment strategy, it is possible to obtain an estimate of the associated ruin probability.