THE ANNUITY PUZZLE REVISITED:A DETERMINISTIC VERSION WITH LAGRAGIAN METHODS

Pierre DEVOLDER
Donatien HAINAUT

(Université Catholique de Louvain- UCL)

Outline

- 1. Introduction
- 2. The classical approach
- 3. Optimal consumption with constraint
- 4. Optimal level of annuitization
- 5. Conclusion

1. Introduction

-Increasing interest for the annuity market these last years :

- Financial problems of the classical *pay as you go* pension systems …a major threat for the future; need for funding;
- Protection against longevity risk;
- Development of Funded Defined Contribution plans and sharing of the financial risk;

1. Introduction

-But the volontary annuity markets remain thin everywhere

- In Belgium , annuities are completely confidential in the third pillar of pension and are present in the second pillar only when compulsory (pension funds)....

- Same situation in other countries : annuities are well developed in the pension structure only when there is no other choice !!!

1. Introduction

-Opposite theoretical result : YAARI (1965):

Annuitization is an optimal rule for a rational individual !!!!

… under some assumptions:

- only risk is longevity risk
- only one risk free asset on the market
- no bequest motive
- annuity market is actuarially fair (no loading)

1. Introduction

Principal idea :

MORTALITY CREDIT IN THE RETURN OF A LIFE ANNUITY

$$\text{return} \approx r + \mu_x$$

Since then, more general models have been developed in order to address this so called « *annuity puzzle* » and to explain why full annuitization is not popular at all.

1. Introduction

- Richard (1974)
- Milevsky (1998, 2001)
- Milevsky / Young (2003)
- Purcal/ Piggott (2004)

1. Introduction

Purpose of this presentation :

- introducing some constraint on the level of wealth (*positiveness* ; *no possibility of borrowing)*

Effect on the optimal consumption profile ???

Effect on the optimal annuitization level ???

1. Introduction

Tool : incorporation of the wealth constraint in the Bellman's equation using Lagrangian multipliers in a deterministic framework

Main result in terms of consumption: decomposition in 2 periods :

Period 1 : constraint inactive
Period 2 : constraint active

Main result in terms of annuitization : full annuitization is no more optimal

2. The classical approach

NOTATIONS :

-initial age : x (retirement)
-initial wealth : W_0
-part invested in a life annuity : α
-part invested in the financial market : $1-\alpha$
-level of annuity : B
-consumption at time t : $c(t)$
-assets at time t : $F(t)$
-utility function : U

2. The classical approach

Level of annuity :

$$B = \frac{\alpha W_0}{(1-\varepsilon)\, a_x}$$

loading

Dynamic of the assets :

$$dF(t) = (r\,F(t) + B - c(t))\,dt$$

Risk free rate

$$(F(0) = (1-\alpha)\,W_0)$$

2. The classical approach

First Optimization problem :optimal consumption

$$v((1-\alpha)W_0,0) = \max_c \; E\int_0^T e^{-\rho s}\; U(c(s))\,{}_s p_x \; ds$$

$$\rho = \text{subjective discount factor}$$

$$U = \text{utility function} = \frac{c^\gamma}{\gamma} \quad (\gamma < 1)$$

Second Optimization problem :optimal annuitization

$$\max_\alpha \; v((1-\alpha)W_0,0)$$

2. The classical approach

Value function :

$$v(F(t),t) = \max_c \int_t^T e^{-r(s-t)} \frac{(c(s))^\gamma}{\gamma} \, {}_{s-t}p_{x+t} \, ds$$

HAMILTON JACOBI BELLMAN Equation :

$$\frac{\partial v}{\partial t} - (r + \mu(x+t))v + \sup_c((rF(t) + B - c(t))\frac{\partial v}{\partial F} + \frac{(c(t))^\gamma}{\gamma}) = 0$$

2. The classical approach

Optimal consumption :

$$c^*(t) = \left(\frac{\partial v}{\partial F} \right)^{\frac{1}{\gamma - 1}}$$

Candidate value function :

$$v(F(t),t) = b(t)\frac{(F(t)+a(t))^{\gamma}}{\gamma}$$

2. The classical approach

First case : no annuitization possible (α=0) (MERTON)

$$c^*(t) = \frac{F(t)}{\int_t^T \exp\left(-\int_t^s \left(r + \frac{\mu(x+z)}{(1-\gamma)}\right)dz\right) ds} < F(t)$$

↓

The fund remains positive

↓

acceptable solution

2. The classical approach

Second case : annuitization possible

Optimal consumption:

$$c^*(t) = \frac{F(t) + \int_t^T B \exp(-r(s-t))\, ds}{\int_t^T \exp\left(-\int_t^s \left(r + \frac{\mu(x+z)}{(1-\gamma)}\right) dz\right) ds}$$

Wealth

Present value of future annuities

2. The classical approach

Optimal annuitization :

$$v((1-\alpha)W_0,0) = k\ \ ((1-\alpha)W_0 + \alpha W_0\ \ \psi)^{\gamma}$$

With:

$$\psi = \frac{1}{1-\varepsilon}\frac{\int_0^T \exp(-r\,s)\,ds}{\int_0^T \exp(-r\,s - \int_0^s \mu^t(x+z)\,dz)\,ds}$$

2. The classical approach

Optimal rule :

if $\psi>1$ (normal case) : full annuitization is optimal

$$\alpha^* = 1$$

if $\psi<1$ (extreme case) : « no annuitization » is optimal

$$\alpha^* = 0$$

→ *Partial annuitization is never optimal*

2. The classical approach

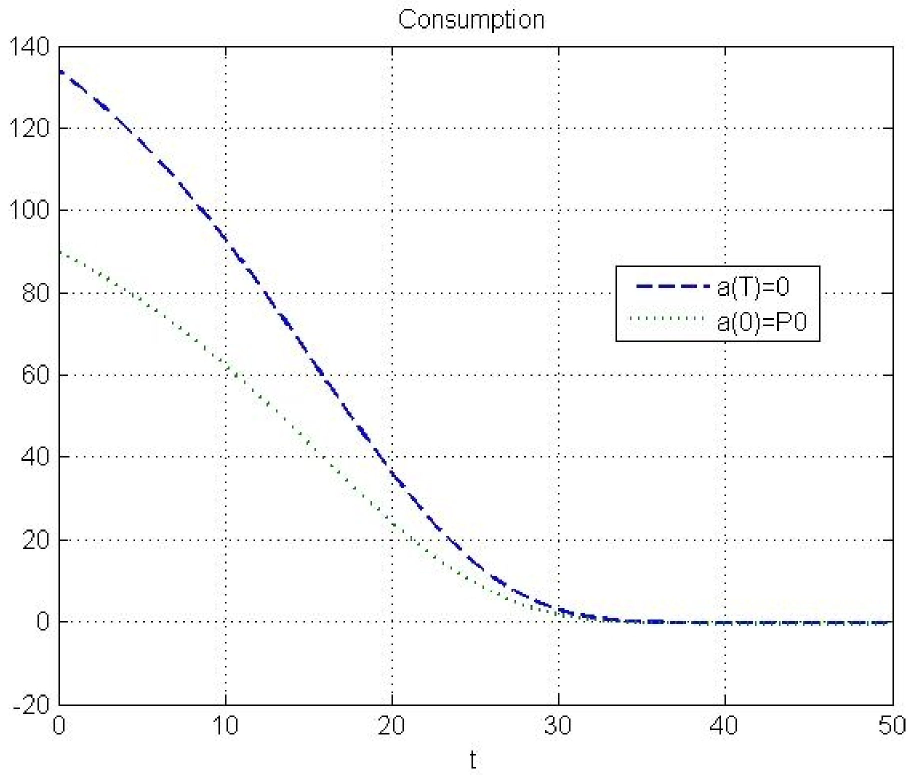
… But the fund becomes negative after a few years even if it goes back to zero at maturity (t=T) …

Example :

$$x = 60$$
$$W_0 = 1000$$
$$\alpha = 75\%$$
$$r = 3.25\%$$
$$\gamma = 0.60$$
$$\varepsilon = 0$$
$$\text{Belgian mortality rates } (T = 50)$$

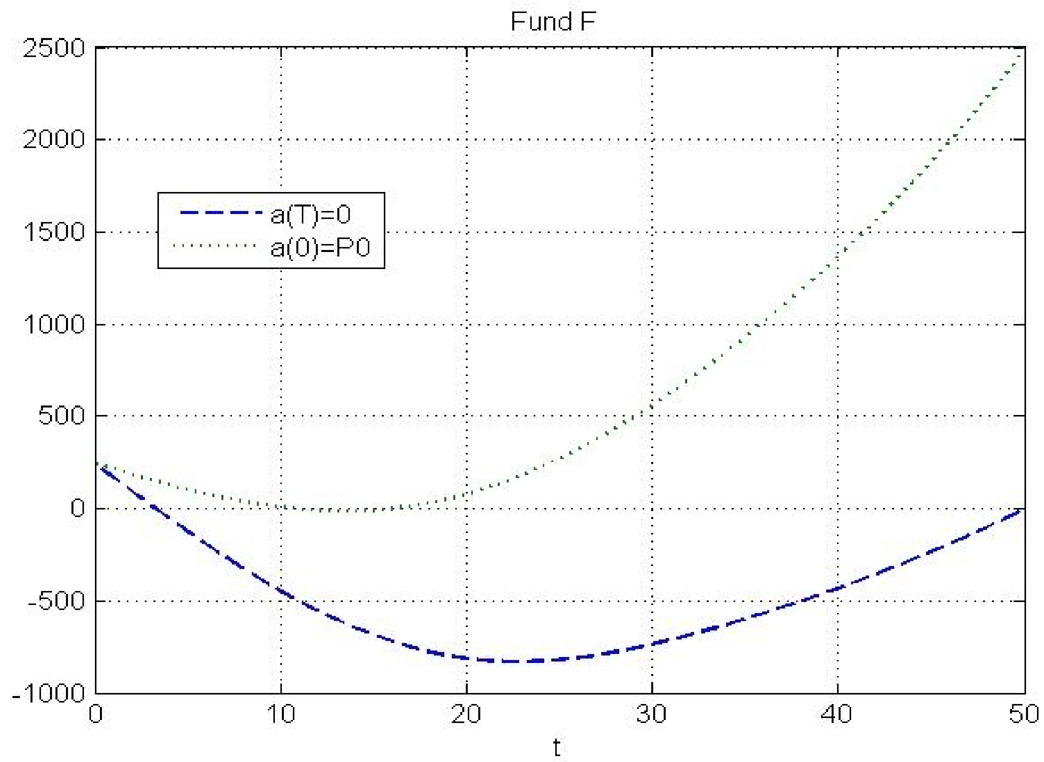
2. The classical approach

Evolution of the consumption :



2. The classical approach

Evolution of the fund :



3. Optimal consumption with constraint

Additional constraint :

The fund F(t) has to remain positive during the whole life of the agent.

$$v((1-\alpha)W_0,0) = \max_c \; E\int_0^T e^{-\rho s} \; U(c(s)) \, {}_s p_x \; ds$$

Subject to :

$$F(t) \geq 0$$

3. Optimal consumption with constraint

Introduction of a Lagrangian multiplier:

$$\Lambda(F(t), t)$$

>0	$=0$
if constraint active $(F(t)=0)$	if constraint not active $(F(t)>0)$

3. Optimal consumption with constraint

HAMILTON JACOBI BELLMAN Equation :

$$\frac{\partial v}{\partial t} - (r + \mu(x+t))\, v + \sup_c ((rF(t) + B - c(t))\frac{\partial v}{\partial F} + \frac{(c(t))^{\gamma}}{\gamma})$$

$$+ \Lambda(F(t), t)\; F(t) = 0$$

3. Optimal consumption with constraint

Candidate value function:

$$v(F(t),t) = b(t)\frac{(F(t)+a(t))^{\gamma}}{\gamma}$$

Candidate Lagrangian multiplier :

$$\Lambda(F(t),t) = \lambda(t)\ b(t)\ (F(t)+a(t))^{\gamma-1}$$

3. Optimal consumption with constraint

We obtain 2 ODE :

$$\text{(i)}\quad b'(t)-(r(1-\gamma)+\mu(x+t)-\lambda(t)\gamma)\,b(t)-b^{\frac{\gamma}{\gamma-1}}=0$$

$$\text{(ii)}\quad a'(t)-r\,a(t)+B-\lambda(t)\,a(t)=0$$

1° period : $\lambda=0$ and $F>0$ (cf. without constraint)

2° period : $\lambda>0$ and $F=0$ (effect of the constraint)

3. Optimal consumption with constraint

First case : full annuitization :

- F(0)=0
- the discount rate of the objective (r+ μ) is bigger than the rate of return of the cash (r)

$$c^*(t) = B$$

$$\lambda(t) = \mu(x+t)$$

$$v(F(t),t) = \frac{B^\gamma}{\gamma} a_{x+t}$$

3. Optimal consumption with constraint

Second case : partial annuitization :

- *first period* : **t<t*** : F(t)>0 : consumption of the fund

  (*same kind of solution as without constraint*)

- *second period* : **t>t*** : F(t)=0 : consumption equal to the annuity B and lagrangian equal to μ

  (*same kind of solution as full annuitization*)

3. Optimal consumption with constraint

No analytical expression for the transition time t* !!!

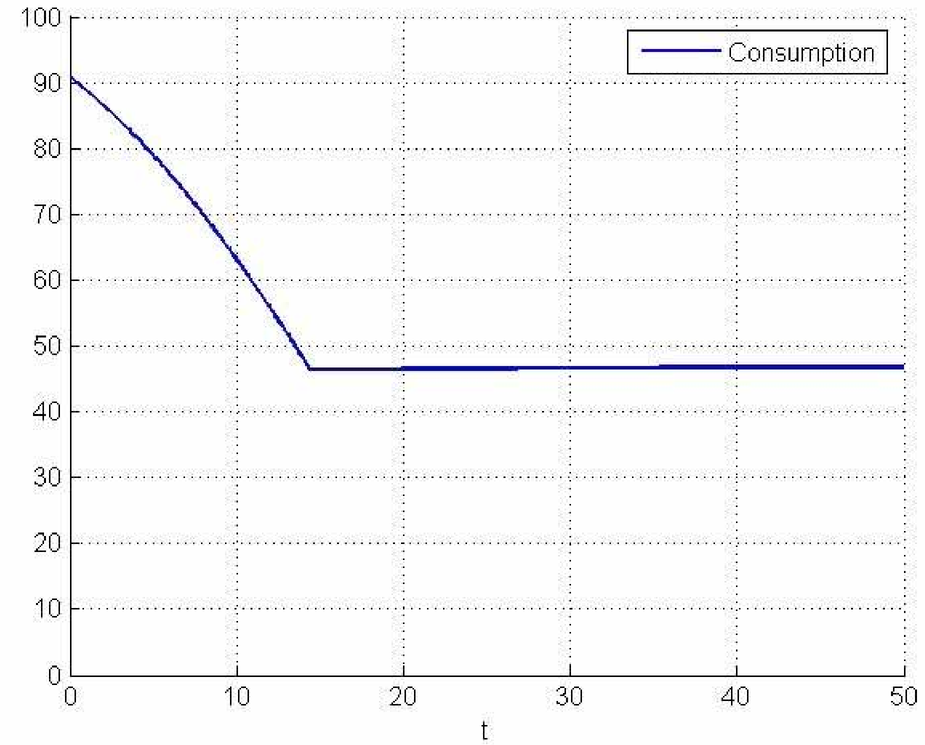
(numerical algorithm)

EXAMPLE of section 2 :

$t^* = 13.35$

3. Optimal consumption with constraint

Evolution of the consumption :



3. Optimal consumption with constraint

Possible generalization :

- cash return different from the guarantee of the life insurance (r):

$$r_f = r + \theta$$

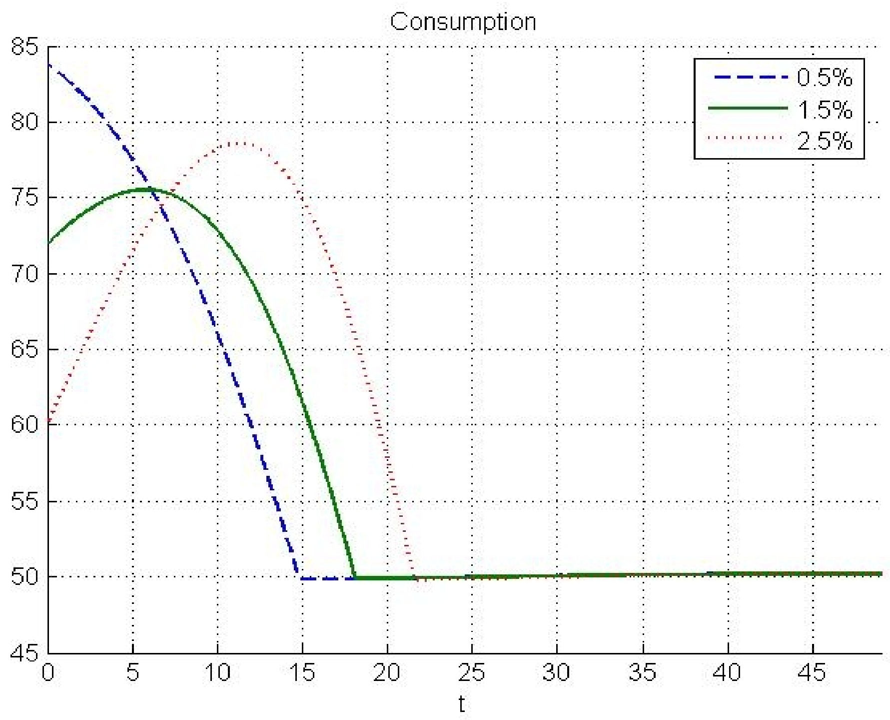
- discount rate different from the guarantee of the life insurance :

$$\rho = r + \phi$$

3. Optimal consumption with constraint

Example of section 2 with various positive θ :

Evolution of the consumption :



4. Optimal level of annuitization

Problem without constraint :

Partial annuitization is never optimal

↓

?? *Effect of the wealth constraint* ??

$$\alpha^* = \arg\sup_{\alpha} v((1-\alpha)W_0, 0)$$

4. Optimal level of annuitization

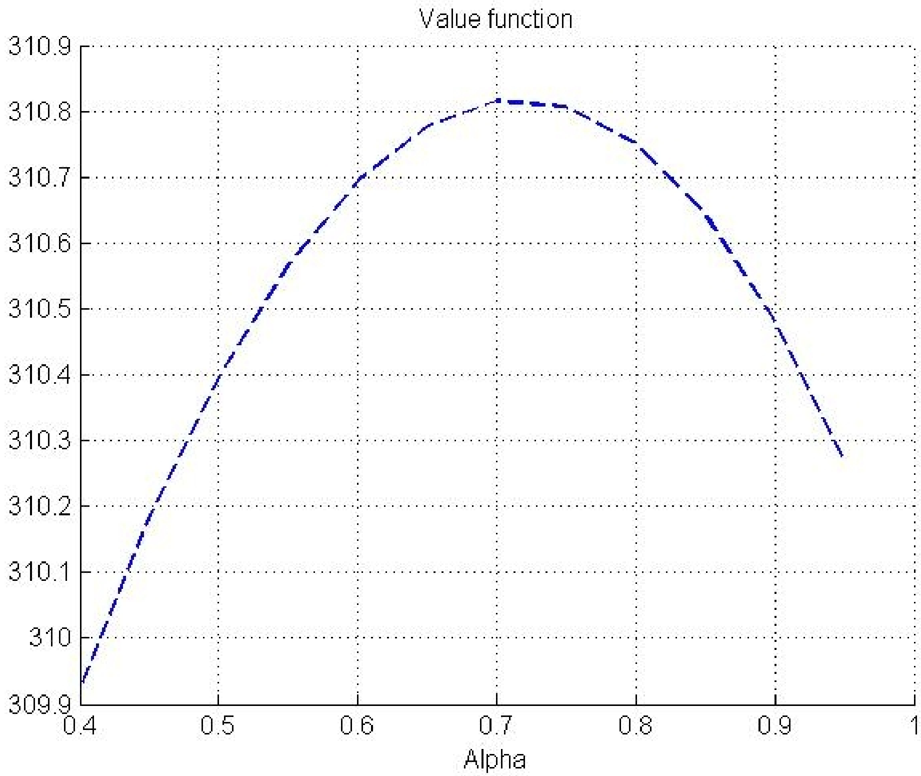
The solution with constraint is now a *partial annuitization* .

The level of the optimal ratio α^* is function of the age and of the spread between the cash return and the guarantee.

EXAMPLE OF SECTION 2 with a spread of 2%

4. Optimal level of annuitization

Evolution of the value function :



→ $\alpha^* = 70\%$

5.Conclusion

Future research :

same kind of model in a stochastic financial environment with 2 assets (risky and riskless).

devolder@fin.ucl.ac.be