# Backward Stochastic Differential Equations with Enlarged Filtration

*Option hedging of an insider trader in a financial market with Jumps*

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**Abstract**

Insider trading consists in having an additional information, unknown from the common investor, and using it on the financial market. Mathematical modeling can study such behaviors, by modeling this additional information within the market, and comparing the investment strategies of an insider trader and a non informed investor. Research on this subject has already been carried out by A. Grorud and M. Pontier since 1996 (see [8], [9], [10] et [12]), studying the problem in a wealth optimization point of view. This work focuses more on option hedging problems. We have chosen to study wealth equations as backward stochastic differential equations (BSDE), and we use Jeulin's method of enlargement of filtration (see [6]) to model the information of our insider trader. We will try to compare the strategies of an insider trader and a non insider one. Different models are studied: at first prices are driven only by a Brownian motion, and in a second part, we add jump processes (Poisson point processes) to the model.

**Keywords:** Enlargement of filtration, BSDE, option hedging, insider trading, asymmetric information, martingale representation.

# 1 Mathematical Model

Let $W$ be a standard $d$-dimensional Brownian motion, and $(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ a filtered probability space, with $\Omega = C([0,T];\mathbf{R}^d)$ and $(\mathcal{F}_t)_{t\in[0,T]}$ the natural filtration of Brownian motion $W_t$. We consider a financial market with $k$ risky assets, whose prices are driven by:

$$S_t^i = S_0^i + \int_0^t S_s^i b_s^i ds + \int_0^t S_s^i(\sigma_s^i, dW_s)\ ,\ 0 \leq t \leq T, \tag{1}$$

and a bond (or riskless asset) evolving as: $S_t^0 = 1 + \int_0^t S_s^0 r_s ds$. Parameters $b, \sigma, r$ are supposed to be bounded on $[0,T]$, $\mathcal{F}$-adapted, and take values respectively on $\mathbf{R}^d, \mathbf{R}^{d\times k}, \mathbf{R}$. Matrix $\sigma_t$ is invertible $dt \otimes dP$-a.s. This is the usual conditions to have a complete market. A financial agent has a positive $\mathcal{F}_0$-measurable initial wealth $X_0$ at time $t = 0$ ($X_0$ constant a.s. as $\mathcal{F}_0$ is trivial). His consumption $c$ is a nonnegative $\mathcal{Y}$-adapted process verifying $\int_0^T c_s ds < \infty$,$P$-a.s. He gets $\theta^i$ parts of $i^{th}$ asset. His wealth at time $t$ is $X_t = \sum_{i=0}^k \theta_t^i S_t^i$. We consider the standard self-financing hypothesis:

$$dX_t = \sum_{i=0}^k \theta_t^i dS_t^i - c_t dt \tag{2}$$

It means that the consumption is only financed with the profits realized by the portfolio, and not by outside benefits. Then, the wealth of the agent satisfies the following equation:

$$dX_t = \theta_t^0 S_t^0 r_t dt + \sum_{i=1}^k \theta_t^i S_t^i b_t^i dt + \sum_{i=1}^k \theta_t^i S_t^i (\sigma_t^i, dW_t) - c_t dt \tag{3}$$

Then, we denote by $\pi_t^i = \theta_t^i S_t^i$ the amount of wealth invested in the $i^{th}$ asset for $i = 1, ..., k$, and we notice that $\theta_t^0 S_t^0 = X_t - \sum_1^k \pi_t^i$. We denote also by $\pi_t = (\pi_t^i, i = 1, .., k)$ the portfolio (or strategy), and so the total wealth can be written as a solution of a stochastic differential equation:

$$dX_t = (X_t r_t - c_t)dt + (\pi_t, b_t - r_t\mathbf{1})dt + (\pi_t, \sigma_t dW_t)\ ,\ X_0 \in L^0(\mathcal{F}_0) \tag{4}$$

where $\mathbf{1}$ is the vector with all coordinates equal to 1. The previous line can also be rewritten by integrating from $t$ to $T$:

$$X_T - X_t = \int_t^T (X_s r_s - c_s)ds + \int_t^T (\pi_s, b_s - r_s\mathbf{1})ds + \int_t^T (\pi_s, \sigma_s dW_s) \text{ a.s.} \tag{5}$$

so:

$$X_t = X_T - \int_t^T \underbrace{[(X_s r_s - c_s) + (\pi_s, b_s - r_s\mathbf{1})]}_{-f(s,X_s,Z_s)} ds - \int_t^T (\underbrace{\sigma_s^* \pi_s}_{Z_s}, dW_s) \text{ a.s.} \tag{6}$$

It is the form under which we will study the wealth process, as a solution of a backward stochastic differential equation. We consider an option-hedging problem, represented by a pay-off $\xi$, to be reached at maturity $T$. As a transcription, we have a problem of portfolio duplication: we look for the initial wealth $X_0$ and the portfolio $\pi$ such that $X_T = \xi$. The reason why BSDEs are interesting in our case is that they allow us to model such a problem of option hedging with a unique equation (see El Karoui, Peng and Quenez [7]).
BSDEs are stochastic differential equations of the form:

$$X_t = \xi + \int_t^T f(s, X_s, Z_s)ds - \int_t^T (Z_s, dW_s) \ , \forall 0 \leq t \leq T \tag{7}$$

- $\xi \in L^2(\Omega)$ is the final wealth, a goal to reach,
- $f : \Omega \times [0,T] \times \mathbf{R}^k \times \mathbf{R}^{k\times d} \longrightarrow \mathbf{R}^k$ is a drift function,
- $X_t$ is the total wealth of the portfolio at time $t$
- $Z_t$ represents the portfolio investments at time $t$

One of the fundamental results about BSDEs is a theorem given by E.Pardoux (see [19]), which gives the existence and uniqueness of the solution of a BSDE under some Lipschitz hypotheses on the drift function.

**Theorem 1.1** *(Pardoux I) Suppose $f(.,y,z)$ is $\mathcal{F}$-prog. measurable, and*

1. $\exists \phi : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ *increasing such that* $|f(t,y,0)| \leq |f(t,0,0)| + \phi(|y|), \forall t, y$ *a.s.*
2. $\mathbf{E}_P(\int_0^T |f(t,0,0)|^2) < \infty$
3. $f$ *is globally Lipschitz w.r.t.* $z$ *and continuous w.r.t.* $y$
4. $\langle y - y', f(t,y,z) - f(t,y',z)\rangle \leq \mu |y-y'|^2, \forall t, y, y', z,$ *a.s.*

*Then the BSDE has a unique solution* $(X,Z)$ *such that* $\mathbf{E}_P \int_0^T \parallel Z_t \parallel^2 dt < \infty$

From now on, we suppose that the financial agent is an insider trader: he has an additional information compare to the standard normally informed investor. To model it, we use the method of enlargement of filtration. We will suppose in all this paper that $r = 0$, which means that we don't have interest rates, because we will only consider small investors, who do not influence interest rates. In this model, we introduce an insider, who has an information at time 0 denoted by $L \in \mathcal{F}_{T'}$. $L$ is $\mathcal{F}_{T'}$-measurable, which means that it will be public at time $T'$. To model the insider space, we enlarge the initial filtration with $L$, in order to obtain the filtration of the insider trader probability space:

$$\mathcal{Y}_t = \bigcap_{s>t} (\mathcal{F}_t \vee \sigma(L)) \tag{8}$$

Since the discounted asset prices are martingales in the initial probability space under a risk-neutral probability, it would be interesting and natural that they still have similar properties in the larger space. So the main problem is under which condition do we have the following useful property:

**Hypothesis 1 (H')** *If $(M_t)_{0\leq t\leq T'}$ is a given $(\mathcal{F}_t, P)$-martingale (or semi-martingale), then $(M_t)$ is a $(\mathcal{Y}_t, P)$-semi-martingale.*

This problem has been developed by Jeulin [6] and Yor, and by Jacod [13], who shows that this assertion is true under the following hypothesis:

**Hypothesis 2** $(\mathbf{H}'')$ *The conditional probability law of $L$ knowing $\mathcal{F}_t$ is absolutely continuous with respect to the probability law of $L$, $\forall t < T'$.*

**Remark:** if $L$ is $\mathcal{F}_{T'}$-measurable, and if its conditional probability law given $\mathcal{F}_{T'}$ $(\delta_L)$ is absolutely continuous with respect to the distribution of $L$, it implies that $\sigma(L)$ is atomic (see for a deeper study Meyer [17]). But this is not the case in this article, because we will suppose $L \in \mathcal{F}_{T'}$ and a terminal point of view of our problem $T < T'$.
Under hypothesis $(\mathbf{H}'')$, Jacod gives the expected decomposition: one split a $(\mathcal{F}_t, P)$-martingale (the Brownian motion $W_t$ in our example) into a $(\mathcal{Y}_t, P)$-martingale part and a finite variation part as $W_t = B_t + \int_0^t l_s ds$ where $B_t$ is a $(\mathcal{Y}_t, P)$-martingale (a $(\mathcal{Y}, P)$-Brownian motion in case of $W_t$ Brownian motion), and $l$ is $\mathcal{Y}$-adapted. This property is also verified under a stronger hypothesis, for which we have stronger results, and which has been developed by Amendinger [1], Jeulin [6], Grorud and Pontier [10] :

**Hypothesis 3** $(\mathbf{H_3})$ *There exists a probability $Q$ equivalent to $P$ under which $\mathcal{F}_t$ and $\sigma(L)$ are independent, $\forall t < T$.*

Among the remarkable consequences of this hypothesis, we can notice that $W_t$ is a $(\mathcal{Y}, Q)$-Brownian motion. This article will successively study the existence and uniqueness of the BSDE on the enlarged probability space under $(\mathbf{H_3})$ and under $(\mathbf{H}'')$.
**Remark:** Before the study of hypothesis $(\mathbf{H}')$, $(\mathbf{H}'')$ and $(\mathbf{H_3})$, Bremaud and Yor [5] studied hypothesis $(\mathbf{H})$ under which $(\mathcal{F}, P)$-(local) martingales are still $(\mathcal{Y}, P)$-(local) martingales. This hypothesis is not currently used in insider models with initial enlargement of filtrations. In the case of initial enlargement, $(\mathbf{H_3})$ implies $(\mathbf{H})$. In fact, $(\mathbf{H_3})$ implies the existence of a probability $Q$ under which $(\mathbf{H})$ is verified (see also Amendinger [2]). Conversely, it is easy to prove that if $(\mathbf{H})$ is true under $P$, and if $\mathcal{F}_0$ is trivial, then $(\mathbf{H_3})$

is true. In a practical and financial sense, it means that it is not realistic to consider that the "natural" probability makes the information and the market independent. Nevertheless, hypothesis (**H**) appears to be relevant and useful in default risk models and progressive enlargement of filtrations.

# 2 BSDE under hypothesis $(\mathbf{H_3})$

## 2.1 Existence and Uniqueness Theorem

Let $(\mathbf{H_3})$ be verified. We denote by $Q$ the new probability. As $(\mathbf{H_3})$ can not hold until $T$ exactly, but only for $t < T$, we chose $L \in \mathcal{F}_{T'}$ and we consider a problem of maturity $T < T'$. So we can enlarge our filtration until $T$. We suppose also that the BSDE with parameter $(\xi, f)$ has a unique solution in the non insider space: we will suppose that the hypotheses of Pardoux's existence Theorem 1.1 are verified. To simplify the proof, we will even suppose that $f$ is globally Lipschitz with respect to $y$ and $z$. For the non insider investor, the initial BSDE is verified:

$$\begin{cases} X_t = \xi + \int_t^T f(s, X_s, Z_s)ds - \int_t^T (Z_s, dW_s) \ , \forall 0 \leq t \leq T \\ (\Omega, (\mathcal{F}_t)_{0 \leq t \leq T}, P), \xi \in L^2(\Omega, \mathcal{F}_T, P) \end{cases} \tag{9}$$

As $(W_t)_{0 \leq t \leq T}$ is still a Brownian motion under $((\mathcal{Y}_t)_{0 \leq t \leq T}, Q)$ thanks to hypothesis $(\mathbf{H_3})$ (cf Jacod [13]), the equation becomes in the insider space:

$$\begin{cases} \tilde{X}_t = \xi + \int_t^T f(s, \tilde{X}_s, \tilde{Z}_s)ds - \int_t^T (\tilde{Z}_s, dW_s) \ , \forall 0 \leq t \leq T \\ (\Omega, (\mathcal{Y}_t)_{0 \leq t \leq T}, Q), \xi \in L^2(\Omega, \mathcal{Y}_T, Q) \end{cases}$$

where a solution $(\tilde{X}, \tilde{Z})$ is a couple of $(\mathcal{Y})$-adapted processes. We also suppose that $\xi \in L^2(\Omega, \mathcal{Y}_T, Q)$, such that the problem is correctly given in the insider space. We have then the following result:

**Theorem 2.1** *Under hypothesis of Theorem 1.1, and if* $E_Q(\int_0^T |f(t,0,0)|^2 dt) < \infty$ *then the backward equation has a unique solution in the insider space.*

**Proof:** The hypotheses of Pardoux's Theorem 1.1 can be checked. The filtration is not the natural filtration of the Brownian motion any more. We will have to cope with this problem. $f(., y, z)$ is $\mathcal{F}_t$-progressively measurable $\forall y, z$ and $\mathcal{F}_t \subset \mathcal{Y}_t$, so $f(., y, z)$ is $\mathcal{Y}_t$-progressively measurable. Moreover, as $P \sim Q$ the $P$-null sets are the same as the $Q$-null sets. So we still have point 1, 3 and 4 $Q$-a.s. For point 2, under new probability $Q$, the expected value is not finite any more, so we have to suppose this point true in the hypotheses of Theorem 2.1. The last problem we have to cope with is the new filtration which is not any more the natural Brownian filtration. This is annoying because the proof of Pardoux's Theorem 1.1 uses Itô martingale representation theorem, which supposes that the filtration is the natural

Brownian filtration. Nevertheless, as the new filtration $\mathcal{Y}$ is generated by $L$ and by the Brownian motion, we still have a martingale representation result in the case of a filtration generated by the Brownian motion and $\mathcal{H}_0$ an initial $\sigma$-algebra (see [14] Theorem $III.4.33$ p.189). And so Pardoux's proof can be adapted to our case. To simplify our proof, we suppose $f$ globally Lipschitz in $y$.
Let $\mathcal{B}^2 = (\mathcal{M}^2(0,T))^k \times (\mathcal{M}^2(0,T))^{k\times d}$. We will define a function $\Phi : \mathcal{B}^2 \to \mathcal{B}^2$ such that $(X,Z) \in \mathcal{B}^2$ is a solution of the BSDE if it is a fixed point of $\Phi$. Let $(U,V) \in \mathcal{B}^2$, and $(X,Z) = \Phi(U,V)$ with:

$$X_t = \mathbf{E}_Q \left[ \xi + \int_t^T f(s,U_s,V_s)ds \middle| \mathcal{Y}_t \right] \ , 0 \leq t \leq T \ , \ X_T = \xi.$$

Then $\{Z_t, 0 \leq t \leq T\}$ is obtained by using Jacod and Shiryaev [14] generalized martingale representation theorem, applied to the martingale $E_Q \left[ \xi + \int_0^T f(s,U_s,V_s)ds | Y_t \right]_{t\in[0,T]}$.
So we obtain:

$$\xi + \int_0^T f(s,U_s,V_s)ds = \mathbf{E}_Q \left( \xi + \int_0^T f(s,U_s,V_s)ds \middle| \sigma(L) \right) + \int_0^T (Z_s, dB_s)$$

In this last equality, conditional expectation is taken with respect to $\mathcal{Y}_t$ and so $\forall t \leq T$:

$$X_t + \int_0^t f(s,U_s,V_s)ds = X_0 + \int_0^t (Z_s, dB_s)$$

$$\text{Which implies} \quad X_0 = \xi + \int_0^T f(s,U_s,V_s)ds - \int_0^T (Z_s, dB_s)$$

$$\text{and consequently} \quad X_t = \xi + \int_t^T f(s,U_s,V_s)ds - \int_t^T (Z_s, dB_s) \qquad (10)$$

This proves that $(X,Z) \in \mathcal{B}^2$ is solution of the BSDE if it is a fixed point of $\Phi$.
As $f$ is globally Lipschitz with respect to $U, V$ and using Davis-Burkholder-Gundy's inequality, we deduce:

$$\mathbf{E}_Q \left( \sup_{0\leq t\leq T} |X_T|^2 \right) < \infty$$

consequently $\{\int_0^t (X_s, Z_s dB_s), 0 \leq t \leq T\}$ is a martingale.
Let $(U,V), (U',V') \in \mathcal{B}^2$, $(X,Z) = \Phi(U,V), (X',Z') = \Phi(U',V')$,
$(\bar{U},\bar{V}) = (U-U', V-V')$ and $(\bar{X},\bar{Z}) = (X-X', Z-Z')$.
Then, from Itô formula, $\forall \gamma \in \mathbf{R}$, we have:

$$\begin{aligned} e^{\gamma t}\mathbf{E}_Q|\bar{X}_t|^2 \ &+ \ \mathbf{E}_Q \textstyle\int_t^T e^{\gamma s}(\gamma|\bar{X}_s|^2 + \parallel \bar{Z}_s \parallel^2)ds \\ &\leq \ 2K\mathbf{E}_Q \textstyle\int_t^T e^{\gamma s}|\bar{X}_s|(|\bar{U}_s| + \parallel \bar{V}_s \parallel)ds \\ &\leq \ 4K^2\mathbf{E}_Q \textstyle\int_t^T e^{\gamma s}|\bar{X}_s|^2 ds + \frac{1}{2}\mathbf{E}_Q \int_t^T e^{\gamma s}(|\bar{U}_s|^2 + \parallel \bar{V}_s \parallel^2)ds \end{aligned} \qquad (11)$$

We chose $\gamma = 1 + 4K^2$, and obtain:

$$\mathbf{E}_Q \int_0^T e^{\gamma t}(|\bar{X}_t|^2 + \parallel \bar{Z}_t \parallel^2)dt \leq \frac{1}{2}\mathbf{E}_Q \int_0^T e^{\gamma t}(|\bar{U}_t|^2 + \parallel \bar{V}_t \parallel^2)dt \qquad (12)$$

Then $\Phi$ is a strict contraction on $\mathcal{B}^2$ with norm
$|||(X,Z)|||_\gamma = (\mathbf{E}_Q \int_0^T e^{\gamma t}(|X_t|^2 + \parallel Z_t \parallel^2)dt)^{\frac{1}{2}}$.
We deduce that $\Phi$ has a unique fixed point and we conclude that the BSDE has a unique solution.

□

## 2.2 Comparison of the solutions

We first look at an intuitive example. Suppose $L = S_T$: the agent knows the final price (he deduces it for instance from an information on a former financial operation, as a takeover). Suppose also that he wants to hedge a digital option $\mathbf{1}_{S_T \leq K}$. The insider will then have two possible investments: invest on the risky asset if $S_T \leq K$, or doing nothing otherwise. He has obviously a different strategy from the non insider agent. Moreover, in this special case, there is an arbitrage opportunity.
In the general case, it is not so easy to determine whether the insider will have a different investment strategy from the non insider or not, especially when information is at time $T' > T$. So we have two questions: will the insider invest differently from the non insider? Is there an arbitrage in the insider space? Answering these questions can give us other clues: is the information relevant? Is it useful? Moreover, when the insider has a very different strategy from the non insider, it will be possible to detect the former through statistical tests. This could be useful for market fraud detection agencies, as the French A.M.F. We can recall that in a wealth optimization point of view (see Grorud and Pontier [9]), the insider will immediately have a completely different strategy from the non insider. Is it the same in our hedging problem? We compare first the strategies of the two agents (comparison of the solutions of the two BSDE's), before studying viability and completeness of the insider market.

**Corollary 2.1** *Suppose that $\xi \in L^2(\Omega, \mathcal{Y}_T, Q) \cap L^2(\Omega, \mathcal{F}_T, P)$, so that the financial problem has a sense in the insider space as in the non insider space. We denote by $(X,Z)$ and $(X',Z')$ the solutions of the two BSDE's. Then, if $\mathbf{E}_Q \int_0^T \parallel Z_t \parallel^2 dt < \infty$, the solution of the insider's BSDE is the same as the non insider's one: $(X,Z) = (X',Z')$.*

**Proof:** according to Theorem 2.1, in the insider space $(\Omega, (\mathcal{Y}_t)_{0\leq t\leq T}, Q)$ the BSDE has a unique solution $(X'_t, Z'_t)$. But the non insider BSDE solution $(X_t, Z_t)$ is $(\mathcal{F}_t)_{0\leq t\leq T}$-progressively measurable, and so is it with respect to

$(\mathcal{Y}_t)_{0\leq t\leq T}$. As the BSDE is the same in both spaces, we have

$$X_t = \xi + \int_t^T f(s, X_s, Z_s)ds - \int_t^T (Z_s, dB_s)$$

So $(X_t, Z_t)$ is a solution of the insider BSDE. As $\mathbf{E}_Q \int_0^T \| Z_t \|^2 \, dt < \infty$, we conclude that it is the unique solution of the insider BSDE. □

**Remarks**: Intuitively, as $L \in (\mathcal{F}_{T'})$, $T < T'$ and $L \perp \xi$, from hypothesis $(\mathbf{H_3})$, we can understand that if under $Q$ the objective is independent from the insider information, he will not have a different strategy, as soon as this strategy is admissible in the insider space. In a certain sense, the information is useless. In this case, there is no arbitrage opportunity, and the insider market is viable. We have a hedging problem in a complete initial market, so there exists a price for the option, and a strategy for hedging the risk. What is the use of the information? Either to create an arbitrage, which is impossible under $(\mathbf{H_3})$ (see next paragraph), or to propose a different price for the option in the market. But then two problems appear: first, who would buy such an option? and second, proposing a different price from the market means exhibiting the fact that we have an information... which is uninteresting from the insider point of view considering that using the information is a fraud.

## 2.3 Viability and completeness of the insider market

We try to translate our results in term of viability and completeness of the market. The main point is to know if there is an arbitrage opportunity, and if the insider market is complete.

**Theorem 2.2** *Suppose that the insider market is viable, and let $Q^*$ be a risk-neutral probability. If $\xi \in L^2(\mathcal{F}, P) \cap L^2(\mathcal{Y}, Q)$, then $E_{P^*}(\xi) = E_{Q^*}(\xi)$. So the information does not create any arbitrage opportunity: prices are the same in both spaces.*

**Proof:** By a Girsanov transformation, risk-neutral probabilities allows us to remove drift in price processes, keeping volatility. So in the insider space as in the non insider space, we obtain $dS_t = S_t(\sigma_t, dW_t)$ where $W_t$ is a $(\mathcal{F}, P)$ and a $(\mathcal{Y}, Q)$-Brownian motion. Then price processes under the two risk neutral probabilities follow the same diffusion processes, and prices on both markets are the same. □

In general, the insider market is incomplete, but has a particular property:

**Theorem 2.3** *Let $R_1$ and $R_2$ be two risk neutral probabilities in the insider space. Let $Y \in L^1_{R_1}(Q) \cap L^1_{R_2}(Q)$, then prices are equal: $E_{R_1}(Y) = E_{R_2}(Y)$.*

**proof:** See Grorud [8]. □

The insider market may have several risk neutral probabilities. It is not necessarily complete, nevertheless it is always "pseudo-complete", in the sense that all prices calculated under different risk neutral probabilities are the same. It could be interpreted by the fact that prices in the insider market will only depend on information $L$ and on the non insider market: as the non insider has a unique risk neutral probability, there is only one price in the insider market.

Finally, following Amendinger [2] and Grorud and Pontier [10] we have the following result:

**Theorem 2.4** *Under* $(\mathbf{H_3})$*, if the non insider market is viable, then the insider market is also viable. Financially speaking the information $L$ does not create any arbitrage opportunity.*

On the other hand, completeness of the non insider market does not necessarily imply completeness of the insider market. The enlarged space may have several risk neutral probabilities, but which will have property of pseudo-completeness of Theorem (2.3).

# 3 BSDE under hypothesis $(\mathbf{H}'')$

## 3.1 Existence and Uniqueness Theorem

In this section $(\mathbf{H}'')$ is supposed to hold: the conditional probability law of $L$ knowing $\mathcal{F}_t$ is absolutely continuous with respect to the law of $L$, $\forall t \leq T$. We still take $L \in \mathcal{F}_{T'},\ T < T'$. Let's recall the non insider BSDE:

$$\begin{cases} X_t = \xi + \int_t^T f(s, X_s, Z_s) ds - \int_t^T (Z_s, dW_s) \ , \forall 0 \leq t \leq T \\ (\Omega, (\mathcal{F}_t)_{0 \leq t \leq T}, P) \end{cases} \tag{13}$$

$(\mathbf{H}')$ holds, say every $(\mathcal{F}_t, P)$-martingale $(M_t)_{0 \leq t \leq T}$ is a $(\mathcal{Y}_t, P)$-semi-martingale. So the Brownian motion $W_t$ can be written: $W_t = B_t + \int_0^t l_s ds$ where $B_t$ is a $(\mathcal{Y}, P)$-Brownian motion and $l$ is a $\mathcal{Y}$-adapted process. We deduce the new backward equation in the insider space:

$$\begin{cases} X_t = \xi + \int_t^T [f(s, X_s, Z_s) - (Z_s, l_s)]\, ds - \int_t^T (Z_s, dB_s) \ , \forall 0 \leq t \leq T \\ (\Omega, (\mathcal{Y}_t)_{0 \leq t \leq T}, P) \end{cases} \tag{14}$$

If we take $\xi \in L^1(\Omega, \mathcal{Y}_T, P)$ in the insider space, we have a new BSDE with a new drift, deduced from the previous drift according to the formula:

$$g(\omega, t, y, z) = f(\omega, t, y, z) - (z, l(\omega, t)).$$

Let's consider Pardoux's existence and uniqueness Theorem 1.1. The filtration is not generated by the Brownian motion any more. So we don't have any martingale representation theorem. $f(.,y,z)$ and $l_t$ are $\mathcal{Y}_t$-progressively measurable, so the new drift $g(.,y,z)$ is $\mathcal{Y}_t$-progressively measurable. As $g(t,y,0) = f(t,y,0)$, the condition on $f$ stands also on $g$, so $|g(t,y,0)| \leq |g(t,0,0)| + \phi(|y|)$, $\forall y, z$ $P$-a.s. Identically, as $g(t,0,0) = f(t,0,0)$ we still have $\mathbf{E}_P(\int_0^T |g(t,0,0)|^2 dt) < \infty$. On the other hand, $g$ is not globally Lipschitz, because:

$$|g(t,y,z)-g(t,y,z')| = |f(t,y,z)-f(t,y,z')+l(t)(z-z')| \leq (K+l_t) \parallel z-z' \parallel$$

So if $l_t$ is a.s. bounded, then $g$ is globally Lipschitz with respect to $z$, but if $l_t$ is not bounded, this property does not hold. Moreover, as $g(y) = f(y) + constant$, we still have $< y-y', g(t,y,z) - g(t,y',z) >=< y-y', f(t,y,z) - f(t,y',z) >\leq \mu|y-y'|^2$. As $f$, $g$ is also continuous with respect to $y$, $\forall t, z$ a.s. Finally, all conditions are verified for the enlarged BSDE in the insider space, as soon as we suppose $l_t$ bounded. But we need a martingale representation theorem. If $l_t$ is almost surely bounded, then $E_P(\mathcal{E}(-l.B)) = 1$, $\forall t < T$. Then, according to proposition 4.2 of Grorud and Pontier [12], hypothesis $(\mathbf{H_3})$ is verified. We are in the previous case : under hypothesis $(\mathbf{H_3})$, we have a martingale representation theorem, and we can conclude similarly to Theorem 2.1 (and without a change of probability). We obtained the following result:

**Theorem 3.1** *Under $(H'')$ and hypotheses of Theorem 1.1, if $l_t$ is a.s. bounded in the enlarged space $(\Omega, (\mathcal{Y}_t)_{0\leq t\leq T}, P)$, then we deduce the existence and uniqueness of the solution of the enlarged BSDE.*

**Remark:** It will be useful to study what happens on examples for which $l$ is not bounded, and $(\mathbf{H_3})$ does not hold. But a problem is that we do not know any example of $L$ in a continuous model for which $(\mathbf{H''})$ holds but not $(\mathbf{H_3})$. And if $(\mathbf{H_3})$ holds, we have the result of previous section, and the problem is solved. This is the reason why it seems natural to introduce jump processes into our model, in order to have examples of $L$ for which we have $(\mathbf{H''})$ but not $(\mathbf{H_3})$.

# 4 Introduction of Jump processes

## 4.1 Extended model

We add jump processes in the price dynamics studied in the previous section. $W$ is still a $m$-dimensional standard Brownian motion on $(\Omega^W, \mathcal{F}^W, P^W)$ and $(\mathcal{F}_t^W)_{t\in[0,T]}$ its completed natural filtration. We denote by $(\Omega^N, \mathcal{F}^N, P^N)$ another probability space where $N = (N^1, .., N^n) : \Omega^N \to \mathbf{R}^n$ is a $n$-dimensional multivariate Poisson process, with intensity $\lambda_t$, $t \in [0,T]$. We denote by $M_t = N_t - \int_0^t \lambda_s ds$ the compensated multivariate Poisson process. $N$ is denoted as a vector $(N^k)_{k=1,..,n}$ of unidimensional multivariate Poisson processes with intensity $(\lambda^k)_{k=1,..,n}$, $\mathcal{F}_0^N$-measurable. $\mathcal{F}^N$ is generated by $\mathcal{F}_0^N$ and the jump times of $N$. So the global probability space is:

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P) = (\Omega^W \times \Omega^N, \mathcal{F}^W \otimes \mathcal{F}^N, (\mathcal{F}_t^W \otimes \mathcal{F}_t^N)_{t\in[0,T]}, P^W \times P^N).$$

The market model still contains a bond and $d = m + n$ risky assets whose prices $(S_t^i)_{i=1,..,d}$ follow a diffusion run by $W$ and $N$:

$$\begin{array}{rcl} dS_t^0 & = & S_t^0 r_t \ , \quad S_0^0 = 1 \\ dS_t^i & = & S_t^i b_t^i dt + S_t^i(\sigma_t^i, dW_t) + S_t^i(\rho_t^i, dM_t) \ , \quad S_0^i = x^i, \quad i = 1, .., d. \end{array} \tag{15}$$

We suppose the following, so that the market is viable and complete:

- $b, r, \sigma, \rho$ are predictable and globally bounded processes,
- $\lambda$ is a nonnegative $\mathcal{F}_0$-measurable process, which does not meet any neighborhood of 0, $\rho_t^{i,k} > -1, \ \forall i, k, t$,
- $\Phi_t^* \Phi_t$ is uniformly elliptic, where $\Phi_t = [\sigma_t \rho_t]$
- Let $\theta = \Phi^{-1}(b - r\mathbf{1})$, then $\theta^k < \lambda^k, \ \forall k = 1, .., n$.

We consider again an insider in this new market with jumps. The insider still has information $L \in L^1(\Omega, \mathcal{F}_{T'}, P)$ taking its values in $\mathbf{R}^k$, and the new filtration on the insider space is $\mathcal{Y}_t = \cap_{s>t}(\mathcal{F}_s \vee \sigma(L)), \ \forall t \in [0, T], \ T < T'$. We have the same hypothesis on wealth process and investment strategy, and we study self-financing strategies $dX_t = \sum_{i=0}^d \theta_t^i dS_t^i - c_t dt$, so the wealth process of the trader on this market satisfies:

$$\begin{array}{rcl} X_t & = & X_0 + \int_0^t \theta_s^0 S_s^0 r_s ds - \int_0^t c_s ds \\ & & + \sum_{i=1}^d \left[ \int_0^t \left( \theta_s^i S_s^i b_s^i ds + \theta_s^i S_s^i (\sigma_s^i, dW_s) + \theta_s^i S_s^i (\rho_s^i, dM_s) \right) \right] \end{array}$$

As in the continuous model, we obtain the following BSDE for the wealth process:

$$\begin{array}{rcl} X_t & = & X_T - \int_t^T \underbrace{[(X_s r_s - c_s) + (\pi_s, b_s - r_s\mathbf{1})]}_{-f(s, X_s, Z_s, U_s)} ds \\ & & - \int_t^T (\underbrace{\sigma_s^* \pi_s}_{Z_s}, dW_s) - \int_t^T (\underbrace{\rho_s^* \pi_s}_{U_s}, dM_s) \text{ a.s.} \end{array}$$

## 4.2 BSDE with jumps

In this model with jumps, and even in a more general model with Poisson point processes (see further), Barles, Buckdahn and Pardoux [4] developed an existence theorem for the solution of BSDEs with jumps. We denote by $\mathcal{B}^2 = \mathcal{S}^2 \times L_m^2(P) \times L_n^2(P)$ where:

- $\mathcal{S}^2$ is the set of $k$-dimensional $\mathcal{F}_t$-adapted càdlàg processes $\{Y_t\}_{0\le t\le T}$ such that $||Y||_{\mathcal{S}^2} = \| \sup_{0\le t\le T} |Y_t| \, \|_{L^2(\Omega)} < \infty$

- $L^2_m(P)$ the set of all $k \times m$-dimensional $\mathcal{F}_t$-progressively measurable processes $\{Z_t\}_{0\leq t\leq T}$ such that $||Z||_{L^2_m(P)} = \left(E_P \int_0^T |Z_t|^2 dt\right)^{\frac{1}{2}} < \infty$
- $L^2_n(P)$ the set of all $k \times n$-dimensional $\mathcal{F}_t$-progressively measurable processes $\{U_t\}_{0\leq t\leq T}$ such that $||U||_{L^2_n(P)} = \left(E_P \int_0^T |U_t|^2 dt\right)^{\frac{1}{2}} < \infty$

We have the following theorem (see Barles et al. [4]):

**Theorem 4.1** *(Pardoux II)*
*Let* $\xi \in L^2(\Omega, \mathcal{F}_T, P)^k$ *and* $f : \Omega \times [0,T] \times \mathbf{R}^k \times \mathbf{R}^{k\times m} \times \mathbf{R}^{k\times n} \longrightarrow \mathbf{R}^k$.
*If* $f$ *is measurable, if* $E_P \int_0^T |f_t(0,0,0)|^2 dt < \infty$ *and if* $\exists K$ *such that:*
$|f_t(y,z,u) - f_t(y',z',u')| \leq K(|y-y'| + ||z-z'|| + ||u-u'||), \forall t \leq T, y, y', z, z', u, u'$
*then there exists a unique triple* $(X, Z, U) \in \mathcal{B}^2$ *solution of the BSDE:*

$$X_t = \xi + \int_t^T f_s(X_s, Z_s, U_s) ds - \int_t^T (Z_s, dW_s) - \int_t^T (U_s, dM_s), \quad 0 \leq t \leq T$$

**Proof**: The proof is the same as Pardoux's Theorem 1.1 proof: constructing a strict contraction and using a martingale representation theorem. □

### 4.3 Under hypothesis ($\mathbf{H_3}$)

Everything works globally as in the first part of the paper. More precisely:

- **Existence and Uniqueness Theorem**

Thanks to Jacod and Shiryaev ([14] Theorem $III$.4.34 p.189), Grorud ([8] Theorem 3.1 p.648) shows a martingale representation theorem under ($\mathbf{H_3}$) with jumps. With this martingale representation theorem, we can adapt the proof of Pardoux's Theorem (4.1), and as in the continuous case in section 2.1, we have the following result:

**Theorem 4.2** *Under hypothesis of Theorem 4.1 (so the initial BSDE has a unique solution), for* $\xi \in L^2(\Omega, \mathcal{Y}_T, Q)$ *and if* $E_Q\left(\int_0^T |f_t(0,0,0)|^2 dt\right) < \infty$ *then the BSDE in the insider space has a unique solution* $(X_s, Z_s, U_s) \in \mathcal{B}^2$.

- **Comparison of solutions**

We have a similar result as in section 2.2:

**Proposition 4.1** *For* $\xi \in L^2(\Omega, \mathcal{Y}_T, Q) \cap L^2(\Omega, \mathcal{F}_T, Q)$*, we have:*
*if* $E_Q\left(\int_0^T \| Z_t \|^2_{L^2_m(Q)} + \| U_t \|^2_{L^2_n(Q)} dt\right) < \infty$*, then the solution of the enlarged BSDE is the same as the solution of the initial BSDE:* $(X, Z, U) = (X', Z', U')$.

- **Viability and Completeness of the market**

As in the continuous case, if the non insider market is viable, then the insider market is also viable: there is no arbitrage opportunity (see Grorud [8]).

### 4.4 Under hypothesis $(\mathbf{H}'')$

In this case, the new model becomes interesting, because now we have examples of $L$ for which $(\mathbf{H}'')$ holds but not $(\mathbf{H_3})$. We summarize the results we have under this hypothesis before treating an example. We use Jacod's result on enlargement of filtration under $(\mathbf{H}'')$ (see [13]), a bit different from the result in the continuous model (see Grorud [8]).

**Proposition 4.2** *Under hypothesis* $(\mathbf{H}'')$*, we have:*

- *If $Q_t$ is the conditional law of $L$ knowing $\mathcal{F}_t$, then there exists a measurable version of the conditional density $dQ_t : (\omega, t, x) \mapsto p(\omega, t, x)$ which is a martingale and can be written, $\forall x \in \mathbf{R}$ as:*

$$p(t,x) = p(0,x) + \int_0^t (\alpha(s,x), dW_s) + \int_0^t (\beta(s,x), dM_s)$$

  *where $\forall x,\ s \mapsto \alpha(s,x)$ and $s \mapsto \beta(s,x)$ are $\mathcal{F}$-predictable processes. Moreover, $\forall s < T'$, $p(s,L) > 0$ a.s.*

- *If $Y$ is a martingale written as $Y_t = Y_0 + \int_0^t (u_s, dW_s) + \int_0^t (v_s, dM_s)$ then $d < Y, p(.,x) >_t = < \alpha(.,x), u >_t dt + < \beta(.,x), v >_t dt$ a.s. $\forall t$, and:*

$$\bar{Y}_t = Y_t - \int_0^t \frac{(< \alpha(.,x), u >_s + < \Gamma.\beta(.,x), v >_s)|_{x=L}}{p(s,L)} ds, \quad 0 \le t \le T$$

  *is a $(\mathcal{Y}, P)$-local martingale where $\Gamma$ is the diagonal matrix of intensities of $N$: $d < M >_s = \Gamma_s ds$*

We denote by $l_s = \frac{\alpha(s,L)}{p(s,L)}$ and $\mu_s = \frac{\Gamma_s \beta(s,L)}{p(s,L)}$. Then $\bar{W}_t = W_t - \int_0^t l_s ds$ is a $(\mathcal{Y}, P)$-Brownian motion and if $1 + \frac{\beta(t,L)}{p(t,L)} \ge 0$ then $\bar{M}_t = M_t - \int_0^t \mu_s ds$ is a compensated Poisson process with intensity $\lambda_t(1 + \frac{\beta(t,L)}{p(t,L)})$.
Then the wealth process can be written in term of a BSDE in the insider space:

$$\begin{aligned} X_t &= X_T - \int_t^T \underbrace{[(X_s r_s - c_s) + (\pi_s, b_s - r_s \mathbf{1}) + \sigma_s^* \pi_s l_s + \rho_s^* \pi_s \mu_s]}_{-g(s,X_s,Z_s,U_s)} ds \\ &\quad - \int_t^T (\underbrace{\sigma_s^* \pi_s}_{Z_s}, dW_s) - \int_t^T (\underbrace{\rho_s^* \pi_s}_{U_s}, dM_s) \text{ a.s.} \end{aligned}$$

with a new drift $g(s, X_s, Z_s, U_s) = f(s, X_s, Z_s, U_s) - l_s Z_s - \mu_s U_s$.

### 4.5 Study of an example of $L$

For this example, let us take $L = N_T$: the insider trader knows the number of jumps at final time $T$. In order to simplify the problem, we will consider a unidimensional process. The law of $L$ is absolutely continuous with respect to the counting measure on $\mathbb{N}$. We obtain a measurable version of the conditional density:

$$p(t, y) = \exp\left(-\int_t^T \lambda_s ds\right) \frac{(\int_t^T \lambda_s ds)^{y-N_t}}{(y-N_t)!} \mathbf{1}_{[N_t;\infty[}(y).$$

Then it is clear that $(\mathbf{H_3})$ does not hold (non equivalence of the laws), whereas $(\mathbf{H''})$ is verified (law absolutely continuous with respect to the law of $L$). We give an explicit expression of $\beta$ in Proposition 4.2:

$$\beta(s,y) = k_s^y p(s^-, y) \text{ with } k_t^y = \frac{y - N_{t-}}{\int_t^T \lambda_s ds} - 1 \text{ and so } \mu_t = \lambda_t k_t^L = \lambda_t \left(\frac{N_T - N_{t-}}{\int_t^T \lambda_s ds} - 1\right)$$

In the insider space, $\tilde{M}_t = M_t - \int_0^t \lambda_s \left(\frac{N_T - N_{s-}}{\int_s^T \lambda_u du} - 1\right) ds$ is a $\mathcal{Y}_t$-martingale. So $N$ is a $\mathcal{Y}$-Poisson process with intensity $\frac{N_T - N_{t-}}{T-t} \geq 0$, $\forall t \leq T$ with respect to $\mathcal{Y}$. Indeed we should enlarge the initial space until $T'$. Brownian motion does not change because the conditional density is represented only on the Poisson process, because of the independence between Brownian motion and Poisson process. In this case, the enlarged BSDE is:

$$X_T = \xi + \int_t^T \left(f(s, X_s, Z_s, U_s) - \lambda_s \left(\frac{N_T - N_{s-}}{\int_s^T \lambda_u du} - 1\right)\right) ds - \int_t^T Z_s dW_s - \int_0^T U_s d\tilde{M}_s$$

The martingale representation theorem that stands in $(\Omega, \mathcal{F}, P)$ allows us to find a solution to the enlarged BSDE, but we do not have any uniqueness result in this case ($\mu$ is not bounded).

## 5 Introduction of a Poisson measure

Such a model is interesting to develop because its incompleteness allows us to have hypothesis $(\mathbf{H''})$ without $(\mathbf{H_3})$.

### 5.1 The model

In our last section we introduce jump processes where jumps are continuous in time and space, by using a Poisson measure. We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ with $\mathcal{F}$ the completed filtration generated by both $(W_t)_{t\geq 0}$ and $(N_t)_{t \geq 0}$. $(W_t)_{t\geq 0}$ is a standard $m$-dimensional Brownian motion and $(N_t)_{t\geq 0}$ a point process with random Poisson measure $\mu$ on $\mathbb{R}_+ \times E$ and compensator $\nu(dt, de)$ such that $\{\tilde{\mu}([0,t] \times A) = (\mu - \nu)([0,t] \times A)\}_{t \geq 0}$

is a martingale $\forall A \in \mathcal{E}$ satisfying $\nu([0,t] \times A) < \infty$. $E = \mathbb{R}^l \setminus \{0\}$ with its Borel $\sigma$-algebra $\mathcal{E}$. We can write as $N_t = \int_0^t \int_E \mu(ds, de)$ the point process, so $dN_t = \int_E \mu(dt, de)$. And we denote by $\tilde{N}_t = N_t - \int_0^t \int_E \nu(ds, de)$ the compensated process. We use an additional hypothesis on $\nu$: $\nu(dt, de) = dt\lambda(de)$, $\lambda$ supposed to be a $\sigma$-finite measure on $(E, \mathcal{E})$ that satisfies : $\int_E (1 \vee |e|^2)\lambda(de) < +\infty$.
Let $\mathcal{H}$ be a finite-dimensional linear space, and let

- $L^2_{\mathcal{F}}([0,1]; \mathcal{H})$ be the space of all $(\mathcal{F}_t)$-adapted $\mathcal{H}$-valued square integrable processes
- $L^2_{\mathcal{F},P}([0,1]; \mathcal{H})$ be the space of $(\mathcal{F}_t)$-predictable equivalent class versions.

As previously we consider a financial market with one bond and $k$ risky assets, in which asset prices are driven by the following stochastic differential equation $(t \in [0,T], 1 \le i \le k)$ :

$$S_t^i = S_0^i + \int_0^t S_s^i b_s^i ds + \int_0^t S_s^i (\sigma_s^i, dW_s) + \int_0^t S_{s^-}^i \int_E \phi_s^i(e) \mu(ds, de) \tag{16}$$

where $b, \sigma$ and $\phi$ are predictable and globally Lipschitz processes. We rewrite the self-financing equation as a BSDE, and the wealth-investment process is solution of:

$$\begin{aligned} X_t &= X_T - \int_t^T \underbrace{[(X_s r_s - c_s) + (\pi_s, b_s - r_s \mathbf{1})]}_{-f(s, X_s, Z_s)} ds - \int_t^T (\underbrace{\sigma_s^* \pi_s}_{Z_s}, dW_s) \\ &\quad - \int_t^T \int_E \underbrace{(\pi_{s^-}, \phi(s,e))}_{U_s(e)} \mu(ds, de) \text{ a.s.} \end{aligned} \tag{17}$$

As in the previous parts, an insider trader has an information $L \in L^1(\Omega, \mathcal{F}_{T'}, \mathbb{R}^k)$ on the future. $\mathcal{Y}$ is still the insider's natural filtration. In both spaces, we study again existence and uniqueness of the admissible wealth-portfolio processes in order to cover a pay-off represented by $\xi = X_T$.

## 5.2 Existence and uniqueness

We use here two main articles: Barles, Buckdahn and Pardoux [4], and Tang and Li [23]. Let us first define several process spaces.
Let $\mathcal{S}^2(\mathcal{F})$ be the set of all $\mathcal{F}_t$-adapted cadlag $k$-dimensional processes square-integrable $\{Y_t\}_{0 \le t \le T}$ such that $\| Y \|_{\mathcal{S}^2(\mathcal{F})} = \| \sup_{0 \le t \le T} |Y_t| \|_{L^2(\Omega)} < \infty$.
Let $L^2(W)$ be the set of all $\mathcal{F}_t$-progressively measurable $k \times d$-dimensional processes $\{Z_t\}_{0 \le t \le T}$ such that $\| Z \|_{L^2(W)} = \left( E_P \int_0^T |Z_t|^2 dt \right)^{1/2} < \infty$.
Let $L^2(\tilde{\mu})$ be the set of all mappings $U : \Omega \times [0,T] \times E \to \mathbb{R}$ that are $\mathcal{P} \otimes \mathcal{E}$-measurable ($\mathcal{P}$ being the $\sigma$-algebra of $\mathcal{F}_t$-predictable subsets of $\Omega \times [0,T]$)

such that $\| U \|_{L^2(\tilde{\mu})} = \left(E_P \int_0^T \int_E U_t(e)^2 \nu(de,dt)\right)^{1/2} < \infty$.
Finally we define the functional space $\mathcal{B}^2(\mathcal{F}) = \mathcal{S}^2(\mathcal{F}) \times L^2(W) \times L^2(\tilde{\mu})$. Then we have the following result:

**Theorem 5.1** *(Barles et al. [4] Theorem 2.1, and Tang and Li [23] Lemma 2.4) Let $\xi \in (L^2(\Omega, \mathcal{F}_T, P))^k$ and let $f : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \times L^2(E, \mathcal{E}, \nu; \mathbb{R}^k) \to \mathbb{R}^k$ be a $\mathcal{P} \otimes \mathcal{B}_k \otimes \mathcal{B}_{k\times d} \times \mathcal{B}(L^2(E,\mathcal{E},\nu;\mathbb{R}^k))$-measurable function satisfying:*

$$\begin{aligned} \exists K > 0, \quad E_P \int_0^T |f_t(0,0,0)|^2 dt &< K \\ |f_t(y,z,u) - f_t(y',z',u')| &\leq K\left[|y-y'| + |z-z'| + \| u - u' \|\right] \end{aligned} \tag{18}$$

*Then there exists a unique triple $(Y,Z,U) \in \mathcal{B}^2(\mathcal{F})$ solution of the BSDE:*

$$Y_t = \xi + \int_t^T f_s(Y_s,Z_s,U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\tilde{\mu}(ds,de)\ ,\ 0 \leq t \leq T$$

### 5.3 BSDE under $(H_3)$ : Adaptation of the Existence and Uniqueness Theorem

Under hypothesis ($\mathbf{H_3}$), in this model with continuous random jumps, we can also adapt the existence theorem, as in the standard model under the same hypothesis on the drift. An insider with information $L$ verifying ($\mathbf{H_3}$) will have an admissible hedging strategy for an option with pay-off $\xi$. We have the following theorem:

**Theorem 5.2** *Let $\xi \in (L^2(\Omega, \mathcal{Y}_T, Q))^k$ and let $f$ be a drift function verifying hypothesis (18), and such that $E_Q \int_0^T |f_t(0,0,0)|^2 dt < \infty$. Then there exists a unique triple $(Y,Z,U) \in \mathcal{B}^2(\mathcal{Y})$ solution of the BSDE:*

$$Y_t = \xi + \int_t^T f_s(Y_s,Z_s,U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\tilde{\mu}(ds,de)$$

We first prove an important lemma for this proof: a martingale representation theorem in our context, under $(\mathcal{Y}, Q)$ :

**Lemma 5.1** *Let $\mathcal{H}$ be a finite-dimensional space and $M_t$ an $\mathcal{H}$-valued $(\mathcal{Y}_t)$-adapted square integrable martingale.*
*Then there exists $Z^i(.) \in L^2(W), i = 1,..,d$ and $U(.,.) \in L^2(\tilde{\mu})$ such that*

$$M_t = M_0 + \int_0^t Z_s^i dW^i(s) + \int_0^t \int_E U(s,e)\tilde{\mu}(ds,de)$$

**Proof of the Lemma:** $\tilde{N}(ds,de) = N(ds,de) - \lambda(de)ds$ is a local martingale. The couple $(W,N)$ is a Brownian-Poisson process couple, and it is an independent increment process (IIP) on space $(\mathcal{F}, P)$. So $(W,N)$ is the same

Brownian-Poisson process IIP in the enlarged space $(\mathcal{Y}, Q)$, from hypothesis $(\mathbf{H_3})$. Then, Jacod and Shiryaev ( [14] th. $III.4.34$) gives us the expected martingale representation theorem for independent increment processes. $\square$
We can now prove the theorem.

**Proof of the theorem:**
For all $(\bar{Y}(.), \bar{Z}(.), \bar{U}(.,.)) \in \mathcal{B}^2(\mathcal{Y})$, we know from the previous lemma that there exists $Z^i(.) \in L^2(W), i = 1,..,d$ and $U(.,.) \in L^2(\tilde{\mu})$ such that:

$$E_Q^{\mathcal{Y}_t}\left[Y_T + \int_0^T f_s(\bar{Y}_s, \bar{Z}_s, \bar{U}_s)ds\right] = \xi + \int_0^t Z_s dW_s + \int_0^t \int_E U_s(e)\tilde{\mu}(ds, de)$$

This implies:

$$\xi = Y_T + \int_0^T f_s(\bar{Y}_s, \bar{Z}_s, \bar{U}_s)ds - \int_0^T Z_s dW_s - \int_0^T \int_E U_s(e)\tilde{\mu}(ds, de)$$

$$\text{We put} \quad Y_t = E_Q^{\mathcal{Y}_t}\left[Y_T + \int_t^T f_s(\bar{Y}_s, \bar{Z}_s, \bar{U}_s)ds\right]$$

We verify then that for each triple $(\bar{Y}(.), \bar{Z}(.), \bar{U}(.,.))$, the triple $(Y(.), Z(.), U(.,.))$ is characterized by the following equation:

$$Y_t = Y_T + \int_t^T f_s(\bar{Y}_s, \bar{Z}_s, \bar{U}_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\tilde{\mu}(ds, de)$$

which implies:

$$Y_t = Y_0 - \int_0^t f_s(\bar{Y}_s, \bar{Z}_s, \bar{U}_s)ds - \int_0^t Z_s dW_s - \int_0^t \int_E U_s(e)\tilde{\mu}(ds, de)$$

The previous equation defines a mapping $\Lambda : (\bar{Y}(.), \bar{Z}(.), \bar{U}(.,.)) \to (Y(.), Z(.), U(.))$.
We introduce, for $k := (Y(.), Z(.), U(.,.)) \in \mathcal{B}^2(\mathcal{Y})$ the norm defined by:

$$\| k \| := \sup_{0\le t\le T} e^{bt} E_Q|Y_t|^2 + \sup_{0\le t\le T} e^{bt}\left[\int_t^T E_Q|Z_s|^2 ds + \int_t^T \int_E E_Q|U_s(e)|^2 \nu(ds, de)\right]$$

with $b > 0$ a constant to be determined later.
To complete the proof, it is sufficient to prove that $\Lambda$ maps $\mathcal{B}^2(\mathcal{Y})$ onto itself, and is a strict contraction for the previous norm. Let $(\bar{Y}_i(.), \bar{Z}_i(.), \bar{U}_i(.,.)) \in \mathcal{B}^2(\mathcal{Y})$ and $(Y_i(.), Z_i(.), U_i(.,.) := \Lambda(\bar{Y}_i(.), \bar{Z}_i(.), \bar{U}(.,.))$ for $i = 1, 2$.
Then, using Itô's formula and equation (18), we obtain:

$$\begin{aligned}
&E_Q|Y_1(t) - Y_2(t)|^2 + E_Q\int_t^T \sum_{i=1}^d |Z_1^i(s) - Z_2^i(s)|^2 ds \\
&+E_Q\int_t^T \int_E |U_1(s,e) - U_2(s,e)|^2 \nu(ds, de) \\
\le\; &\bar{\gamma}K^2 E_Q\int_t^T |Y_1(s) - Y_2(s)|^2 ds + \frac{1}{\bar{\gamma}}\left[E_Q\int_t^T |\bar{Y}_1(s) - \bar{Y}_2(s)|^2 ds\right. \\
&\left.+E_Q\int_t^T \sum_{i=1}^d |\bar{Z}_1^i(s) - \bar{Z}_2^i(s)|^2 ds + E_Q\int_t^T \int_E |\bar{U}_1(s,e) - \bar{U}_2(s,e)|^2 \nu(ds, de)\right]
\end{aligned}$$

Which implies, from Gronwall inequality:

$$
\begin{aligned}
& E_Q|p_1(t)-p_2(t)|^2 + E_Q\int_t^1 \sum_{i=1}^d |q_1^i(s)-q_2^i(s)|^2 ds \\
& + E_Q\int_t^1\int_E |r_1(s,e)-r_2(s,e)|^2\nu(ds,de) \\
\leq\ & \frac{1}{\bar\gamma}\left[E_Q\int_t^1 |\bar p_1(s)-\bar p_2(s)|^2 ds + E_Q\int_t^1\sum_{i=1}^d |\bar q_1^i(s)-\bar q_2^i(s)|^2 ds\right. \\
& \left. + E_Q\int_t^1\int_E |\bar r_1(s,e)-\bar r_2(s,e)|^2\nu(ds,de)\right] \\
& + K^2\int_t^1 e^{\bar\gamma K^2(s-t)}\left[E_Q\int_t^1 |\bar p_1(\tau)-\bar p_2(\tau)|^2 d\tau + E_Q\int_t^1\sum_{i=1}^d |\bar q_1^i(\tau)-\bar q_2^i(\tau)|^2 d\tau\right. \\
& \left. + E_Q\int_t^1\int_E |\bar r_1(\tau,e)-\bar r_2(\tau,e)|^2\nu(d\tau,de)\right]
\end{aligned}
$$

where $\bar\gamma$ is a positive real number. So we conclude:

$$\parallel (Y_1-Y_2, Z_1-Z_2, U_1-U_2) \parallel \leq \alpha \parallel (\bar Y_1-\bar Y_2, \bar Z_1-\bar Z_2, \bar U_1-\bar U_2) \parallel$$

$$\text{with } \alpha = \max\{\frac{2}{b\bar\gamma}, \frac{4K^2}{\bar\gamma b(b-\bar\gamma K^2)}, \frac{2K^2}{b-\bar\gamma K^2}\}$$

which completes the proof, with an appropriate choice of $\bar\gamma$ and $b$ such that the constant $\alpha$ is strictly majored by 1. It means that $\bar\gamma$ and $b$ has to verify $\bar\gamma(1+\bar\gamma/2) < K^{-2}$ and $b > 2/\bar\gamma$. □

Thanks to this theorem, we have a similar result as in the two other models: under $(\mathbf{H_3})$ we have existence and uniqueness of the solution of the enlarged BSDE. Moreover, as before, if the problem is well defined in both spaces, both solutions are the same.

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## Conclusion

Successively in a continuous process model, in a discrete jump process model and finally in a continuous jump process model, we have studied and compared the strategies of an insider trader and a non informed agent. Under certain hypotheses we proved existence and uniqueness of solutions for their hedging strategies, and arbitrage free model for the insider trader. In fact, with correct hypotheses on the information on a complete initial market, the insider market is viable, and even pseudo-complete.

A limit to these models can be raised: we have only considered small investors. It is perhaps not relevant enough. A further work would be to consider an option hedging problem in a jump process model with a large investor. This would lead us to use Forward-Backward stochastic differential equations, instead of BSDEs.

What is the practical use of such results? It seems difficult to concretely apply them at the moment. However such comparison results between insider and non insider investment strategies could be interesting to establish statistical tests for the detection of insider traders. Applied to market datas, it could help organisms like French A.M.F. determining whether an agent is informed or not. Unfortunately, theories are not yet enough performing to compute such tests, and A.M.F.'s monitoring agents do not use so specialized statistical tests.

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