# The Underlying Dynamics of Credit Correlations

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**Abstract**

We explore the downside risks for large homogeneous portfolios where the dynamics of asset returns is governed by one factor GARCH model. We study the properties of long-term aggregate return distributions generated by such processes and relate their characteristics to persistence, asymmetry, tail risks and other stylized features of periodic returns. Following the latent variable modeling assumption, we then discuss the credit correlation structures generated by such dynamic processes, and argue that one must consider a much richer set of copulas than it is often assumed when evaluating portfolio credit risks and synthetic CDO tranches. We introduce the notion of correlation spectrum as a convenient tool to compare default generating models and show that it can be used to price synthetic CDO tranches .

## 1 Introduction

The credit derivatives market, which exceeds \$4 trillion according to most recent estimates from the British Bankers' Association [2], encompasses a wide range of instruments, from plain vanilla credit default swaps, to credit swaptions, portfolio CDS, and synthetic CDO tranches. Among these, the portfolio credit default swaps (PCDS, also frequently called index swaps) and structured products based on them have seen a particularly impressive growth, especially after their consolidation into the Dow Jones CDX (North America) and iTraxx (Europe) families of indexes backed by all major broker-dealers. The introduction of CDX and iTraxx index swaps allowed market participants to gain a long or short exposure to performance of the broad segments of corporate credit markets and became the most heavily traded securities in the U.S. credit market. Of course, such trading activity provides a fertile ground for development of derivatives linked to these instruments. While in the equity markets the most liquid derivatives are the short-term exchange traded put and call options on

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index futures and ETFs, in the credit markets the so-called correlation products (i.e. synthetic CDO tranches) are far more popular.

As explained in [5] and [6] the synthetic CDO tranches and other correlation products provide an efficient way of leveraging the spread to expected loss rate ratio of the underlying collateral portfolio and are often considered an attractive alternative to unlevered credit investments such as cash bonds with a similar expected loss rate. However, such an evaluation presumes that it is sufficient to compare the mean returns and loss rates of investment alternatives to obtain a reliable relative value estimate. Moreover, it puts a very heavy burden on the accuracy of the loss estimates – while the estimate of the mean return requires one to characterize the shape of the much better known middle part of the return distribution, the estimates of the long-term loss rates rely on the knowledge of the shape of the far tails of the return distribution.

The latter problem is not easy, and is not new either — it is the same problem that risk managers face when estimating an investment portfolio's value-at-risk. This is why the same mathematical tools that proved to be useful in risk management applications also found a wide acceptance in structured credit modeling. There has been a large body of work both in the industry and in academia formalizing the definition of downside portfolio risk beyond the simple mean-variance return assumption. In particular, the copula-based methods received widespread attention thanks to their ability to directly confront the issue of the dependence structure of assets returns separately from the equally important issue of the return distribution of individual assets.

In the market risk management setting, this approach has been advanced by Embrechts *et. al.* [7, 8], who demonstrate the inadequacy of conventional correlation measures for long-term downside portfolio risk measurement. In the actuarial setting, Frees and Valdez [9] have introduced the copula-based methodology as an efficient approach to estimation of long-term risks of losses in insurance portfolios.

In the credit portfolio risk setting, Li [10] has first introduced the copula-based approach by noting in part the similarities between the credit risk modeling and insurance loss modeling [9]. In his original model Li [10] considered a particularly simple Gaussian copula dependence structure of times to default and demonstrated that it is equivalent to the CreditMetrics [4] portfolio loss distribution assumptions which in turn were derived from an adaptation of the Merton's classic structural model of credit risk [1]. In this model, the multi-variate probability distribution of times to default is generated as a transformation of the multi-variate probability distribution of asset returns of portfolio constituents.

The observation that using the Gaussian copula model, which has become the de-facto market standard, is in principle equivalent to using a version of Merton's original model is under-appreciated by many researchers. With this implicit use of Merton's model also come certain well-known drawbacks such as the insufficient probabilities of downside risks for investment grade issuers in the near- and intermediate terms. From the econometric perspective, the main drawbacks of the classic Merton model are its inability to account for a number of well established stylized facts regarding the time series properties of observed equity returns, such as the stochasticity and persistence of volatilities, asymmetry of volatility response to returns well beyond the levels that can be explained by the simple leverage effect, and the presence of fat tails and other non-Gaussian features.

The adaptation of the copula-based methodology to reduced-form models of default risk [11], and its re-interpretation in terms of generic latent variable models [12] have opened the possibility to reconcile the parsimony of the copula methodology with more flexible models of single-name credit risk. In particular, one no longer has to explicitly assume that the latent variable driving the times to default generation is log-normally distributed. We will refer to this approach combining the modeling of the dependence structure via copulas based on latent variable models, with the modeling of the single-name (i.e. marginal) credit loss distributions via reduced-form models, as the hybrid framework. A textbook treatment of the copula approach to credit portfolio risk can be found in [15], [16]. The mathematical introduction to copulas is available in [19], and a more general overview of copula-based modeling in finance covering various markets and applications beyond credit can be found in [18].

Among the important steps towards more realistic modeling of the dependence structure of portfolio risks within the hybrid framework are the multi-factor Gaussian copula models [13], the extension to non-Gaussian copulas and in particular to Student-t copulas [14] reflecting the fat-tailed distribution of asset returns, and the explicit modeling of asymmetric latent variable distributions in [17]. While all of these models can be efficiently implemented in full generality with explicit treatment of each issuer's risk within the portfolio, it is often useful to focus on the most relevant aspects of the models and understand the dependencies of the portfolio risk on the few parameters describing the systematic and idiosyncratic credit loss risks. When the dominant risks are the systematic ones, a very useful approximation is the large homogeneous portfolio (LHP) limit dating back to the original paper by Vasicek (see a recent review in [20] and references therein).

The latest advances in the credit correlation modeling were in part motivated by the growth and sophistication of the so called correlation trading strategies, namely strategies involving various standardized tranches based on the broad market indices such as CDX and iTraxx. The synthetic CDO market in general, and the standard tranche market in particular, allows investors to take rather specific views on the shape of the credit loss distribution of the underlying diversified collateral portfolio. The investor's views on various slices of this distribution is now well exposed through the pricing of liquid standard tranches, which in turn is expressed through the notion of base correlations (see [21] and [6] for more details). The base correlation concept has proven useful because it allowed to map the pricing function of the standard tranches which was a function of two variables (attachment and detachment points) to a one-dimensional pricing function of the base equity tranches which only depends on the detachment point. This mapping is similar to a mapping of the bull/bear spread options with various lower and upper limits onto a sequence of call options with various strikes – with the base equity tranches being analogous to a call option on the survival of the portfolio, and the generic mezzanine tranches being analogous to bull spread options on survival.

A lot of intuition about the shapes of the base correlation can be gained simply by noting that, given a certain level of underlying index spreads, the higher the attachment point K, the farther out-of-money is the senior tranche (i.e. the tranche which is exposed to losses above the K up to 1). In the case of the equity index options the out-of-money put options are typically priced with a higher level of implied volatility which corresponds to a much fatter downside tails of the implied return distribution. Similarly, the senior synthetic CDO tranches are typically prices at a higher level of implied base correlation which corresponds to fatter downside tails of default loss distributions (compare figures 1 and 2, where we have drawn the correlation skew graph in somewhat unusual way, by placing the farther out-of-money top tranches to the left of x-axis to emphasize the similarity with put options).

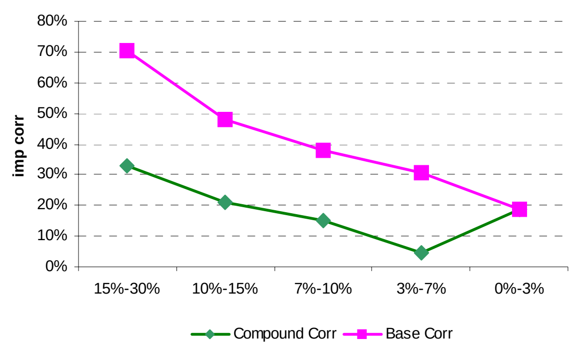



Figure 1: Compounded and base correlation skew for CDX.NA.IG series 3 (March 2005)

Such pricing is commonly attributed to investors' risk aversion to large loss scenarios and correspondingly higher risk premia demanded for securities exposing them to such scenarios. However, we believe that it would be unfair to think of the entire cost premium between various in- and out-of-money tranches as risk premium and that there are real risks which are being compensated by these additional costs, albeit perhaps still accompanied by (relatively smaller) risk premia. To justify this line of thought let us return for a moment to the case of equity index options and recall that the empirical distribution of returns does indeed exhibit significant downside tails, and that a large part of the implied volatility skew can be explained by the properties of the empirical distribution [27].

Let us list some of the key stylized facts that are known to be relevant for explanation of the equity index option pricing:

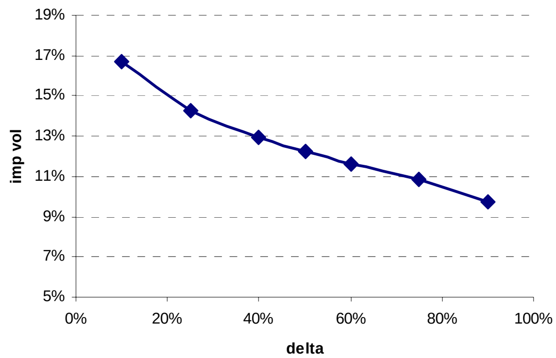



Figure 2: Implied volatility skew for SP500 index options with 1 year expiry (March 2005)

- The fat tails in the return distribution can explain the implied volatility smile.
- The asymmetry in the return distribution is a necessary ingredient for explaining the implied volatility skew.
- The simple leverage effect is typically insufficient to explain the level of the implied skew.
- There exists an implied volatility surface with non-trivial strike and term structure.
- The term dependence of the volatility surface is determined by the long-run aggregated return distribution characteristics which can be significantly different from those the short-term distribution.
- The implied volatility surface has a much more pronounced skew for stock indexes than for individual stocks, reflecting a more important role of the common driving factors compared to idiosyncratic returns in the explanation of the downside risks.

Given the above mentioned analogies between the synthetic CDO tranches and equity index options, it is quite natural to look for similar stylized facts that could explain the nature of the implied base correlation skew and its key dependencies. As we already noted, the standard Gaussian copula framework implicitly relies on the Merton-style structural model for definition of the default correlations. Therefore, if we are to give empirical explanation to the observed base correlation skew we must start by giving an empirical meaning to the parameters of this model. Our working hypothesis in this paper will be that the meaning of the "market factor" in the factor copula framework is the same as the market factor used in equity return modeling. As such, it is often possible to use an observable broad market index such as S&P 500 as a proxy for the economic market factor, with an added convenience that there exists a rich set of equity options data for such indexes from which one can glean an independent information about their return distribution.

This hypothesis is not uncommon in portfolio credit risk modeling – for example, the authors of the paper [14] emphasized the importance of using a fat-tailed distribution of the asset returns in the copula framework in part citing the empirical evidence from equity markets. However most researchers have focused on the single-period return distribution characteristics. In section 2 we argue that the empirical characteristics of the long-run aggregate return distribution are quite distinct from those of the short-term (single-period) returns, and that for sufficiently long horizons (greater than several months) the effects of the stochastic volatility and volatility asymmetry dominate the effects of non-normality of single-period return shocks. Assuming a factor-GARCH model of single-period returns, will derive formulas for the skewness and kurtosis of the aggregated return distributions for a variety of specifications of the single period return process. We then demonstrate that the empirically estimated parameters of the market factor time series do indeed lead to non-Gaussian distribution of aggregated returns for horizons up to 5 or even 10 years.

In section 2.2 we show that there is a simple relationship between the lower tail dependence and pairwise default correlations the latent variable framework, and that our methodology leads to a significantly different dependence of both measures on the risk threshold compared to the previously studied copula models. Both the lower tail dependence and the pairwise default correlations are shown to increase at very low thresholds which is precisely a behavior that would be expected of any model that aims to explain the steep correlation skew growing toward higher attachment points (i.e. lower default thresholds).

In section 3 we lay the groundwork for extending our analysis to portfolio credit risk models by giving a brief introduction to general copula framework, pricing methodology for synthetic CDO tranches, and the large homogeneous portfolio approximation which we adopt in the rest of the paper. The section 3.2 presents the application of our model to pricing of synthetic CDO tranches. In particular, we introduce a notion of the correlation spectrum which both simplifies and extends the definition of base correlations to a framework suitable for comparison of various default loss generating models. We then use the computed correlation spectra for time series models considered in section 4 to study the impact of various stylized characteristics of market factor and idiosyncratic return dynamics on the portfolio credit risk. We are able to discriminate between time series model specification and practically rule out those models which do not have an asymmetric volatility process. We demonstrate that the empirical parameters correspond to a substantial skew, thus confirming that a large portion of the synthetic CDO tranche pricing reflects real risks and not just risk premia. We conclude the section by examining the dependence of the base correlations as function of term to maturity and the level of hazard rates.

The section 5 presents a brief summary and outline of remaining open questions and possible extensions of our methodology. The Appendices present additional proofs and empirical details.

## 2 Time series models of short and long horizon equity returns

A lot of empirical research has been done on time series properties of equity returns and their implications for risk management and option pricing. The levels of equity returns for daily and weekly frequencies have been shown to have little or no predictability. Volatilities and covariances (conditional or latent) of the stock returns on the other hand are predictable and very persistent. Moreover, volatility dynamics is asymmetric (also referred to as "leverage effect") and rises more in response to bad news (negative returns) than good news (positive returns). The pairwise correlations of stock returns vary with time and are often asymmetric: stocks are more correlated when the market goes down.

In many applications we first specify time series properties of stock returns in continuous time framework or for discrete but relatively high frequency time intervals (daily and weekly) and then look at the distribution of stock price changes over longer horizons measured in months or even years. The popular log normal model for example assumes constant mean returns and volatilities, iid Gaussian shocks and in the context of option pricing (Merton and Black-Scholes) the time horizon is associated with the maturity of the option. In this model the distribution of the log stock price changes is Gaussian for any time horizon. More realistic dynamics, such as stochastic volatility and GARCH type of models with persistent and asymmetric volatilities, imply that the distribution[1] of time aggregated returns has fat tails and negative skewness even if we assume Gaussian distribution for the return innovations. The term structure of fat-tailness and skewness of aggregated returns depends on the parametric form chosen for the volatility process.

Volatility and correlation dynamics affect not only the marginal distributions of stock returns but also the distribution of stock co-movements over long horizons or more generally the copula of long horizon returns. The log normal model implies a Gaussian copula for any time horizon whereas multivariate models with more realistic dynamic properties, such as dynamic volatilities and correlations, result in non-Gaussian copulas. In this paper we focus on two non-Gaussian features of long horizon return copulas: tail dependence and asymmetry. In this section we describe a simple one factor model with TARCH(1,1) dynamics that allows us to incorporate persistence and asymmetry in volatility and correlations and yet is tractable enough to derive qualitative and quantitative results for non-Gaussian properties of long horizon return distribution copula.

We start by describing the univariate TARCH(1,1) model and then formulate a multivariate model with TARCH(1,1) dynamics. Let $r_t$ be log return of a

[1] both conditional and unconditional

particular stock or an index such as SP500 from time $t-1$ to time $t$ . $\mathcal{F}_t$ denotes the information set containing realized values of all the relevant variables up to time $t$. We will use the expectation sign with subscript $t$ to denote the expectation conditional on time $t$ information set: $E_t(.) = E(.|\mathcal{F}_t)$ . The time step that we use in the empirical part is 1 day or 1 week. As we already mentioned, predictability of stock returns is negligible over such time horizons and therefore we assume the conditional mean is constant and equal to zero[2]:

$$m_t \equiv E_{t-1}(r_t) = 0$$

## 2.1 Univariate model: TARCH(1,1)

Conditional volatility $\sigma_t^2 \equiv E(r_t^2|\mathcal{F}_{t-1})$ of $r_t$ in TARCH(1,1) has the autoregressive functional form similar to GARCH(1,1) but with an additional asymmetric term

$$\begin{aligned} r_t &= \sigma_t \varepsilon_t \\ \sigma_t^2 &= \omega + \alpha r_{t-1}^2 + \alpha_D r_{t-1}^2 1_{\{r_{t-1} \leq 0\}} + \beta \sigma_{t-1}^2 \end{aligned} \tag{1}$$

We assume that $\{\varepsilon_t\}$ are iid with zero mean, unit variance, finite skewness $s_\varepsilon$ and kurtosis $k_\varepsilon$.We also assume that $\omega > 0$ and $\alpha, \alpha_D, \beta$ are non-negative so that the conditional variance $\sigma_t^2$ is guaranteed to be positive. of The persistence in the model is governed by $\rho$ :

$$\rho \equiv E\left(\beta + \alpha\varepsilon_t^2 + \alpha_D \varepsilon_t^2 1_{\{\varepsilon_t \leq 0\}}\right) = \beta + \alpha + \alpha_D v_\varepsilon^d \tag{2}$$

where $v_\varepsilon^d \equiv E(\varepsilon_t^2 1_{\{\varepsilon_t \leq 0\}})$[3]. Let us introduce the notations for the moments of $\varepsilon_t$ that will be used in some of the formulas.

$$\begin{aligned} m_\varepsilon &\equiv E(\varepsilon_t) = 0 \\ v_\varepsilon &\equiv E\left(\varepsilon_t^2\right) = 1 \\ v_\varepsilon^d &\equiv E\left(\varepsilon_t^2 1_{\{\varepsilon_t \leq 0\}}\right) \\ s_\varepsilon &\equiv E\left(\varepsilon_t^3\right) \\ s_\varepsilon^d &\equiv E\left(\varepsilon_t^3 1_{\{\varepsilon_t \leq 0\}}\right) \\ k_\varepsilon &\equiv E\left(\varepsilon_t^4\right) \\ k_\varepsilon^d &\equiv E\left(\varepsilon_t^4 1_{\{\varepsilon_t \leq 0\}}\right) \end{aligned}$$

We can conveniently rewrite 1 in terms of the increments of the conditional volatility $\Delta\sigma_{t+1}^2 \equiv \sigma_{t+1}^2 - \sigma_t^2$ and the volatility shocks $\eta_t \equiv \alpha\left(\varepsilon_t^2 - 1\right) + \alpha_D\left(\varepsilon_t^2 1_{\{\varepsilon_t \leq 0\}} - v_\varepsilon^d\right)$

$$\begin{aligned} r_t &= \sigma_t \varepsilon_t \\ \Delta\sigma_{t+1}^2 &= (1-\rho)\left(\sigma^2 - \sigma_t^2\right) + \sigma_t^2 \eta_t \end{aligned} \tag{3}$$

[2] for derivatives pricing we will discuss the "risk-neutralization" of the return process.
[3] Note that for $\varepsilon_t^2$ with symmetric distribution $v_\varepsilon^d = 0.5$.

Volatility shocks $\eta_t$ are iid, zero mean and have variance $var(\eta_t)^4$. If $\rho \in [0, 1)$ then conditional variance mean-reverts to its unconditional level $\sigma^2 = E\left(\sigma_t^2\right) = \frac{\omega}{1-\rho}$

$$E_{t-1}\sigma_{t+i}^2 = \sigma^2 + \rho^i\left(\sigma_t^2 - \sigma^2\right) \text{ for } i \geq 0 \tag{4}$$

The speed of mean reversion in volatility is $1-\rho$ and is small when $\rho$ is close to one which is true for daily and weekly equity returns. Persistent and volatile volatility generate fat tails in the unconditional return distribution even for models with Gaussian shocks. It is easy to see from 3 that conditional volatility of $\sigma_{t+1}^2$ is proportional to $\sigma_t^4$ and $var(\eta_t)$

$$var_{t-1}\left(\sigma_{t+1}^2\right) = var_{t-1}\left(\sigma_t^2\eta_t\right) = \sigma_t^4 var(\eta_t) \tag{5}$$

The correlation of returns and future conditional volatility depends on covariance of return and volatility innovations

$$corr_{t-1}\left(r_t, \sigma_{t+1}^2\right) = corr_{t-1}\left(\varepsilon_t, \eta_t\right) = \frac{\alpha s_\varepsilon + \alpha_D s_\varepsilon^d}{\sqrt{var(\eta_t)}} \tag{6}$$

The negative correlation of return and volatility shocks or so-called "leverage effect" is the main source of asymmetry in the return distribution. We can see from formula 6 that negative return-volatility correlation can be achieved either through negative skewness of return innovations $s_\varepsilon < 0$, through asymmetry in volatility process $\alpha_D > 0$ or combination of the two. We call these static and dynamic asymmetry respectively.

In this paper we are interested in the effects of the volatility dynamics on the distribution of long horizon returns. The closed form solution for the probability density function of aggregated returns is not available for TARCH(1,1) but we can still derive some analytical results for the conditional and unconditional moments: volatility, skewness and kurtosis.

#### 2.1.1 Volatility

The conditional variance $V_{t+1,t+T}$ of the normalized log return $R_{t+1,t+T} = \frac{1}{\sqrt{T}}\left(\ln S_{t+T} - \ln S_t\right) = \frac{1}{\sqrt{T}}\sum_{u=t+1}^{t+T} r_u$ from t+1 to t+T can be easily computed using 4

$$V_{t+1,t+T} = E_t R_{t+1,t+T}^2 = \frac{1}{T}E_t\left(\sum_{t+1 \leq u \leq t+T}\sigma_u^2\right) = \sigma^2 + \left(\sigma_{t+1}^2 - \sigma^2\right)\frac{1}{T}\frac{1-\rho^T}{1-\rho}$$

The unconditional volatility is therefore $\sigma^2$

$$V_T = E(V_{t+1,t+T}) = \sigma^2$$

[4] $var(\eta_t) = var(\alpha\varepsilon_t^2 + \alpha_D\varepsilon_t^2 1_{\{\varepsilon_t \leq 0\}}) = (\alpha + \alpha_D\gamma)^2 k_\varepsilon + \alpha_D^2(1-\gamma)\gamma(k_\varepsilon + 1)$

The deviation of the T-horizon conditional volatility $V_{t+1,t+T}$ from its unconditional level $\sigma^2$ depends on the current deviation of the short horizon volatility $\sigma^2_{t+1} - \sigma^2$, aggregation horizon T and the level of volatility persistence $\rho$.

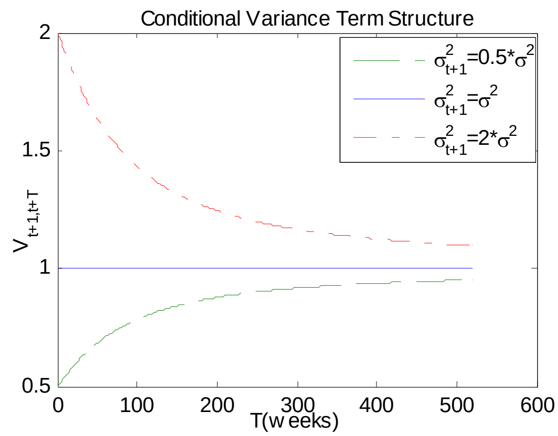



Figure 3: Conditional Volatility Term Structure of time aggregated return $R_{t+1,t+T}$.TARCH(1,1) has persistance coefficient $\rho = 0.98$ and the followng parametrization: $\sigma^2 = 1$, $\alpha = 0.01$, $\alpha_D = 0.10$, $\beta = 0.92$, $\varepsilon_t \sim N(0,1)$.We plotted volatility term structure for three different initial volatilities: $\sigma^2/2$, $\sigma^2$and $2\sigma^2$

### 2.1.2 Skewness

Skewness is a convenient measure of return distribution asymmetry. The following proposition gives the formulas for conditional and unconditional third moments of aggregated returns generated by the TARCH(1,1) model.

**Proposition 1** *Suppose $0 \leq \rho < 1$ and the return innovations have finite skewness, $s_\varepsilon$, and finite "truncated" third moment, $s^d_\varepsilon$, then conditional third moment of $R_{t+1,t+T}$ has the following representation for TARCH(1,1)*

$$E_t R^3_{t+1,t+T} = \frac{1}{T^{3/2}} s_\varepsilon \sum_{u=1}^{T} E_t\left(\sigma^3_{t+u}\right) + \frac{3}{T^{3/2}}\left(\alpha s_\varepsilon + \alpha_D s^d_\varepsilon\right) \sum_{u=1}^{T} \frac{1-\rho^{T-u}}{1-\rho} E_t\left(\sigma^3_{t+u}\right) \tag{7}$$

*In addition if $E\sigma^3_t$ is finite then unconditional skewness of $R_{t+1,t+T}$ is given by*

$$S_T \equiv \frac{ER^3_{t+1,t+T}}{E(R^2_{t+1,t+T})^{3/2}} = \left[\frac{1}{T^{1/2}} s_\varepsilon + 3\frac{1}{T^{3/2}}\left(\alpha s_\varepsilon + \alpha_D s^d_\varepsilon\right) \frac{T(1-\rho) - 1 + \rho^T}{(1-\rho)^2}\right] E\left(\frac{\sigma_t}{\sigma}\right)^3 \tag{8}$$

***Proof.*** *See appendix for the details.* ■

The conditional third moment is a function of the conditional term structure of $\sigma_t^3$, term horizon $T$ and volatility parameters. The conditional skewness can be computed using second and third conditional moments derived above. The asymmetry in the return distribution arises from two sources - skewness of return innovations and asymmetry of the volatility process. Note that the second term in the formulas for conditional and unconditional skewness is directly related to the correlation of return and volatility innovations. If return-volatility correlation is zero ($\alpha s_\varepsilon + \alpha_D s_\varepsilon^d = 0$) then $S_T = \frac{1}{T^{1/2}} s_\varepsilon E\left(\frac{\sigma_t}{\sigma}\right)^3$. If return innovations are symmetric then asymmetric volatility component drives the asymmetry in return distribution. On figure 4 we show conditional and unconditional skewness term structures. For realistic parameters corresponding approximately to parameters of the TARCH(1,1) estimated for weekly SP500 log returns skewness, both conditional and unconditional, is negative, decreases in the medium term, attains the minimum at approximately 2 year point and then decays to zero as T increases. The skewness term structure conditional on the high/low current volatility is above/below the unconditional skewness.

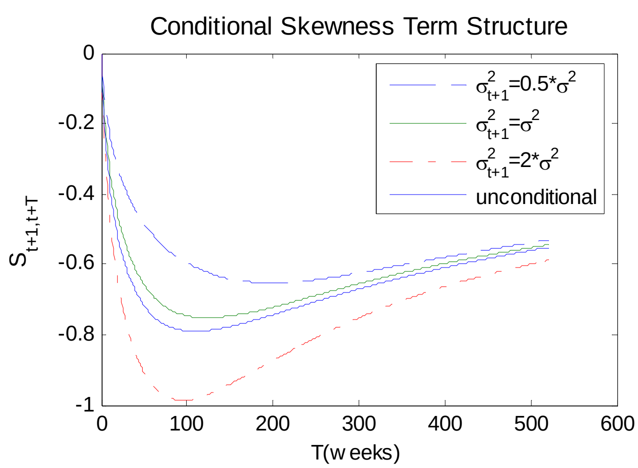



Figure 4: Conditional Skewness Term Structure of time aggregated return $R_{t+1,t+T}$.TARCH(1,1) has persistance coefficient $\rho = 0.98$ and the followng parametrization: $\sigma^2 = 1$, $\alpha = 0.01$, $\alpha_D = 0.10$, $\beta = 0.92$, $\varepsilon_t \sim N(0,1)$.We plotted unconditional skewness term structure and conditional for three different initial volatilities: $\sigma^2/2$, $\sigma^2$and $2\sigma^2$.The term structure of $E_t\sigma_{t+u}^3$was computed from 10,000 independent simulations.

### 2.1.3 Kurtosis

The fourth conditional moment, if exists, describes the fat-tailness of the conditional return distribution and volatility of the return volatility. Formula 5 gives us the conditional volatility of the conditional volatility. The fourth conditional moment of the one period return is proportional to the kurtosis of the return innovation

$$E_t r_{t+1}^4 = \sigma_{t+1}^4 k_\varepsilon$$

For symmetric return shocks and symmetric volatility dynamics ($s_\varepsilon = 0$ and $\alpha_D = 0$) $K_T$ has a simple representation in terms of the model parameters

**Proposition 2** *If the distribution of $\varepsilon_t$ is symmetric and $\alpha_D = 0$ then unconditional kurtosis of $R_T$, if exists, is given by the following formula:*

$$K_T = 3 + \frac{1}{T}(K_1 - 3) + 6\frac{\gamma_1}{T^2}\frac{T(1-\rho) - 1 + \rho^T}{(1-\rho)^2} \textit{for } T > 1 \tag{9}$$

$$K_1 = k_\varepsilon \frac{1-\rho^2}{1-\gamma} \tag{10}$$

*where $k_\varepsilon$ is unconditional kurtosis of $\varepsilon_t$ and*

$$\gamma \equiv E\left(\beta + \alpha\varepsilon_t^2 + \alpha_D \varepsilon_t^2 1_{\{\varepsilon_t \le 0\}}\right)^2 = \beta^2 + \alpha^2 k_\varepsilon + \alpha_D^2 k_\varepsilon^d + 2\alpha\beta + 2\alpha_D \beta v_\varepsilon^d + 2\alpha\alpha_D k_\varepsilon^d.$$
$$\gamma_1 \equiv corr\left(r_{t-1}^2, r_t^2\right) = \alpha\left(k_r - 1\right) + \alpha_D\left(k_r^d - v_r^d\right) + \beta k_r / k_\varepsilon$$

***Proof.*** *See appendix for the details.* ■

## 2.2 Multivariate model: One factor ARCH with TARCH(1,1) volatility dynamics

In this section we describe a simple dynamic factor model for the stock returns of M companies. To concentrate on time dimension of the model we make some homogeneity assumptions for cross-sectional return properties. We assume that factor loadings and volatilities of idiosyncratic terms are constant and the same for all stocks. A homogeneous one factor ARCH model has the following form.

$$\begin{aligned} r_{i,t} &= b\sigma_{m,t}\varepsilon_{m,t} + \sigma\varepsilon_{i,t} \\ \Delta\sigma_{t+1}^2 &= (1-\rho_m)\left(\sigma_m^2 - \sigma_{m,t}^2\right) + \sigma_{m,t}^2 \eta_{m,t} \end{aligned} \tag{11}$$

where

- $b \geq 0$ is the market factor loading and it is the same for all stocks
- $r_{m,t}$ is the market factor with conditional volatility $\sigma_{m,t}^2 \equiv E_{t-1}(r_{m,t}^2)$ and zero conditional mean $E_{t-1}(r_{m,t}) = 0$
- $\sigma\varepsilon_{i,t}$ are idiosyncratic return components with conditional volatilities $\sigma^2$ and zero conditional means $E_{t-1}(\sigma\varepsilon_{i,t}) = 0$
- $\{\varepsilon_{i,t}, \varepsilon_{m,t}\}$ are zero mean shocks with unit variance, mutually independent (for each $i$, $m$ and $t$) and identically distributed for each given $i$ and $m$

In this model conditional volatilities and pairwise conditional correlations of stock returns are time varying and depend on volatility dynamics of market return only.

$$\sigma_{i,t}^2 \equiv Var_{t-1}(r_{i,t}^2) = b^2\sigma_{m,t}^2 + \sigma^2 \tag{12}$$

$$\rho_{(i,j),t} = \frac{Cov_{t-1}(r_{i,t}, r_{j,t})}{\sqrt{\sigma_{i,t}^2\sigma_{j,t}^2}} = \frac{b^2\sigma_{m,t}^2}{\sigma^2 + b^2\sigma_{m,t}^2} \tag{13}$$

Unconditional correlation is $\rho_{(i,j)} = \frac{b^2\sigma_{m,t}^2}{\sigma^2+b^2\sigma_{m,t}^2}$. Pairwise correlation $\rho_{(i,j),t}$ is strictly increasing function of market volatility $\sigma_{m,t}^2$ if $b > 0$ and therefore persistence and asymmetry of the market volatility $\sigma_{m,t+1}^2$ translates to the persistence and dynamic asymmetry of the correlation $\rho_{(i,j),t}$. Because of the simple linear factor structure and constant market loadings time aggregated equity returns $R_{i,T} = \frac{1}{\sqrt{T}}\sum_{u=1}^{T} r_{i,u}$ have one factor representation[5]

$$R_{i,T} = bR_{m,T} + E_{i,T} \tag{14}$$

where $R_{m,T} = \frac{1}{\sqrt{T}}\sum_{u=1}^{T} r_{m,u}$ and $E_{i,T} = \frac{1}{\sqrt{T}}\sigma\sum_{u=1}^{T}\varepsilon_{i,u}$ are independent conditional on $\mathcal{F}_0$. The conditional variance, skewness and kurtosis of $R_{i,T}$ can be easily computed in terms of the corresponding moments of market and idiosyncratic returns.

$$\begin{aligned}
V_{i,T} &= E_0\left(R_{i,T}^2\right) = b^2V_{m,T} + \sigma^2 \\
\Gamma_{(i,j),T} &= corr(R_{i,T}, R_{j,T}|\mathcal{F}_0) = \frac{b^2V_{m,T}}{b^2V_{m,T} + \sigma^2} \\
S_{i,T} &= \frac{E_0\left(R_{i,T}^3\right)}{V_{i,T}^{3/2}} = \Gamma_{(i,j),T}^{3/2}S_{m,T} + \left(1 - \Gamma_{(i,j),T}\right)^{3/2} S_{E,T} \\
K_{i,T} &= \Gamma_{(i,j),T}^2 K_{m,T} + 6\Gamma_{(i,j),T}\left(1 - \Gamma_{(i,j),T}\right) + \left(1 - \Gamma_{(i,j),T}\right)^2 K_{E,T}
\end{aligned}$$

## 2.3 SP500 as a proxy for market return

In this section we discuss estimation results of several TARCH(1,1) specifications for SP500 daily and weekly returns. Value-weighted daily levels of SP500 were downloaded from CRSP. Total number of observations is 10699 and covers period from 07/02/1962 till 12/31/2004. We constructed daily and weekly log returns and estimated parameters of TARCH and GARCH models with Gaussian ,Student-t and Generalized Error Distribution(GED) shocks for 2 samples

[5] To simplify the notations we assume that the initial time $t = 0$ and use only subscipt for the time aggregation horizon $T$.

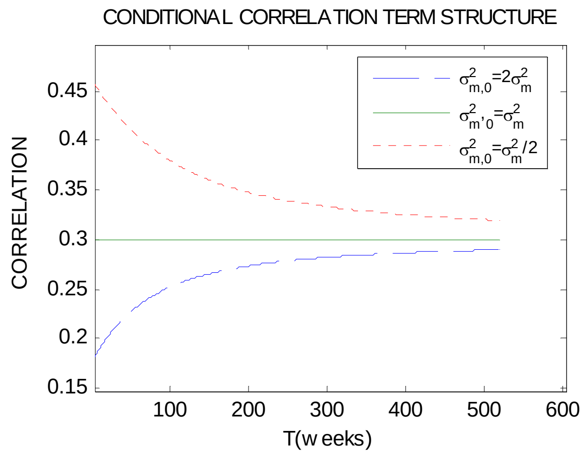



Figure 5: Conditional Correlation Term Structure of time aggregated returns $R_{i,T}$ and $R_{j,T}$.TARCH(1,1) has persistance coefficient $\rho_m = 0.98$ and the followng parametrization: $\sigma_m^2 = 1$, $\alpha_m = 0.01$, $\alpha_{m,D} = 0.10$, $\beta_m = 0.92$, $\varepsilon_{m,t} \sim N(0,1)$.We plotted conditional correlation term structure for three different initial volatilities: $\sigma_m^2/2$, $\sigma_m^2$and $2\sigma_m^2$.

- full and post 1990. Student-t and GED distributions have an additional parameter that adjusts the fat-tails of the error distribution[6]. Tables 1,2,3 in the appendix show estimated parameters and various data statistics. The asymmetric coefficient $\alpha_D$ in the TARCH model is significantly higher than symmetric one for both samples and both daily and weekly frequencies and all types of shock distributions. In Figure 6 we calculated rolling estimate of skewness for returns of different aggregation horizons measured in days. The full sample shows high negative skewness for one day return because of the 1987 crash. On the post 1990 sample negative skewness rises with aggregation horizon up to 1 year and then slowly decays toward zero. Both samples show significant skewness for horizons of several years. We should note that confidence bounds around skewness curves are quite wide due to the persistency and high volatility of the squared returns and serial correlation of the rolling window observations.

To make sure that asymmetry in volatility is not a result of several extreme negative returns like 1987 crash we provide data statistics and re-estimated parameters of TARCH models for trimmed full and post 1990 samples[7]. We can

[6] The Student-T distribution sometimes critiqued as a model for continously componded returns because the expectation of the exponent of Student-T variable is infinite and therefore expected return over one period is also infinite. In practice(estimation) we can think of Studen-T distribution being trancated at far enough tails so that estimation procedure is not changed. This makes the expectation of the exponent finite.

[7] The trimming is done by cutting excess volatilty in extreme 0.5% both positive and negative return observations.

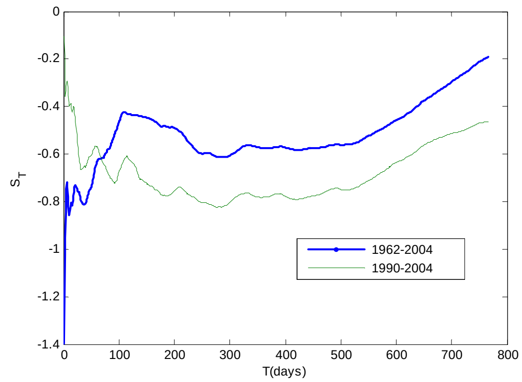



Figure 6: Skewness of SP500

see from Tables 1b and 2b that trimming significantly reduced skewness,$s_r$ ,of daily returns but the volatility of daily returns is still significantly asymmetric. Weekly returns are less affected by trimming both in terms of TARCH parameters and unconditional skewness and kurtosis.

# 3 Modeling Tail Risk and Default Correlation

The dynamic models of aggregate equity returns presented in the previous section can serve as important ingredient for the modeling of tail risks and default correlations. In this paper we are interested in the effects of the return dynamics on the joint distribution of $\mathbf{R}_T = [R_{1,T}, ..., R_{K,T}]'$[8]. Denote

- $F_T(d_i) \equiv P(R_{i,T} \leq d_i | \mathcal{F}_0)$ conditional cdf of $R_{i,T}$
- $G_T(d_i) \equiv P(E_{i,T} \leq d_i | \mathcal{F}_0)$ conditional cdf of $E_{i,T}$
- $F_T(\mathbf{d}) \equiv P(\mathbf{R}_T \leq \mathbf{d} | \mathcal{F}_0)$ joint conditional cdf of $\mathbf{R}_T$
- $C_T(\mathbf{u}) \equiv F_T\left(F_T^{-1}(u_1), ..., F_T^{-1}(u_M)\right)$ conditional copula of $\mathbf{R}_T$

Note than the assumption of one factor structure implies that equity returns $\mathbf{R}_T$ are independent conditional on the market return $R_{m,T}$ and therefore $F_T(\mathbf{d})$ can be computed as expectation of the product of conditional cdfs e.g. for unconditional[9] distribution:

$$\bar{F}_T(\mathbf{d}) = E\left(\prod_{i=1}^{M} \bar{P}(R_{i,T} \leq d_i | R_{m,T})\right) = E\left(\prod_{i=1}^{M} \bar{G}_T(d_i - b_i R_{m,T})\right) \tag{15}$$

[8]the bold letters denote N dimentional vectors e.g. $\mathbf{x} \equiv [x_1, ..., x_N]'$ .

[9]for unconditional distributions and copulas we use the same notations but with bar above the corresonding letter e.g $\bar{F}_T$

where $R_{m,T}$ and $\{E_{i,T}\}_{i=1,M}$ are independent aggregated market and idiosyncratic return components.

Tail dependence coefficient and "default correlation" coefficient are convenient measures of the risk of joint extreme movements for a pair of assets. Suppose $R_{i,T}$ and $R_{j,T}$ are the stock returns for companies $i$ and $j$ over the $[0,T]$ time horizon. The coefficient of lower tail dependence and default correlation coefficient for two random variables with the same continuous cdfs, $F_T(R)$, are defined as

$$\lambda_{i,j}^{D} = \lim_{p\to+0} P\left(R_{i,T} \le d_p | R_{j,T} \le d_p\right) = \lim_{p\to+0} \frac{E\left(1_{\{R_{i,T}\le d_p\}} 1_{\{R_{j,T}\le d_p\}}\right)}{p} \tag{16}$$

$$\rho_{i,j}^{D}(p) = corr(1_{\{R_{i,T}\le d_p\}}, 1_{\{R_{j,T}\le d_p\}}) = \frac{E\left(1_{\{R_{i,T}\le d_p\}} 1_{\{R_{j,T}\le d_p\}}\right) - p^2}{(1-p)p} \tag{17}$$

where $d_p = F_T^{-1}(p)$.Both measures depend only on the bivariate copula of the two random variables

$$\lambda_{i,j}^{D} = \lim_{\alpha\to+0} \frac{P\left(R_{i,T} \le d_p, R_{j,T} \le d_p\right)}{p} = \lim_{p\to+0} \frac{C_T(p,p)}{p}$$
$$\rho_{i,j}^{D}(p) = \frac{P\left(R_{i,T} \le d_p, R_{j,T} \le d_p\right) - p^2}{(1-p)p} = \frac{C_T(p,p) - p^2}{(1-p)p}$$

It is easy to see that asymptotically both measures are equivalent $\lim_{p\to+0} \rho_{i,j}^{D}(p) = \lambda_{i,j}^{D}$. Using the factor decomposition of the returns we can use conditional independence to integrate out idiosyncratic risks

$$\rho_{i,j}^{D} = \frac{EG_T\left(d_p - bR_{m,T}\right)^2 - p^2}{(1-p)p} \tag{18}$$

$$\lambda_{i,j}^{D} = \lim_{p\to+0} \frac{EG_T\left(d_p - bR_{m,T}\right)^2}{p} \tag{19}$$

On Figure 7 we show scatter plots for 2 stocks for the model with Gaussian shocks, unconditional correlation of weekly returns $\rho_{1,2} = 0.3$, market TARCH parameters $\alpha = 0.01$, $\alpha_D = 0.1$, $\beta = 0.92$ and Gaussian idiosyncrasies. We simulated returns of 2 stocks with random initial volatilities so that the joint distribution corresponds to the unconditional distribution. The term structure of skewness for this example is shown on Figure 8 in the previous section (red dash line). We can see from scatterplot and skewness term structure figure that weekly returns are symmetric whereas time aggregated returns for longer horizons(1 and 5 years) are significantly asymmetric and seem to have fatter tails in the lower left corner. Moreover, since fat tails and asymmetry come from the market return only, we can expect that joint downside moves tend to occur for both returns at the same time. To measure the frequency of joint extreme

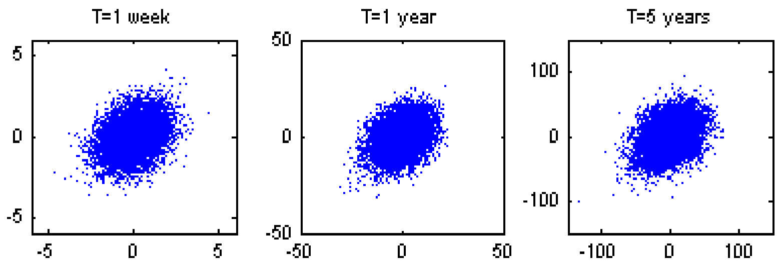



Figure 7: Scatter plot of 2 equity returns in one factor model ($\rho = 0.3$ Market TARCH: $\alpha = 0.01$, $\alpha_D = 0.1$, $\beta = 0.92$,gaussian shocks and idiosyncracies)

negatives returns we plot on Figure 8 default correlation $\lambda_{1,2}^D$ as a function of $p$. As expected the default correlation for 1 and 5 year horizons is higher than for weekly return.

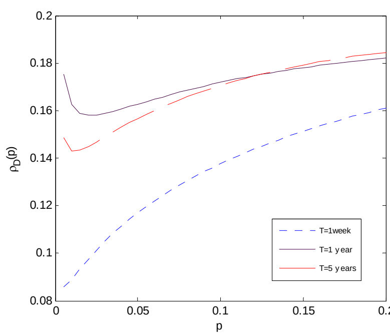



Figure 8: Default correlation as a function of p. we used formula 18 with sample average across 100,000 simulations of market return.

# 4 Modeling Portfolio Credit Risk

## 4.1 General Copula Framework

In this section we describe a semi-dynamic approach to model default correlations in a large homogeneous portfolio of credit exposures. Consider a portfolio of $M$ credit-risky obligors. We start with a static setup with a fixed time horizon $[0, T]$ and to simplify notations skip the time subscript for time dependent variables. At time $t = 0$ all $M$ obligors are assumed to be in non-default state and at time $T$ firm $i$ is in defaults with probability $p_i$.We assume to know the individual default probabilities $\mathbf{p} = [p_1, ..., p_M]'$ (either risk-neutral e.g. inferred from default swap quotes or actual e.g. estimated by rating agencies). Let $\tau_i \geq 0$ be the random default time of obligor $i$ and $Y_i = 1_{\{\tau_i \leq T\}}$ the default dummy variable which is equal to 1 if default happened before $T$ and 0 otherwise.

The loss generated by obligor $i$ conditional on its default is denoted as $l_i > 0$[10]. The loss $l_i$ is a product of the total exposure size $n_i$ and percentage losses in case default occurs $1-\bar{R}_i$ where $\bar{R}_i \in [0,1]$ is the recovery rate. We also assume that $l_i$ is constant (see [17] for discussion on stochastic recoveries). Portfolio loss $L_M$ at time $T$ is the sum of the individual losses for the defaulted obligors

$$L_M = \sum_{i=1}^{M} l_i 1_{\{\tau_i \leq T\}} = \sum_{i=1}^{M} l_i Y_i \tag{20}$$

The mean loss of the portfolio can be easily calculated in terms of individual default probabilities:

$$E\left(L_M\right) = \sum_{i=1}^{M} l_i E\left(Y_i\right) = \sum_{i=1}^{M} l_i p_i \tag{21}$$

Risk management and pricing of derivatives contingent on the loss of the credit portfolio, such as CDO tranches, require knowing not only the mean but the whole distribution of portfolio loss $F_L(x) = P\left(L_M \leq l\right).$ Portfolio loss distribution $F_L(l)$ depends on the joint distribution of default indicators $\mathbf{Y} = [Y_1, ..., Y_M]'$ and in a static setup can be conveniently modeled using latent variables approach (see e.g. [12]). Particularly, to impose structure on the joint distribution of default indicators we assume that there exists a vector of $M$ real-valued random variables $\mathbf{R} = [R_1, ..., R_M]'$ and $M$ dimensional vector of non-random default thresholds $\mathbf{d} = [d_1, ..., d_M]'$ such that

$$Y_i = 1 \Longleftrightarrow R_i \leq d_i \text{ for } i = 1, ..., M \tag{22}$$

Denote $F : \mathbb{R}^M \rightarrow [0,1]$ as a cdf of $\mathbf{R}$ and assume that it is a continuous function with marginal cdf $\{F_i\}_{i=1}^{M}$ For each obligor $i$ default threshold $d_i$ is then calibrated to match the obligor's default probability $p_i$ by inverting cdf of $R_i : d_i = F_i^{-1}(p_i)$. The continuity assumption implies that $F$ can be uniquely decomposed[11] into marginal cdf $\{F_i\}_{i=1}^{M}$ and $M$ dimensional copula $C : [0,1]^M \Rightarrow [0,1]$

$$F(\mathbf{d}) = C\left(F_1(p_i), .., F_M(p_M)\right) \tag{23}$$

Several popular copula choices are Gaussian copula model (Gupton et al.[1997] and see [10]), Student-t(...) and Clayton:

- Gaussian copula

$$C^G(\mathbf{p}; \Sigma) = \Phi_\Sigma\left(\Phi^{-1}(p_1), ..., \Phi^{-1}(p_M)\right)$$

[10] Note that loss $l_i = n_i\left(1 - \bar{R}_i\right)$ where $n_i$ is the exposure size and $\bar{R}_i \in [0,1]$ is the recovery rate.

[11] see Sklar's theorem

- Student-t copula

$$C^T(\mathbf{p};\Sigma,v) = t_{\Sigma,v}(t_v^{-1}(p_1),...,t_v^{-1}(p_M))$$

- Clayton

$$C^{Cl}(\mathbf{p}) = \max\left(1 - M + \sum_{i=1}^{M} p_i^{-\beta}\right)^{\beta}$$

The choice of copula $C$ defines the joint distribution of default indicators whereas information about marginal cdfs is only used on the threshold calibration step. For example, the joint default probability for any subset of obligors $\{i_1,...,i_k\} \subset \{1,...,M\}$ is given by

$$P(Y_{i_1} = 1,...,Y_{i_k} = 1) = P(R_{i_1} \leq d_{i_1},...,R_{i_k} \leq d_{i_k}) = C_{i_1,...,i_k}(p_{i_1},...,p_{i_k})$$

where $C_{i_1,...,i_k}$is corresponding $k$ dimensional margin of $C$.

The number of names in the portfolio can be big and therefore calibration of the copula parameters can be problematic. To reduce the number of parameters some form of symmetry is usually imposed on the distribution of the default indicators. Gordy [25] and Frey McNeil [12] discuss the mathematics behind the modeling of credit risk in homogeneous groups of obligors and equivalence of the homogeneity assumption to the factor structure of default generating variables. Conditional on the factors, defaults are independent and the conditional joint distribution of default indicators can be easily calculated using multinomial distribution. To simplify the calculations even more large homogenous portfolio (LHP) approximation can be used to approximate multinomial distribution with a finite number of obligors. Schonbucher and Shubert [11] and Vasicek [20] show that LHP approximation is very accurate for upper tail of the loss distribution even for mid-sized portfolios of about 100 names. We will use one factor LHP setup in this paper for analytical tractability.

**Assumption 1**: *Assume that loss given default $l_i = (1-\bar{R}_i)n_i$, individual default probabilities $p_i$ are the same for all $M$ names in the portfolio and the latent variables admit symmetric linear one factor representation:*

$$n_i = n \tag{24}$$

$$\bar{R}_i = \bar{R} \tag{25}$$

$$p_i = p$$

$$R_i = bR_m + \sqrt{1-b^2}E_i \text{ with -1} \leq \text{b} \leq 1$$

*where $R_m$ and $E$ are independent zero mean, unit variance random variables. $E_i's$ are identically distributes with cdf $G(.)$.*

Suppose we increase the number of names in the portfolio while keeping the total exposure size of the portfolio constant so that $n_i = N/M$. Conditional on $R_m$ the loss of the portfolio contains the mean of independent identically distributed random variables, $L_M = \left(1-\bar{R}\right)N\frac{1}{M}\sum_{i=1}^{M}1_{\{R_i \leq d\}}$, which a.s. converges to its conditional expectation as $M$ increases to infinity. We use $L$ without subscript to denote the loss cdf under LHP assumption.

**Proposition 3** ***(LHP approximation)*** *Under Assumption 1*

$$L \equiv \lim_{M\to\infty}\left[\left(1-\bar{R}\right)N\frac{1}{M}\sum_{i=1}^{M}1_{\{R_i\leq d\}}\right] = \left(1-\bar{R}\right)NE\left(1_{\{R_i\leq d\}}|R_m\right) = \tag{26}$$
$$= \left(1-\bar{R}\right)NG\left(\frac{d-bR_m}{\sqrt{1-b^2}}\right) \text{ a.s. for any } R_m \in supp\left(G\right)$$

**Proof.** see proposition 4.5 in [12] ∎

Based on 26 cdf of $L$ can be expressed in terms of the cdf of $R_m$

$$P\left(L\leq l\right) = P\left(R_m \geq d_1\left(l\right)\right)$$
$$d_1\left(l\right) = \frac{d}{b} - \frac{\sqrt{1-b^2}}{b}G^{-1}\left(\frac{l}{\left(1-\bar{R}\right)N}\right)$$

The probability of a small loss in a diversified portfolio is high when the probability of "market return", $R_m$, falling below barrier $d_1$ is low. In other words a small loss corresponds to the right tail of the market return distribution. The left tail of the market return distribution corresponds to the large portfolio loss - the thicker the left tail the more probable large loss is. $d_1$ depends on the single name default barrier $d$, market factor loadings and the loss-per-obligor parameters. Note that $d_1$ is not necessarily monotonic function of $b$. Only for small losses, such that $l < \left(1-\bar{R}\right)NG(0)$, it is increasing in $b$. For Gaussian copula we have familiar formula derived by Vasicek([20])

$$L^G = \left(1-\bar{R}\right)N\Phi\left(\frac{\Phi^{-1}\left(p\right)-bR_m}{\sqrt{1-b^2}}\right) \tag{27}$$
$$P\left(L\leq l\right) = 1-\Phi\left(d_1^G\left(l\right)\right) \tag{28}$$
$$d_1^G\left(l\right) = \frac{\Phi^{-1}\left(p\right)}{b} - \frac{\sqrt{1-b^2}}{b}\Phi^{-1}\left(\frac{l}{\left(1-\bar{R}\right)N}\right) \tag{29}$$

## 4.2 Generating Portfolio Loss Distributions in Dynamic Models

Because of the one factor structure of the model we can use LHP setup described in proposition 1 to calibrate the loss of a large homogeneous portfolio using the distribution of the aggregated market return generated by TARCH(1,1) model. The latent variables are assumed to have symmetric one factor structure with the factor following TARCH(1,1) model. We calibrate the loss of the portfolio using the one factor GARCH model described in and the formula 26 for LHP loss

$$L_T = \left(1 - \bar{R}\right)\Phi\left(\frac{d_T - bR_{m,T}}{\sqrt{var(R_{m,T})}\sqrt{1-b^2}}\right)$$

where

- $R_{m,T} = \frac{1}{\sqrt{T}}\sum_{u=1}^{T} r_{m,u}$ is return over horizon T generated using aggregation of simulated TARCH(1,1) returns with unconditional volatility equal to 1
- $d_T$ is calibrated so that the probability of $R_{i,T} = bR_{m,T} + \sqrt{1-b^2}E_T$ hitting $d_T$ is equal to single name default probability $p_T$

$$P\left(bR_{m,T} + \sqrt{1-b^2}E_T \leq d_T\right) = p_T$$

- $b$ is the factor loading that is chosen to match a given unconditional linear correlation $\zeta = b^2$

To calculate the expected tranche losses generated by the model and to calibrate $d_T$ we use $N = 100,000$ independent Monte Carlo simulations of the factor and then use corresponding sample moments:

$$d_T \text{ solves } \frac{1}{N}\sum_{n=1}^{N}\Phi\left(\frac{d_T - bR_{m,T}^{(n)}}{\sqrt{var(R_{m,T})}\sqrt{1-b^2}}\right) = p_T$$

$$EL_{(0,K]} = \frac{1}{N}\sum_{n=1}^{N} f_{(0,K]}\left(\left(1-\bar{R}\right)\Phi\left(\frac{d_T - bR_{m,T}^{(n)}}{\sqrt{var(R_{m,T})}\sqrt{1-b^2}}\right)\right)$$

## 4.3 From Loss Distributions to Correlation Spectrum

To compare the effects of the copula choice on the loss distribution we define the loss tranches. Let $(K_d, K_u]$ denotes a tranche with attachment point $K_d$ and detachment point $K_u$ expressed as fractions of the reference portfolio notional so that $0 \leq K_d < K_u \leq 1$. The notional of the tranche, $N_{(K_d,K_u]}$, is $N(K_u - K_d)$ where $N$ is the notional of the portfolio. The loss $L_{(K_d,K_u]}$ of the tranche is the fraction of $L$ that falls between $K_d$ and $K_u$. For simplicity assume that total notional is normalized to 1.

$$L_{(K_d,K_u]} = f_{(K_d,K_u]}(L) \tag{30}$$

$$f_{(K_d,K_u]}(x) \equiv (x - K_d)_+ - (x - K_u)_+ \tag{31}$$

Tranches with zero attachment point, $(0, K_u]$, and unit detachment point, $(K_d, 1]$, are called equity and senior tranches correspondingly. Loss of any tranche can be decomposed into losses of two equity tranches $L_{(K_d,K_u]} = L_{(0,K_u]} -$

$L_{(0,K_d]}$. Expected loss of the equity tranche $(0,K]$ depends on the portfolio loss distribution and in LHP setup can be computed using only the distribution of the market factor

$$EL_{(0,K]} = Ef_{(0,K]}(L) = E\left[G\left(\frac{d-bR_m}{\sqrt{1-b^2}}\right)1_{\{R_m \geq d_1(K)\}}\right] + KP\left(R_m < d_1(K)\right) \tag{32}$$

Expectation in 32 can be computed by Monte Carlo simulation or numerical integration if we know $G$ and distribution of $R_m$. For Gaussian copula the integral can be taken in closed form.

$$E^G L_{(0,K]} = \left(1-\bar{R}\right)\Phi\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right) + K\Phi(d_1) \tag{33}$$
$$d_1 = \frac{1}{\sqrt{\rho}}\Phi^{-1}(p) - \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\Phi^{-1}\left(\frac{K}{1-\bar{R}}\right)$$

Because of its analytical tractability, it is convenient to use Gaussian copula as a benchmark model when comparing different choices of copulae. In option pricing implied volatility is used as a way to use lognormal distribution as a benchmark model. For the Gaussian copula, the correlation matrix plays a similar role. The volatility of the portfolio loss increases with the correlation parameter $\rho = b^2$.

$$Vol(L) = EL^2 - \left(1-\bar{R}\right)^2 p^2 = \left(1-\bar{R}\right)^2\left[\Phi\left(\Phi^{-1}(p), \Phi^{-1}(p); \rho/\sqrt{1-\rho}\right) - p^2\right]$$

We take the Gaussian model as the base model and define the correlation spectrum $\rho(K,p,\bar{R})$ to be the level of the single correlation parameter in this Gaussian model that gives the same expected tranche loss for a given width of the equity tranche $(0,K]$, given horizon $T$, and given the single-issuer default probability $p$.

**Definition 4** *Suppose the loss distribution of a large homogeneous portfolio is generated by a model* $\left\{C,p,\bar{R}\right\}$ *with copula* $C$*, equal individual default probabilities* $p$ *and recovery rate* $\bar{R}$*. Let* $L_{(0,K]} \in \left[pf_{(0,K]}\left(1-\bar{R}\right),\ f_{(0,K]}\left(\left(1-\bar{R}\right)p\right)\right]$ *be the expected loss of the equity tranche* $(0,K]$. *We define the **correlation spectrum*** $\rho(K,p,\bar{R})$ *of the model* $\left\{C,p,\bar{R}\right\}$ *as the correlation parameter of the Gaussian copula that produces the same expected loss* $EL_{(0,K]}$ *for the tranche* $(0,K]$ :

$$\rho(K,p,\bar{R}) \text{ solves } E^G L_{(0,K]}(\rho) = EL_{(0,K]} \text{ for all } K \in [0,1] \tag{34}$$
$$\text{where } EL_{(0,K]} \text{ is expected loss of the tranche}$$
$$(0,K] \text{ generated by model } \left\{C,p,\bar{R}\right\}$$

*where* $E^G L_{(0,K]}$ *is defined in 33*

To insure that correlation spectrum is well defined we need to prove that solution of 34 exists and unique. Proposition below shows that for Gaussian copula expected loss of an equity tranche is monotonically decreasing function of $\rho$ and attains its maximum(minimum) when correlation is 0(1).

**Proposition 5** *For the Gaussian copula $E^G_\rho L_{(0,K]} < 0$ for any $\rho \in (0,1)$. The expected loss monotonically decreases from $f_{(0,K]}\left(\left(1-\bar{R}\right)p\right)$ to $pf_{(0,K]}\left(1-\bar{R}\right)$ as $\rho$ increases from 0 to 1.*

***Proof.*** *using 33 and the properties of Gaussian distribution*[12]

$$E^G_\rho L_{(0,K]} \equiv \frac{\partial}{\partial\rho} E^G L_{(0,K]} =$$

$$-\left(1-\bar{R}\right)\left[\frac{1}{2\sqrt{\rho}}\Phi_3\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right) + \Phi_2\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right)\frac{\partial}{\partial b}d_1\right] +$$

$$+K\phi(d_1)\frac{\partial}{\partial b}d_1 = -\frac{1-\bar{R}}{2\sqrt{\rho}}\phi\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right) < 0 \text{ for any } \rho \in (0,1)$$

■

By transforming the loss distribution to the correlation spectrum we do not lose any information about the loss distribution. The next proposition shows how to calculate the loss cdf using the correlation spectrum and its slope along the K-dimention.

**Proposition 6** *Suppose $\rho(K, p, \bar{R})$ is the correlation spectrum for model $\left\{C, p, \bar{R}\right\}$ and the probability distribution function of the portfolio loss is continuous function then the loss cdf can be computed from the correlation spectrum:*

$$P(L \le K) = P^G(L \le K) + \rho_K(K, p, \bar{R}) E^G_\rho L_{(0,K]} \tag{35}$$

[12] The following properties of 2 dimentional Gaussian cdf are used in the calculation

$$\Phi_2(x, y; \rho) = \phi(y)\Phi\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right)$$

$$\frac{\partial}{\partial\rho}\Phi(x, y; \rho) = \phi(x, y; \rho)$$

where $\phi(.)$ denotes, depending on the number of agruments, pdf of standard Normal distribution and pdf of bivariate Normal with standard Normal marginals and correlation coefficient as a third argument. Numerical subscript denotes the partial derivative with respect to the corresponding argument. First formula is straitforward. The proof of the second can be found in Vasicek([26])

*where*

$$P^G(L \le K) = 1 - \Phi(d_1)$$
$$E^G_\rho L_{(0,K]} = \left(1 - \bar{R}\right) \frac{1}{2\sqrt{\rho}} \phi\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right)$$
$$d_1 = \frac{1}{\sqrt{\rho}} \Phi^{-1}(p) - \frac{\sqrt{1-\rho}}{\sqrt{\rho}} \Phi^{-1}\left(\frac{K}{1-\bar{R}}\right) \textit{ and } \rho = \rho(K, p, \bar{R})$$

***Proof.*** *first note that the derivative with respect to $K$ of the expected tranche's loss under true copula $C$ is related to the cdf of the loss*

$$\frac{d}{dK} E L_{(0,K]} = \frac{d}{dK} E\left(L - (L-K)_+\right) = -\frac{d}{dK} E(L-K)_+ = E1_{\{L-K\ge 0\}} = 1 - P(L \le K)$$

*therefore*

$$P(L \le K) = 1 - \frac{d}{dK} E L_{(0,K]} = 1 - E^G_K L_{(0,K]} - \rho_K(K,p,\bar{R}) E^G_\rho L_{(0,K]} =$$
$$1 - \Phi(d_1) + \frac{1-\bar{R}}{2\sqrt{\rho}} \phi\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right) \rho_K(K,p,\bar{R}) =$$
$$P^G(L \le K) + \left(1-\bar{R}\right) \frac{1}{2\sqrt{\rho}} \phi\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right) \rho_K(K,p,\bar{R})$$

*where $E^G_K L_{(0,K]}$ is computed as*

$$E^G_K L_{(0,K]} \equiv \frac{\partial}{\partial K} E^G L_{(0,K]} = -\left(1-\bar{R}\right) \Phi_2\left(\Phi^{-1}(p), -d_1; -\sqrt{\rho}\right) \frac{\partial}{\partial K} d_1 +$$
$$+ K\phi(d_1) \frac{\partial}{\partial K} d_1 + \Phi(d_1) = \Phi(d_1)$$

■

We defined the correlation spectrum for the loss distribution with a fixed time horizon. In the next section we illustrate the pricing of portfolio tranche swap contracts and show that the value of the swap depends on the whole term structure of expected tranche losses up to the maturity of the swap.

### 4.4 Pricing of Synthetic CDO Tranches

In this section we briefly define the payoff structure of synthetic CDO tranche contracts and their pricing. Consider a synthetic CDO with fixed maturity T written on a synthetic portfolio. The loss $L_{(K_d,K_u]}(t)$ of the tranche $(K_d, K_u]$ at time $t \le T$ is a fraction of portfolio loss $L(t)$ that falls between $K_d$ and $K_u$.

$$L_{(K_d,K_u]}(t) = f_{(K_d,K_u]}(L(t)) \tag{36}$$

The swap contract for a particular tranche is swap of cash flows between the "premium leg" and the "protection leg". The issuer of the premium leg, the insured, agrees to pay a fixed fee $s$ to the insurer in the proportion to the survived notional of the tranche. The issuer of the protection leg, the insured, compensates the tranche losses to the insured until the maturity of the contract. Since the swap contract is a contingent claim on the portfolio loss it can be priced using the risk-neutral distribution of the portfolio losses. We assume that interest rate risk is not correlated with credit risk and $D(0,t)$ denotes the price at time 0 of a zero coupon bond maturing at time $t$. The payoff structure of both premium and protection legs is linear in the tranche's loss and therefore to price these legs when the interest rates are not correlated with default risk we only need to know the term structure of expected tranche losses. Introduce the tranche's default probability $P_{(K_d,K_u]}(t)$ as expected fraction of the tranche's notional that is lost due to defaults by time $t$.

$$P_{(K_d,K_u]}(t) = \frac{E_0\left[L_{(K_d,K_u]}(t)\right]}{N_{(K_d,K_u]}} \tag{37}$$

For simplicity of formula presentations assume that time is continuous. The value of the protection leg at time 0

$$V_0^{protection} = E_0\left(\int_{t=0}^{T} D(0,t)\, dL_{(K_d,K_u]}(t)\right) = N_{(K_d,K_u]}\int_{t=0}^{T} D(0,t)\, dP_{(K_d,K_u]}(t) \tag{38}$$

Assuming that protection fee is paid in $\Delta q$ intervals e.g. quarterly the value of the premium leg at time 0

$$V_0^{premium} = E_0\left(\sum_{q=1}^{T/\Delta q} D(0,q\Delta q)\left[N_{(K_d,K_u]} - L_{(K_d,K_u]}(q\Delta q)\right] s\Delta q\right) = \tag{39}$$

$$= N_{(K_d,K_u]} s\Delta q \sum_{q=1}^{T/\Delta q} D(0,q\Delta q)\left[1 - P_{(K_d,K_u]}(q\Delta q)\right]$$

The par spread of the swap contract is the spread that makes the values of protection and premium legs equal. As we already mentioned swap contract cash flows are linear functions of the tranche losses and therefore the values of the both legs depend only on the tranche's expected losses. Because timing of the losses is important when the interest rates are not zero we need the whole term structure of the tranche's expected losses up to the maturity of the swap to price the contract.

The portfolio loss $L(t)$ at time $t$ in the LHP copula framework depends on time $t$ only through the single-issuer default probability $p_t$[13]. As we showed in proposition 2 the correlation spectrum is the equivalent representation of the loss distribution for a fixed time horizon. The dependence of the correlation spectrum $\rho(K,p,\bar{R})$ on $t$ is achieved through the second argument - single-issuer default probability $p$. For example the expected tranche loss at time $t$

[13] The portfolio loss generating copula does not change with T because it corresponds to the copula of time to default distribution which by definition does not depend on T.

is the expected tranche loss of the Gaussian model with correlation parameter $\rho\left(K, p_t, \bar{R}\right)$ and single-issuer default probability $p_t$:

$$EL_{(0,K]}(t) = E^G L_{(0,K]}\left(\rho\left(K, p_t, \bar{R}\right), p_t\right) \tag{40}$$

The expected tranche loss that happens between $t$ and $t + dt$ can therefore be computed from the correlation spectrum using its level and the slope in the p-dimention:

$$dEL_{(0,K]}(t) = \frac{dE^G L_{(0,K]}(t)}{dp_t} dp_t = \left(E_p^G L_{(0,K]}(t) + \rho_p\left(K, p_t, \bar{R}\right) E_\rho^G L_{(0,K]}(t)\right) dp_t \tag{41}$$

where

$$E_\rho^G L_{(0,K]}(t) = -\frac{1-\bar{R}}{2\sqrt{\rho}} \phi\left(\Phi^{-1}(p_t), -d_1; -\sqrt{\rho}\right) \tag{42}$$

$$E_p^G L_{(0,K]}(t) \equiv \frac{\partial}{\partial p} E^G L_{(0,K]}(t) = \left(1-\bar{R}\right) \Phi\left(\frac{-d_1 + \sqrt{\rho}\Phi^{-1}(p_t)}{\sqrt{1-\rho}}\right)$$

$$d_1 = \frac{1}{\sqrt{\rho}} \Phi^{-1}(p_t) - \frac{\sqrt{1-\rho}}{\sqrt{\rho}} \Phi^{-1}\left(\frac{K}{1-\bar{R}}\right) \text{ and } \rho = \rho\left(K, p_t, \bar{R}\right) \tag{43}$$

Both legs of the swap contract can be priced using the correlation spectrum surface, single-issuer default probability term structure and formulas 37 - 41.

# 5 Comparing Portfolio Loss Generating Models

In section 3 we demonstrated that dynamic models such as GARCH and TARCH can produce significant pairwise default correlation even for very low default thresholds. Thus, one can hope that these models should also be able to capture the important aspects of multi-variate default losses in a diversified portfolio setting. But in order to discriminate between these models and to understand which of their characteristics are the most important from a credit modeling perspective, one must have a good measure that makes such comparisons not only possible but hopefully apparent and intuitive. The market standard measure is the base correlation. However, this measure is best suited for comparison of pricing of similar tranches rather than comparison of different models. In particular, base correlation implicitly depends on the level and term structure of interest rates, as well as conventions such as coupon payment frequency, up-front pricing, etc.

Our goal in this paper is not so much to price a specific set of tranches under given market conditions as to provide a general framework for judging the versatility of various dynamic portfolio credit risk models. All such models, whether defined via dynamic multivariate returns model like in this paper or in various versions of the static copula framework ([10], [12], [11], [14], [13], [17]), can be characterized by the full term structure of loss distributions. Thus, without loss of generality, we can refer to all models of credit risk as loss generating models, with an implicit assumption that any two models that produce identical loss distributions for all terms to maturity are considered to be equivalent. The correlation spectrum, introduced in section 4.3, conveniently transforms specific choice of a loss generating model into a two dimensional surface $\rho(K, T)$ of the Gaussian copula correlation parameter, with the main dimensions being the loss threshold (detachment level) $K$ and the term to maturity $T$. All other inputs such as the recovery rate $R$, the term structure of (static) hazard rates $h$, the level of linear asset correlation $\zeta$, the Student-t degrees of freedom $\nu$, various GARCH model coefficients, etc. – are considered as model parameters upon which the two-dimensional correlation spectrum itself depends. Note that in the previous sections we have expressed the correlation spectrum as a function of detachment level and the underlying portfolio's cumulative expected default probability $p$ rather than the term to maturity $T$. Given our assumption of the static term structure of the hazard rates $h$ these two formulations are equivalent. In this section we prefer to emphasize the dependence on maturity horizon in order to facilitate the comparison with base correlation models and also to analyze the dependence on the level of hazard rates separately from the term to maturity dimension.

In this framework, we can compare various dynamic and static loss generating models by comparing their correlation spectra, as well as the characteristic dependencies of the correlation spectra on changes of model parameters. Of course, the correlation spectrum of a static Gaussian copula model [10] is a flat surface with constant correlation across both detachment level $K$ and term to maturity $T$. Any deviation from a flat surface is therefore an indication of a non-trivial loss generating model, and we can judge which features of the model are the important ones by examining how strong a deviation from flatness do they lead to.

## 5.1 Single-Period (Static) Loss Generating Models

Let us begin with the analysis of one of the popular static loss generation models. On figure 9 we show the correlation spectrum computed for the Student-t copula with linear correlation $\zeta = 0.3$ and $\nu = 12$ degrees of freedom. Student-t copula is in the same elliptic family as the Gaussian copula but has non-zero tail dependence governed by the degrees of freedom parameter. As a model of single-period asset returns the Student-t distribution has been shown to provide a significantly better fit to observations than the standard normal [14].

However, from the figure 9 we can see that the static Student-t copula does not generate a significant skew in the direction of detachment level $K$. The main reason for this is the symmetric nature of the model **???? we need more explanation here ????**. On the other hand, we can see that the correlation spectrum is downward sloping with increasing maturity $T$ and cumulative default rate $p$ while remaining flat throughout. Under assumptions of static copula model, the bigger $p$ corresponds to the default boundary being closer to the center of the Student-t distribution and therefore leads to smaller effects of tail-dependence and lower correlation spectrum. These conclusions are in broad agreement with the analysis in [?], and can be taken as an indication of the general inadequacy of symmetric static models for purposes of explanation of the correlation skew.

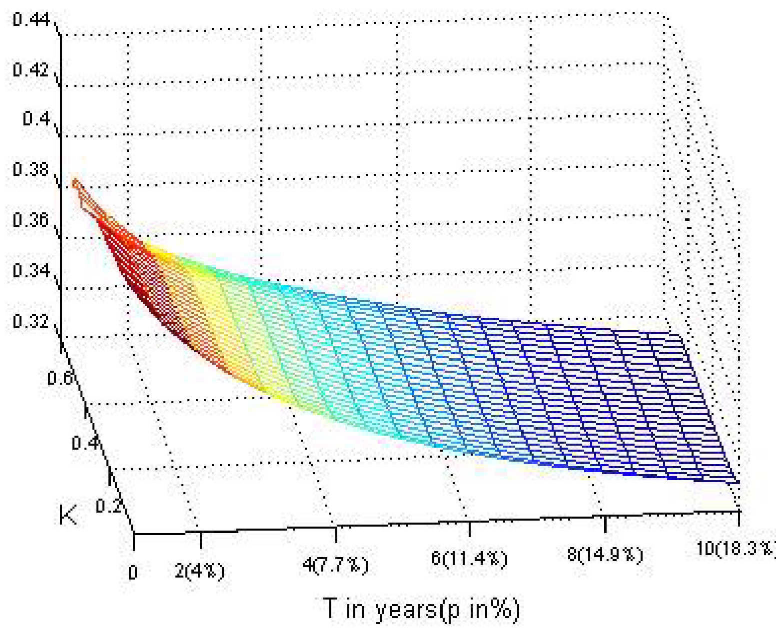



Figure 9: Correlation spectrum for Student T Copula with 12 degrees of freedom and linear correlation 0.3. The x-axes is labled with both time and default probability corresponding to the time horison. The term structure of sinle-issuer default probability is assumed to be $1-(1-p)^T$ where one year default probability $p = 0.02$

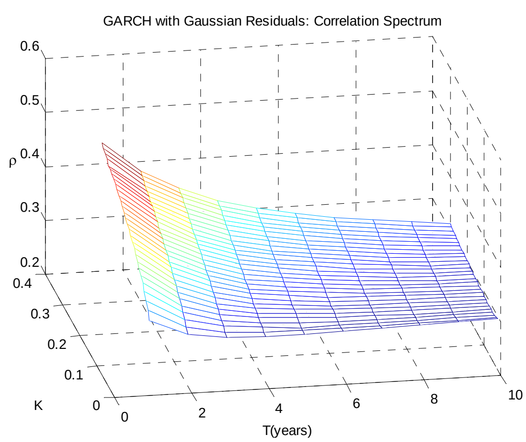
Here we might wish to show the correlation spectrum of Anderson-Sidenius model (the one with analytical results – static model with random factor loadings). It would give us a good comparison with skew and possibly a different nature of term structure vs. TARCH model.

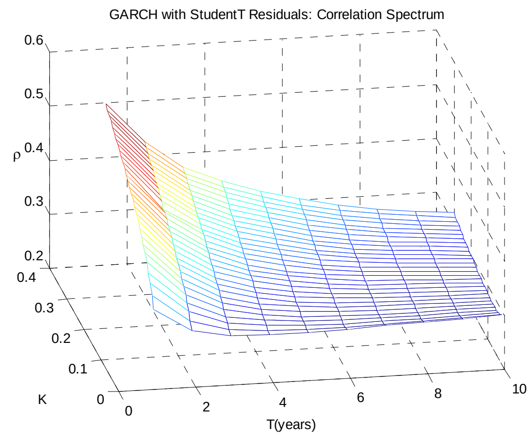
Also, for a separate paper on correlation spectrum, I would present the corr spectrum for Gregory-Laurent factor copula model.

## 5.2 Multi-Period (Dynamic) Loss Generating Models

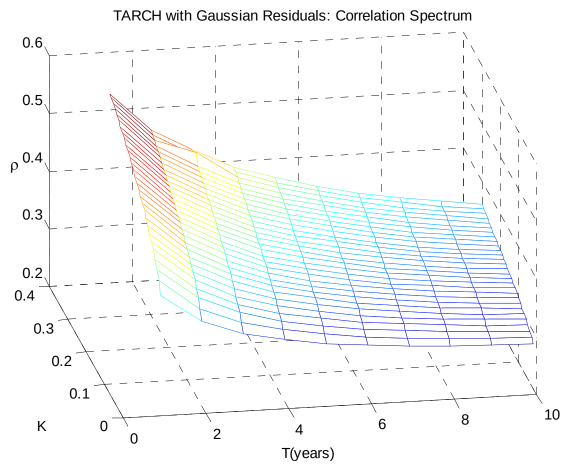
Let us now turn to loss generating models based on latent variables with multi-period dynamics. On figure **??** we show the correlation spectrum computed for a loss generating model based on GARCH dynamics with Gaussian residuals, with a linear correlation set to $\zeta = 0.3$, and GARCH model parameters taken from the weekly estimates in table **??**. As we can see, this model does exhibit a visible deviation from the flat correlation spectrum for short maturities. However, as we already noted in section 2, the distribution of aggregate returns for the symmetric GARCH model quickly converges to normal. Therefore, it is not surprising to see that the correlation spectrum also flattens out fairly quickly and becomes virtually indistinguishable from a Gaussian copula for maturities beyond 5 years. Thus, we conclude that the symmetric GARCH model with Gaussian residuals is inadequate for description of liquid tranche markets where one routinely observes steep correlation skews at maturities as long as 7 and 10 years.

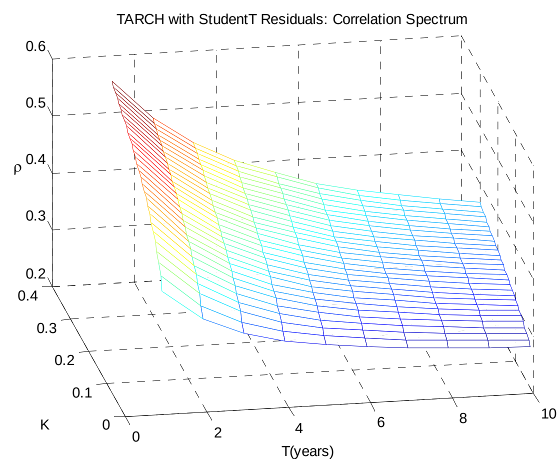
Based on the empirical results of 2.2 we know that a GARCH model with Student-t residuals provides a better fit to historical time series of equity returns. A natural question is whether allowing for such volatility dynamics can lead to a persistent correlation skew commensurate with the levels observed in synthetic CDO markets. The results of section 2.2 suggest that the additional kurtosis of the single-period returns represented by the Student-t residuals does not matter very much for aggregate return distributions are sufficiently long terms. Indeed, figure **??** shows that the GARCH model with Student-t residuals exhibits a correlation skew that is quite a bit steeper at the short maturities, yet is almost as flat and features at the long maturities as its non-fat-tailed counterpart – there is a small amount of skew at 10 years, but it is too small compared to the steepness observed in the liquid tranche markets. Thus, we conclude that one has to focus on the dynamic features of the market factor process in order to achieve the desired correlation skew effect.






Our next candidates are the TARCH models with either Gaussian or Student-t residuals. We have seen in 2 that the asymmetric volatility dynamics of these models leads to a much more persistent skewness and kurtosis of aggregated equity returns that actually grow rather than decay at very short horizons, and survive for as long as 10 years for the range of parameters corresponding to the post-1990 sample of SP500 weekly log-returns. Hence, our hypothesis is that a latent variable model with TARCH dynamics might be capable of producing a non-trivial credit correlation skew for up to 10 year maturity. The figures **??** and **??** show the correlation spectra for the TARCH-based loss generating models. The most immediate observation is that both versions of the model produce a rather persistent correlation skew. Although the correlation spectrum surface flattens out with growing term to maturity, the steepness of the skew is still quite significant even at 10 years. The next observation is that the marginal effect of the fat-tailed residuals is even less evident for the TARCH model than it was for the symmetric GARCH. This is simply a reflection of the relative importance of the two components of the factor model which we already stated above – the impact of the single-period residual dynamics is far less important than the impact of the asymmetry in market factor dynamics.



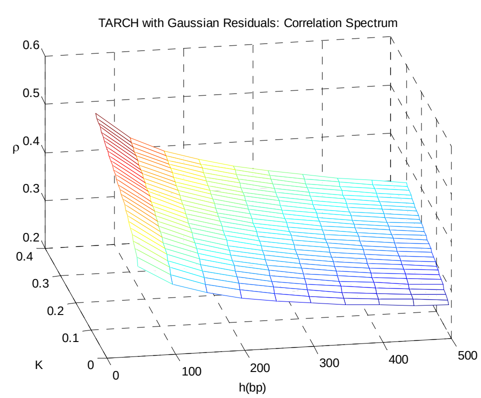

## 5.3 Dependence on Model Parameters

As explained in the beginning of this section, we consider the correlation spectrum surface as an embodiment of the particular loss generating model. Each such model contains various parameters some of which are empirically estimated (e.g. the degrees of freedom of the Student-t distribution) and some of which are calibrated to a particular problem at hand (e.g. the level of hazard rates and expected default probabilities for the collateral portfolio underlying the synthetic CDO tranches under question). While the empirically estimated parameters are not likely to change, the calibrated ones will do so quite frequently as the market conditions change.

In particular, the implied hazard rates can and do change quite significantly even for investment grade credit portfolios. Therefore, the analysis of the dependence of the correlation spectrum on the level of hazard rates has not only an academic relevance as a matter of investigation of the model's range of applicability, but also a practical importance due to reliance of many practitioners on the base correlation methodology which normally takes the correlation skew as an exogenous input and does not incorporate correlation skew adjustments as the market spreads and implied hazard rates change. By contrast, the dynamic multi-period models introduced in this paper produce the correlation spectrum as an output of the model, and therefore can give a specific prediction regarding the way the correlation skew is supposed to change when the model parameters move.

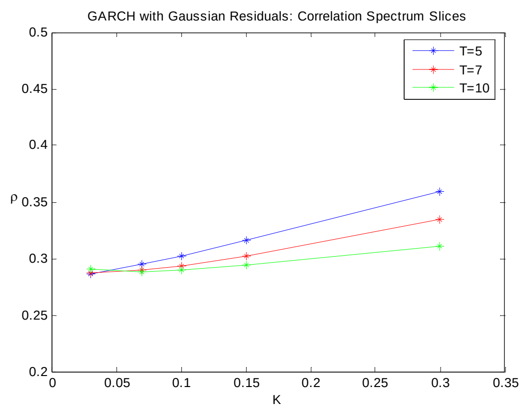
As an example of such predictive behavior of the model consider the correlation spectrum dependence on the hazard rates depicted in figure. We have shown a particular maturity slice, the 5-year skew, as a function of hazard rates. From the visual comparison of figures 5.3 and **??** it appears that the dependence of a correlation skew for a fixed term to maturity but varying level of hazard rates is very similar to the dependence of the correlation spectrum on the term to maturity. The similarity is natural, as the first order effect is the dependence on the level of the cumulative default probability which depends on the product of $h \cdot T$ rather than on the hazard rate or the term to maturity separately. For each level of this product, we get a specific level of the default threshold in the latent variable credit risk model. The higher this threshold, the closer is the sampled region to the center of the latent variables distribution and the less it is affected by the tail risk – thus leading to a lower level of the credit correlation.

However, there is a second order effect which makes these two dependencies somewhat different, and it is related to the shape of the distribution of aggregate returns for the market factor. Assuming that the parameters of the GARCH process are the same in both cases, we can deduce that the dependence on the term to maturity with fixed hazard rate should exhibit a faster flattening of the correlation spectrum than the dependence on the hazard rate with fixed term to maturity because the increasing aggregation horizon for market factor returns leads to gradual convergence of its distribution towards normal and, as a consequence, to even flatter correlation skew.




The dependence of the 5-year correlation skew on the level of the hazard rates.

To make the visual comparison easier, we note that the effect of flattening of the skew while going from a 5-year horizon to the 10-year horizon must be compared against the flattening of the skew while going from 100bp hazard rate to 200bp hazard rate. The following figure contains a direct comparison of the three particular slices of the correlation spectrum surface.




This should actually be a figure with 3 slices: 1) T=5-yr h=100bp, 2) T=10-yr h=100bp, 3) T=5-yr h=200bp

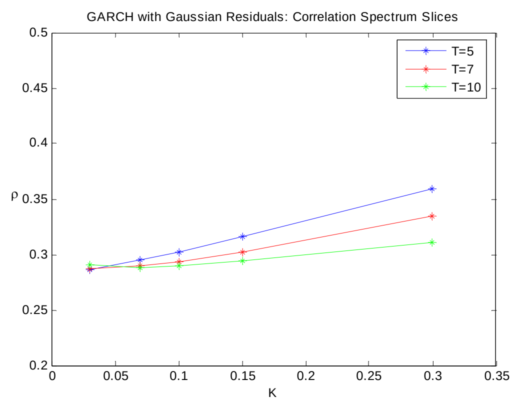
The next figure depicts the dependence of the correlation skew on the changes of the parameters of the dynamic GARCH model itself. We show the dependence of the 10-yr skew with the hazard rate set to 100bp, on the levels of the asymmetry parameter $\alpha_D$ (while keeping the persistence constant by adjusting the level of the symmetric $\alpha$), and on the levels of the persistence parameter $\beta + \alpha + \alpha_D v_\varepsilon^d$ (while keeping the asymmetry constant).

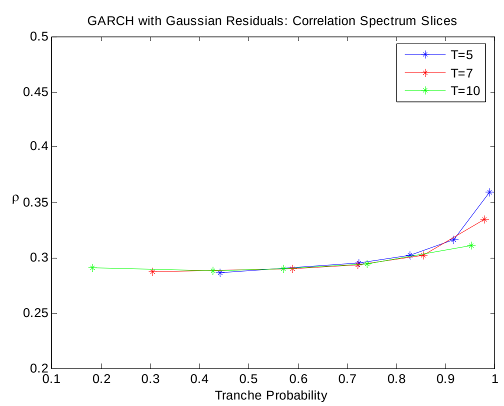
**Here should be two figures showing the slices with changing asymmetry and persistence.**

## 5.4 Dependence on Choice of Variables

The discussion in the previous section of the dependence of the correlation spectrum on the hazard rates revealed the important first order effect that the correlation spectrum most strongly depends on the cumulative default probability rather than hazard rate itself. One could ask a similar question regarding the detachment level which we heretofore have been using as the standard variable for the description of the correlation spectrum. Could it be that the bulk of the flattening of the correlation spectrum with the growing term to maturity is related to simply the fact that the constant detachment level together with the growing cumulative default probability of the collateral portfolio corresponds to more "in-the-money" equity tranches, or in other words to a higher probability of loss for an equity tranche? Indeed, as the figures **??** and **??** attest, for the GARCH model with Gaussian residuals the correlation skews expressed as a function of the tranche probability rather than detachment level do actually coincide until quite high levels of tranche probability (detachment level).

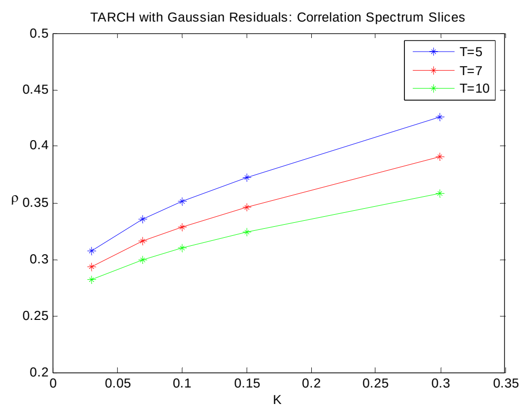
The markers on the figures correspond to industry-standard detachment points (3%, 7%, 10%, 15% and 30%). From **??** it becomes apparent that the 5-year 0-3% tranche corresponds to roughly the same point on the "uniform" correlation skew as the 10-year 0-7% tranche. Same can be said about 7-year 0-7% tranche and 10-year 0-10% tranche. They skews do diverge eventually for the senior tranches, exhibiting a clear flattening with term structure when using the tranche probability variable as well. Thus, while the tranche probability variable does not completely explain away all the features of the correlation skew term structure dependence, it goes a long way towards it.

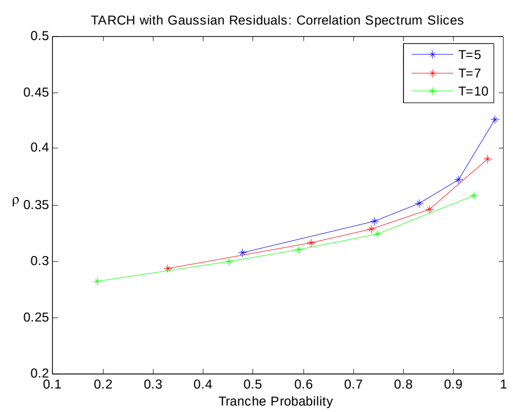



Revisiting the same question in the case of the asymmetric TARCH model with Gaussian residuals demonstrates that the skew generated by this model is more "genuine", in a sense that it is clearly visible even when using the tranche probability variables instead of the detachment level. Still, while the skew does not disappear and remains fairly steep, it does become a lot more consistent across the maturities, with both the steepness and the level of the correlation spectrum slices being very close for all but the most senior (least likely to get hit) tranches.

This is very similar to the well known fact from equity options markets that the implied volatility surface as a function of option delta exhibits a lot less dependence on the term to expiry than when the same surface is expressed as a function of the option strike or moneyness. The Black-Scholes deltas are very close to risk-neutral probability of hitting the particular option strike, and therefore the analogy with our tranche probability variable is very close indeed.




# 6 Summary and Conclusions

In this paper we have introduced and studied a new class of credit correlation models defined via a dynamic loss generation process within a latent variable approach where the latent variable follows a factor-ARCH with either asymmetric TARCH volatility dynamics. We have shown this model to be superior to alternative simpler characterizations of the time series process including symmetric GARCH volatility dynamics with Gaussian or Student-t residuals when it comes to an ability to produce a significant and persistent correlation skew commensurate with the levels observed in the liquid synthetic CDO tranche markets.

We have studied the time aggregation properties of the multivariate dynamic models of equity returns. We showed that dynamics of equity return volatilities and correlations leads to significant departures from the Gaussian distribution even for horizons measured in several years. The asymmetry appears to "survive aggregation" longer than fat tails do based on the parameters estimated from the real data. The main source of skewness and kurtosis of the return distribution for long horizons is the dynamic asymmetry or so called leverage effect.

We introduced the notion of the correlation spectrum as a tool for comparing the loss generating models, whether defined via a single-period (static) copula, or via multi-period (dynamic) latent-variable framework, and for simple and consistent approach to non-parametric pricing of CDO tranches. We showed that for a portfolio loss distributions with smooth pdf the loss distribution can be easily reconstructed from the correlation spectrum using its level and slope along the K-dimention.

To summarize our findings, let us return to some of the stylized facts and hypothesis which we put forward in the introduction and see what have we learned about them:

- **Hypotheis:** There exists an implied correlation surface with non-trivial strike (detachment level) and term structure.
- **Observation:** We have confirmed this hypothesis and furthermore have shown that there are certain (though not exact) relationships between the dependence on the term to maturity and the dependence on the hazard rates, as well as the dependence on the detachment level and the dependence on term to maturity. We have demonstrated that the tranche probability is a better variable than the detachment level when it comes to defining a robust correlation spectrum surface.
- **Hypotheis:** The fat tails in the single-period return distribution may be important in explaining the correlation skew.
- **Observation:** We have seen much less evidence for the importance of the fat tails for credit correlations than the previous studies have shown for equity options. This is at least partially explained by the much longer horizons that interest us here, namely 5 to 10 years, compared to the typical horizons encountered in the equity options studies, namely 3 to 12 months. As we have learned, the effects of the fat tails and other single-period deviations from normality do not survive for sufficiently long horizons, while the effects of dynamic asymmetry do.
- **Hypotheis:** The asymmetry in the return distribution is a necessary ingredient for explaining the correlation skew.
- **Observation:** We have confirmed this hypothesis in several ways, both by examining the term structure dependence of the correlation spectra

generated by the GARCH and TARCH models, and also by comparing the correlation spectra between the GARCH and TARCH – with the latter being able to produce a much more pronounced and persistent correlation skew.

Importantly, in our dynamic loss generating model framework, the correlation spectrum is not only explained, but predicted – based on empirical parameters of the TARCH process and the parameters describing the reference credit portfolio. The model also predicts a specific sensitivity of the correlation spectrum to changes in various such parameters, including the hazard rate. The structural inability of the static models to incorporate the changes in the base correlation have been at the heart of the recent difficulties faced by these models during the synthetic CDO market dislocation in April/May of 2005. While our model is not likely to have given all the answers in such turbulent market conditions either, its ability to accommodate the changes in the correlation skew could help the practitioners get a better handle on the fast moving markets.

One of the possible directions for generalization of our model is to move from a single market factor to a multi-factor framework. The well-documented importance of both macro and industry factors for explanation of equity returns suggests that the same factors could be instrumental in getting a more accurate model of credit correlations as well.

Whether in a single factor or a multi-factor setting, many of our conclusions reflect the limitations of the large homogeneous portfolio approximation which we have adopted in this paper. In particular, it is clear that even deterministic but heterogenous idiosyncrasies, market factor loadings and hazard rates could lead to significant changes in portfolio loss distribution and consequently to the correlation spectrum of the model. An extension of our model to such heterogenous case is possible, although the computational efforts will increase very significantly. Still, the promising features demonstrated by our approach even in the LHP approximation suggest that despite the computational difficulties, such extensions might be a worthy effort. In particular, the explicit modeling of the heterogeneous reference portfolio would have been absolutely necessary if one were to attempt to explain the tranche pricing during significant market dislocations.

Another important simplification which we have made when discussing the results of our model with regard to the correlation spectra is that we have only considered the unconditional return distributions and have not explored the effects of the initial shocks to either TARCH returns of volatility. From the perspective of a credit investor this means that we have described the "equilibrium" (in a loose sense of that word) state of the tranche market, but not the effects related to the relaxation towards the equilibrium. One could expect to find interesting results in this line of research, which would be that much more relevant given the credit market's propensity to undergo unexpected short-term dislocations as we have witnessed several times over the past couple of years.

Among the more practical questions that remain for future investigation are the calculation of the deltas or hedge ratios of synthetic CDO tranches within our framework, defining relative value measures for tranches reflecting the model's ability to produce the "fair" or "predicted" correlation spectrum.

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# A Kurtosis and Skewness of Aggregated TARCH Returns

In this notes we analyze kurtosis and skewness of aggregated returns $R_T = \frac{1}{\sqrt{T}}\sum_{t=1}^{T} r_t$ when $r_t$ is assumed to follow TARCH(1,1) process

$$\begin{aligned} r_t &= \sigma_t \varepsilon_t \\ \sigma_t^2 &= (1-\rho)\,\sigma^2 + \alpha r_{t-1}^2 + \alpha_D r_{t-1}^2 1_{\{r_{t-1} \leq 0\}} + \beta \sigma_{t-1}^2 \end{aligned}$$

where returns innovations $\varepsilon_t$ are assumed to be iid, have zero mean and unit variance. We are interested in variance, skewness and kurtosis of time aggregated returns. To make sure that those moments are finite we need corresponding moments of the return innovations to be finite. Particulaly, we assume that

$\varepsilon_t$ has finite kurtosis. Let us introduce the following notations for the central and truncated moments of $\varepsilon_t$

$$
\begin{aligned}
m_\varepsilon &\equiv E\left(\varepsilon_t\right) = 0 \\
v_\varepsilon &\equiv E\left(\varepsilon_t^2\right) = 1 \\
v_\varepsilon^d &\equiv E\left(\varepsilon_t^2 1_{\{\varepsilon_t \leq 0\}}\right) \\
s_\varepsilon &\equiv E\left(\varepsilon_t^3\right) \\
s_\varepsilon^d &\equiv E\left(\varepsilon_t^3 1_{\{\varepsilon_t \leq 0\}}\right) \\
k_\varepsilon &\equiv E\left(\varepsilon_t^4\right) \\
k_\varepsilon^d &\equiv E\left(\varepsilon_t^4 1_{\{\varepsilon_t \leq 0\}}\right)
\end{aligned}
$$

**Lemma 7** *The following recursions hold for TARCH(1,1) model*

$$
\begin{aligned}
cov_{t-1}\left(r_t^k, r_{t+u}^2\right) &= \rho cov_{t-1}\left(r_t^k, r_{t+u-1}^2\right) \text{ for } u>1 \\
cov_{t-1}\left(r_t^k r_{t+1}^2\right) &= \alpha var_{t-1}\left(r_t^{k+2}\right) + \alpha_D var_{t-1}\left(r_t^{k+2} 1_{\{r_t \leq 0\}}\right)
\end{aligned}
$$

***Proof.***

$$
\begin{aligned}
&cov_{t-1}\left(r_t^k, r_{t+u}^2\right) = cov_{t-1}\left(r_t^k \left[(1-\rho)\,\sigma^2 + \alpha r_{t+u-1}^2 + \alpha_D r_{t+u-1}^2 1_{\{r_{t+u-1} \leq 0\}} + \beta \sigma_{t+u-1}^2\right]\right) = \\
&0 + \alpha cov_{t-1}\left(r_t^k, r_{t+u-1}^2\right) + \alpha_D cov_{t-1}\left(r_t^k, r_{t+u-1}^2 1_{\{r_{t+u-1} \leq 0\}}\right) + \beta cov_{t-1}\left(r_t^k, \sigma_{t+u-1}^2\right)
\end{aligned}
$$

*if u>1 then*

$$
\begin{aligned}
cov_{t-1}\left(r_t^k, r_{t+u-1}^2 1_{\{r_{t+u-1} \leq 0\}}\right) &= v_\varepsilon^d cov_{t-1}\left(r_t^k, r_{t+u-1}^2\right) \\
cov_{t-1}\left(r_t^k, \sigma_{t+u-1}^2\right) &= cov_{t-1}\left(r_t^k, r_{t+u-1}^2\right)
\end{aligned}
$$

*If u=1 then*

$$
cov_{t-1}\left(r_t^k, \sigma_{t+u-1}^2\right) = 0
$$

■

**Proposition 8** *Suppose* $0 \leq \rho < 1$ *and the return innovations have finite skewness,* $s_\varepsilon$*, and finite "truncated" third moment,* $s_\varepsilon^d$*, then conditional third moment of* $R_{t+1,t+T}$ *has the following representation for TARCH(1,1)*

$$
E_t R_{t+1,t+T}^3 = \frac{1}{T^{3/2}} s_\varepsilon \sum_{u=1}^{T} E_t\left(\sigma_{t+u}^3\right) + \frac{3}{T^{3/2}}\left(\alpha s_\varepsilon + \alpha_D s_\varepsilon^d\right) \sum_{u=1}^{T} \frac{1-\rho^{T-u}}{1-\rho} E_t\left(\sigma_{t+u}^3\right)
$$

*In addition if* $E\sigma_t^3$ *is finite then unconditional skewness of* $R_{t+1,t+T}$ *is given by*

$$
S_T \equiv \frac{E R_{t+1,t+T}^3}{E(R_{t+1,t+T}^2)^{3/2}} = \left[\frac{1}{T^{1/2}} s_\varepsilon + 3\frac{1}{T^{3/2}}\left(\alpha s_\varepsilon + \alpha_D s_\varepsilon^d\right) \frac{T(1-\rho) - 1 + \rho^T}{(1-\rho)^2}\right] E\left(\frac{\sigma_t}{\sigma}\right)^3
$$

***Proof.*** *Using Lemma # we have*

$$E_t\left(\sum_{u=t+1}^{t+T} r_u\right)^3 = E_t\left(\sum_{t+1\le t_1\le t_2\le t_3\le t+T} r_{t_1}r_{t_2}r_{t_3}\right) = \sum_{u=1}^{T} E_t r_{t+u}^3 + \sum_{t+1\le t_1<t_2\le t+T} 3E_t\left(r_{t_1}r_{t_2}^2\right) =$$

$$\sum_{u=1}^{T} E_t\left(r_{t+u}^3\right) + 3\sum_{t+1\le t_1<t_2\le t+T} \rho^{t_2-t_1-1}\left(\alpha E_t\left(r_{t_1}^3\right) + \alpha_D E_t\left(r_{t_1}^3 1_{\left\{r_{t_1}\le 0\right\}}\right)\right) =$$

$$\sum_{u=1}^{T} E_t\left(r_{t+u}^3\right) + 3\sum_{u=1}^{T}\frac{1-\rho^{T-u}}{1-\rho}\left(\alpha E_t\left(r_{t+u}^3\right) + \alpha_D E_t\left(r_{t+u}^3 1_{\{r_{t+u}\le 0\}}\right)\right) =$$

$$s_\varepsilon\sum_{u=1}^{T} E_t\left(\sigma_{t+u}^3\right) + \left(\alpha s_\varepsilon + \alpha_D s_\varepsilon^d\right)\sum_{u=1}^{T}\frac{1-\rho^{T-u}}{1-\rho}E_t\left(\sigma_{t+u}^3\right)$$

*Using the law of iterated expectations*

$$E\left(\sum_{u=t+1}^{t+T} r_u\right)^3 = E\left(E_t\left(\sum_{u=t+1}^{t+T} r_u\right)^3\right) = \left[Ts_\varepsilon + 3\left(\alpha s_\varepsilon + \alpha_D s_\varepsilon^d\right)\frac{T(1-\rho)-1+\rho^T}{(1-\rho)^2}\right]E\left(\sigma_t\right)^3$$

*$S_T$ is then computed using the simple formula for the unconditional variance $E(R_{t+1,t+T}^2) = \sigma^2$.* ■

To derive unconditional kurtosis we define the following unconditional autocorrelations

$$\begin{aligned}
\gamma_n &= \gamma_{-n} = corr(r_{t-n}^2, r_t^2)\\
\varphi_n &= corr(r_{t-n}, r_t^2) \text{ for } n \ge 1\\
\psi_{i,j} &\equiv E\left(r_{t-i}r_{t-j}r_t^2\right) \text{ for } 1 \le j < i
\end{aligned}$$

**Lemma 9** *$\gamma_n$, $\varphi_n$ and $\psi_{i,j}$ decay exponentially as $n$ and $i-j$ increase*

$$\begin{aligned}
\gamma_n &= \rho\gamma_{n-1} = \rho^{n-1}\gamma_1 \textit{ for } n \ge 1\\
\varphi_n &= \rho\varphi_{n-1} = \rho^{n-1}\varphi_1 \textit{ for } n \ge 1\\
\psi_{i,j} &= \rho\psi_{i-1,j-1} = \rho^{j-1}\psi_{i-j+1,1} \textit{ for } 1 \le j < i
\end{aligned}$$

*where $\gamma_1$, $\varphi_1$ and $\psi_{k,1}$ are given by*

$$\begin{aligned}
\gamma_1 &= \alpha\left(k_r - 1\right) + \alpha_D\left(k_r^d - v_r^d\right) + \beta k_r/k_\varepsilon\\
\varphi_1 &= \alpha s_r + \alpha_D s_r^d\\
\psi_{k,1} &= \alpha E\left(r_{t-k+1}r_t^3\right) + \alpha_D E\left(r_{t-k+1}r_t^3 1_{\{r_t\le 0\}}\right)
\end{aligned}$$

*with $v_\varepsilon^d = \frac{E\left(r_t^2 1_{\{r_t\le 0\}}\right)}{Er_t^2}$, $s_r = \frac{E\left(r_t^3\right)}{(Er_t^2)^{3/2}}$, $s_r^d = \frac{E\left(r_t^3 1_{\{r_T<0\}}\right)}{(Er_t^2)^{3/2}}$, $k_r = \frac{E\left(r_t^4\right)}{(Er_t^2)^2}$ and $k_r^d = \frac{E\left(r_t^4 1_{\{r_t\le 0\}}\right)}{(Er_t^2)^2}$.*

**Proposition 10** *If*

$$\rho \equiv E\left(\beta + \alpha\varepsilon_t^2 + \alpha_D\varepsilon_t^2 1_{\{\varepsilon_t \le 0\}}\right) = \beta + \alpha + \alpha_D v_\varepsilon^d < 1$$
$$\gamma \equiv E\left(\beta + \alpha\varepsilon_t^2 + \alpha_D\varepsilon_t^2 1_{\{\varepsilon_t \le 0\}}\right)^2 = \beta^2 + \alpha^2 k_\varepsilon + \alpha_D^2 k_\varepsilon^d + 2\alpha\beta + 2\alpha_D\beta v_\varepsilon^d + 2\alpha\alpha_D k_\varepsilon^d < 1$$

*then unconditional kurtosis of $r_t$, $K_1$, is finite and*

$$K_1 \equiv \frac{Er_t^4}{\left(Er_t^2\right)^2} = k_\varepsilon \frac{1-\rho^2}{1-\gamma}$$

***Proof.*** *If the 4th moment of $r_t$ exists then the following equation must hold*

$$Er_t^4 = E\left(\varepsilon_t^4\right) E\left(\sigma_t^4\right) = k_\varepsilon E\left((1-\rho)\,\sigma^2 + \alpha r_{t-1}^2 + \alpha_D r_{t-1}^2 1_{\{r_{t-1}\le 0\}} + \beta\sigma_{t-1}^2\right)^2 =$$
$$k_\varepsilon((1-\rho)^2\sigma^4 + 2(1-\rho)\,\sigma^2 E\left(\alpha r_{t-1}^2 + \alpha_D r_{t-1}^2 1_{\{r_{t-1}\le 0\}} + \beta\sigma_{t-1}^2\right) +$$
$$\left(\alpha r_{t-1}^2 + \alpha_D r_{t-1}^2 1_{\{r_{t-1}\le 0\}} + \beta\sigma_{t-1}^2\right)^2) = k_\varepsilon\left((1-\rho)^2\sigma^4 + 2(1-\rho)\,\rho\sigma^4 + \gamma E\sigma_{t-1}^4\right)$$

*Therefore $Er_t^4$ nessecerily solves*

$$Er_t^4 = k_\varepsilon\left(1-\rho^2\right)\sigma^4 + \gamma Er_t^4$$

■

**Proposition 11** *If the distibution of $\varepsilon_t$ is symmetric and $\alpha_D = 0$ then unconditional kurtosis of $R_T$, if exists, is given by the following formula:*

$$K_T = 3 + \frac{1}{T}(K_1 - 3) + 6\frac{\gamma_1}{T^2}\frac{T(1-\rho) - 1 + \rho^T}{(1-\rho)^2} \textit{for } T > 1 \tag{44}$$
$$K_1 = k_\varepsilon\frac{1-\rho^2}{1-\gamma} \tag{45}$$

*where $k_\varepsilon$ is unconditional kurtosis of $\varepsilon_t$ and*

$$\gamma \equiv E\left(\beta + \alpha\varepsilon_t^2 + \alpha_D\varepsilon_t^2 1_{\{\varepsilon_t \le 0\}}\right)^2 = \beta^2 + \alpha^2 k_\varepsilon + \alpha_D^2 k_\varepsilon^d + 2\alpha\beta + 2\alpha_D\beta v_\varepsilon^d + 2\alpha\alpha_D k_\varepsilon^d.$$
$$\gamma_1 \equiv corr\left(r_{t-1}^2, r_t^2\right) = \alpha\left(k_r - 1\right) + \alpha_D\left(k_r^d - v_r^d\right) + \beta k_r / k_\varepsilon$$

***Proof.***

$$E\left(\sum_{u=t+1}^{t+T} r_u\right)^4 = \sum_{u=1}^{T} E\left(r_{t+u}^4\right) + 6\sum_{t+1\le t_1<t_2\le t+T} E\left(r_{t_1}^2 r_{t_2}^2\right) =$$
$$\sum_{u=1}^{T} E\left(r_{t+u}^4\right) + 6\sum_{t+1\le t_1<t_2\le t+T}\left[cov\left(r_{t_1}^2, r_{t_2}^2\right) + E\left(r_{t_1}^2\right)E\left(r_{t_2}^2\right)\right] =$$
$$TE\left(r_t^4\right) + 6\frac{T(T-1)}{2}E\left(r_t^2\right)^2 + 6cov\left(r_{t-1}^2, r_t^2\right)\sum_{t+1\le t_1<t_2\le t+T}\rho^{t_2-t_1-1} =$$
$$TE\left(r_t^4\right) + 6\frac{T(T-1)}{2}E\left(r_t^2\right)^2 + 6cov\left(r_{t-1}^2, r_t^2\right)\frac{T(1-\rho) - 1 + \rho^T}{(1-\rho)^2}$$

*substituting the derived 4th moment into the definition of the kurtosis* $K_T = E\left(\sum_{u=t+1}^{t+T} r_u\right)^4 / E\left(r_t^2\right)^2$ *completes the proof.* ■

# B Estimation Results for SP500

Table 1a: SP500 moments.

| Sample period | Daily | | | | Weekly | | | |
|---|---|---|---|---|---|---|---|---|
| | $s_r$ | $s_r^d$ | $k_r$ | $k_r^d$ | $s_r$ | $s_r^d$ | $k_r$ | $k_r^d$ |
| 1962-2004 | -1.40 | -2.43 | 39.83 | 0.53 | -0.55 | -1.35 | 7.01 | 0.55 |
| 1990-2004 | -0.11 | -1.14 | 6.67 | 0.51 | -0.64 | -1.36 | 6.10 | 0.56 |

Table 1b: SP500 moments(After trimming 0.1% of extreme positive and negative returns )

| Sample period | Daily | | | | Weekly | | | |
|---|---|---|---|---|---|---|---|---|
| | $s_r$ | $s_r^d$ | $k_r$ | $k_r^d$ | $s_r$ | $s_r^d$ | $k_r$ | $k_r^d$ |
| 1962-2004 | 0.05 | -1.03 | 5.95 | 0.50 | -0.39 | -1.18 | 5.26 | 0.54 |
| 1990-2004 | 0.04 | -1.01 | 5.56 | 0.50 | -0.64 | -1.36 | 6.10 | 0.56 |

Table 2a

Estimated parameters of GARCH(1,1)/TARCH(1,1) with Gaussian/GED shocks on daily(D) and weekly(W) SP500 returns for.[01/01/1990-12/31/2004].

| | D | W | D | W | D | W | D | W |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 0.056 | 0.044 | 0.007 | 0.007 | 0.05 | 0.044 | 0.006 | 0.007 |
| $\alpha_D$ | - | - | 0.100 | 0.112 | - | - | 0.095 | 0.099 |
| $\beta$ | 0.941 | 0.953 | 0.933 | 0.918 | 0.948 | 0.951 | 0.939 | 0.924 |
| $v$ | - | - | - | - | 1.40 | 1.46 | 1.45 | 1.50 |

Table 2a_old

Estimated parameters of GARCH(1,1)/TARCH(1,1) with Gaussian/Student-T shocks on daily(D) and weekly(W) SP500 returns for.[01/01/1990-12/31/2004].

| | D | W | D | W | D | W | D | W |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 0.056 | 0.044 | 0.007 | 0.007 | 0.047 | 0.045 | 0.006 | 0.004 |
| $\alpha_D$ | - | - | 0.100 | 0.112 | - | - | 0.095 | 0.094 |
| $\beta$ | 0.941 | 0.953 | 0.933 | 0.918 | 0.951 | 0.948 | 0.941 | 0.927 |
| $\nu$ | - | - | - | - | 7.25 | 7.77 | 8.21 | 8.31 |

Table 2b(After trimming 0.1% of extreme positive and negative returns )

Estimated parameters of GARCH(1,1)/TARCH(1,1) with Gaussian/GED shocks on daily(D) and weekly(W) SP500 returns for.[01/01/1990-12/31/2004].

| | D | W | D | W | D | W | D | W |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 0.062 | 0.094 | 0.023 | 0.032 | 0.047 | 0.047 | 0.004 | 0.009 |
| $\alpha_D$ | - | - | 0.067 | 0.112 | - | - | 0.091 | 0.095 |
| $\beta$ | 0.935 | 0.896 | 0.941 | 0.895 | 0.950 | 0.948 | 0.944 | 0.925 |
| $\nu$ | - | - | - | - | 1.44 | 1.51 | 1.49 | 1.55 |

Table 2b_old(After trimming 0.1% of extreme positive and negative returns )
Estimated parameters of GARCH(1,1)/TARCH(1,1) with Gaussian/Student-T shocks on daily(D) and weekly(W) SP500 returns for.[01/01/1990-12/31/2004].

| | D | W | D | W | D | W | D | W |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 0.062 | 0.094 | 0.023 | 0.032 | 0.062 | 0.088 | 0.022 | 0.030 |
| $\alpha_D$ | - | - | 0.067 | 0.112 | - | - | 0.073 | 0.104 |
| $\beta$ | 0.935 | 0.896 | 0.941 | 0.895 | 0.936 | 0.90 | 0.938 | 0.90 |
| $\nu$ | - | - | - | - | 8.82 | 10.64 | 9.63 | 11 |

Table 3a
Estimated parameters of GARCH(1,1)/TARCH(1,1) with Gaussian/GED shocks on daily(D) and weekly(W) SP500 returns for.[01/01/1962-12/31/2004].

| | D | W | D | W | D | W | D | W |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 0.076 | 0.107 | 0.029 | 0.037 | 0.068 | 0.098 | 0.028 | 0.036 |
| $\alpha_D$ | - | - | 0.081 | 0.136 | - | - | 0.076 | 0.122 |
| $\beta$ | 0.923 | 0.886 | 0.928 | 0.877 | 0.93 | 0.891 | 0.932 | 0.884 |
| $\nu$ | - | - | - | - | 1.41 | 1.56 | 1.44 | 1.61 |

Table 3a_old
Estimated parameters of GARCH(1,1)/TARCH(1,1) with Gaussian/Student-T shocks on daily(D) and weekly(W) SP500 returns for.[01/01/1962-12/31/2004].

| | D | W | D | W | D | W | D | W |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 0.076 | 0.107 | 0.029 | 0.037 | 0.064 | 0.09 | 0.027 | 0.032 |
| $\alpha_D$ | - | - | 0.081 | 0.136 | - | - | 0.072 | 0.106 |
| $\beta$ | 0.923 | 0.886 | 0.928 | 0.877 | 0.934 | 0.897 | 0.934 | 0.894 |
| $\nu$ | - | - | - | - | 7.86 | 9.26 | 8.44 | 10.19 |

Table 3b(After trimming 0.1% of extreme positive and negative returns )
Estimated parameters of GARCH(1,1)/TARCH(1,1) with Gaussian/GED shocks on daily(D) and weekly(W) SP500 returns for.[01/01/1962-12/31/2004].

| | D | W | D | W | D | W | D | W |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 0.051 | 0.044 | 0.004 | 0.007 | 0.062 | 0.091 | 0.022 | 0.032 |
| $\alpha_D$ | - | - | 0.093 | 0.112 | - | - | 0.071 | 0.108 |
| $\beta$ | 0.945 | 0.953 | 0.940 | 0.918 | 0.936 | 0.898 | 0.940 | 0.897 |
| $\nu$ | - | - | - | - | 1.48 | 1.64 | 1.51 | 1.68 |

Table 3b_old(After trimming 0.1% of extreme positive and negative returns )
Estimated parameters of GARCH(1,1)/TARCH(1,1) with Gaussian/Student-T shocks on daily(D) and weekly(W) SP500 returns for.[01/01/1962-12/31/2004].

| | D | W | D | W | D | W | D | W |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 0.051 | 0.044 | 0.004 | 0.007 | 0.046 | 0.045 | 0.004 | 0.004 |
| $\alpha_D$ | - | - | 0.093 | 0.112 | - | - | 0.096 | 0.094 |
| $\beta$ | 0.945 | 0.953 | 0.940 | 0.918 | 0.952 | 0.947 | 0.941 | 0.926 |
| $\nu$ | - | - | - | - | 7.78 | 7.77 | 9.011 | 8.31 |

## C Generalized Error Distribution(GED)

The density of the GED has the following form

$$
\begin{aligned}
g\left(x; v\right) &= A e^{-\frac{1}{2}\left|\frac{x}{\eta}\right|^{v}} \\
A &= \frac{v}{2^{1+1/v}\eta\Gamma\left(1/v\right)} \\
\eta &= 2^{-1/v}\sqrt{\frac{\Gamma\left(1/v\right)}{\Gamma\left(3/v\right)}}
\end{aligned}
$$