BSDE with enlarged filtration

Option hedging of an insider trader

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- Market Model
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- Results
- Influent investor and FBSDE

Problem Setting

- Insider Trading = Additional Information on the market
- Most often treated point of view : wealth optimization with asymmetrical information (ex: Grorud, Pontier, Amendinger, Becherer, Schweizer, Imkeller, Föllmer among others)
- Different point of view : hedging problem
- In the present model, prices are driven by both a Brownian motion, and Jump processes (Poisson point processes)
- Two different cases studied : *small investors* ($\longrightarrow$BSDE), and also *large investors* (influent insider trader) ($\longrightarrow$FBSDE)

- *Basic Example* :
  - $L = S_T$ the insider knows final stock price S_T.
  - He wants to hedge a digital option $\mathbf{1}_{S_T \leq K}$
  - He has 2 possible investments: invest on risky asset if $S_T \leq K$ and do nothing otherwise
  - He has an obvious different strategy from the non insider trader, and even an arbitrage opportunity.
- *Natural questions arise* :
  - Is the hedging strategy identical to the non informed trader?
  - Is it unique? Are there more hedging strategies?
  - Market completeness/incompleteness? Arbitrage opportunities?

Financial Problem

- Option Hedging, represented by a payoff ξ to reach at maturity T
- Transcription: portfolio duplication, look for initial wealth X_0 and the portfolio π such that final wealth $X_T = \xi$.
- One agent has an information L at time 0 concerning time $T' > T$.
- Will he have different investment from the uninformed agent?
- How does the market differ from a market with symmetrical information?

Main Tools of the Model

- Introducing an insider in a well-known market model
- Comparing insider and non informed strategies
- To model financial strategies of the agents : BSDE
- To model the additional information of the insider : Enlargement of Filtration

Market model

- Prices driven by both W Brownian motion and μ Poisson measure, $(\mathcal{F}_t)_{t\in[0,T]}$ natural filtration of $(W, \tilde{N})$,
- k risky assets, 1 riskless asset,
- no Arbitrage opportunities (AOA)

An insider in the Market

- **Strong initial information**: insider trader has at time 0 the information L, unknown from the common agent.
- $L \in \mathcal{F}_{T'}$, with $T' > T$: it will be public at time T'.
- There are 2 different spaces: the non insider space, and the insider space, with L added.
- New **enlarged filtration**: the smallest right-continuous filtration that contains initial filtration and information L:

$$\mathcal{Y}_t = \bigcap_{s>t} (\mathcal{F}_t \vee \sigma(L))$$

Initially enlarged filtration : Usual hypotheses

- Adding L to the initial filtration :

$$\mathcal{Y}_t = \bigcap_{s>t} (\mathcal{F}_t \vee \sigma(L))$$

- Studies on the subject (Jacod 1989, Jeulin 1985, Grorud-Pontier 1998)
  - **Hypothesis$(\mathbf{H_3})$** : There exists a probability Q equivalent to P under which $\mathcal{F}_t$ and $\sigma(L)$ are independent, $\forall t < T'$.

- Fundamental Properties:
  - Under new probability Q, W_t is a $(\mathcal{Y}, Q)$-Brownian motion, and $\tilde{N}_t$ a $(\mathcal{Y}, Q)$-compensated Poisson process.
  - Martingale Representation Theorem under $(\mathcal{Y}, Q)$.

Translating the problem to a BSDE (1)

- Hedging problem, payoff ξ to be reached at maturity $T < T'$.
- Wealth equation, standard self-financing hypothesis:

$$dX_t = X_t r_t dt + (\pi_t, b_t - r_t \mathbf{1})dt + (\pi_t, \sigma_t dW_t) + \int_E (\pi_{t-}, \phi(t,e))\mu(dt,de)$$

- And integrating from t to T, it follows :

$$X_t = X_T - \int_t^T \underbrace{[(X_s r_s - c_s) + (\pi_s, b_s - r_s \mathbf{1})]}_{-f(s,X_s,Z_s,U_s)} ds - \int_t^T (\underbrace{\sigma_s^* \pi_s}_{Z_s}, dW_s)$$
$$- \int_t^T \int_E \underbrace{(\pi_{s-}, \phi(s,e))}_{U_s(e)} \tilde{\mu}(ds,de)$$

- Solving the hedging problem means finding (X,Z,U) solution of the BSDE:

$$X_t = \xi + \int_t^T f(s,X_s,Z_s,U_s)ds - \int_t^T (Z_s, dW_s) - \int_t^T \int_E U_s(e)\tilde{\mu}(ds,de)$$

Results under H_3

$$X_t = \xi + \int_t^T f(s, X_s, Z_s, U_s)ds - \int_t^T (Z_s, dW_s) - \int_t^T \int_E U_s(e)\tilde{\mu}(ds, de)$$

- From Barles, Buckdahn and Pardoux (1997) and Tang and Li (1994), if f is globally Lipschitz in x,z and u, then there exists a unique triplet (X, Z, U) solution of the BSDE.
- The existence and uniqueness theorem can be adapted in the enlarged space.
- **Existence and Uniqueness Theorem** Our BSDE in the enlarged space has a unique solution (X', Z', U').
  - Thanks to Jacod and Shiryaev, we prove a martingale representation theorem under the enlarged filtration (Using independence of $\mathcal{F}$ and $\sigma(L)$ under Q to state a martingale representation property for (W, N) IIP on $(\mathcal{Y}, Q)$).
  - Constructing a strict contraction to obtain a unique solution.

Viability and completeness of the insider market : Brownian case

- If σ invertible, direct consequence of the existence and uniqueness result : the insider trader has a unique admissible strategy
- Comparison of the strategies : the hedging strategy is the same for both agents.
- **Complete** non insider market
  - Insider market is viable : information L does not create any arbitrage opportunities.
  - Insider market may have several risk-neutral probabilities (as in Grorud 1998), but all prices computed under different risk-neutral probabilities are the same.

FBSDE for a large investor

- **Limit** : In the previous model, the insider is a small investor, whose investment strategy does not influence asset prices.

  Not always realistic. $\longrightarrow$ Large/Influent investor hypothesis.

  $\longrightarrow$ FBSDE to solve in the enlarged space.

$$\left\{ \begin{array}{l} P_t = P_0 + \int_0^t b(s, P_s, X_s, Z_s)ds + \int_0^t < \sigma(s, P_s, X_s, Z_s), dW_s > \\ X_t = X_T - \int_t^T f(s, P_s, X_s, Z_s)ds - \int_t^T < Z_s, dW_s > \end{array} \right. \qquad (1)$$

- Influence hypothesis : the informed investor may influence asset prices.
  - It is a large investor : his wealth X may influence prices
  - and he is influent : his portfolio π influences prices

Existence and Uniqueness of solution

- Under Lipschitz, linear growth and integrability conditions on b, σ, f
- 3 cases where Pardoux and Tang obtained results :
  - *Weak influence* : b and σ weakly depend on X and Z
  - The agent wants to hedge a finite value a.s. : g constant
  - The portfolio does not influence volatility of prices : σ indépendent of Z.
- Our Result under $(\mathbf{H_3})$, in complete market
  - Using our theorem on BSDE under enlarged filtration, we state that the enlarged FBSDE has a unique solution, under the same hypotheses as Pardoux and Tang.
  - The influent agent, in one of the 3 influence cases, has a unique admissible strategy.
  - Complete insider market, as previously.

Incomplete market for the non informed agent

- Consider a non informed agent investing on this market. For a given contingent claim, his wealth and portfolio will satisfy a BSDE on $(\Omega, \mathcal{F}^P, P)$.
  There is no necessarily exact hedging solution. And if there is a solution, it will be a priori different from the insider one.
- The non informed agent will invest on a market with an incomplete information : his information is represented by the filtration generated by asset prices. New filtration to deal with.
- $\longrightarrow$ Hedging under Incomplete information (Föllmer and Schweizer 91)

Conclusion

- Incomplete market from a non insider point of view. Less information. Incomplete information. Different strategy $\mathcal{F}^P$-adapted.
  - Study of the strategy of the non informed agent.
  - Quantifying the "loss" of the non informed agent, due to the lack of information.
- Model with jumps (incomplete markets).

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