THE ANNUITY PUZZLE REVISITED: A DETERMINISTIC VERSION WITH LAGRANGIAN METHODS.

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Abstract. In this paper, the classic problem of the annuity puzzle is revisited. We determine the optimal consumption strategy of an agent who invests his wealth both in an annuity and in a cash account. The constraint, which specifies that the individual doesn't borrow to consume, is introduced in the Bellman's equation via Lagrangian multipliers. The planning of consumption is subdivided in two periods and the transition time is calculated numerically. Finally, we show the influence of the cash return and of the age on the optimal sharing of the initial wealth between the life annuity and the cash account.

1. Introduction.

During the last years, life annuitization has become a serious option for retirees. Reasons explaining this trend are the announced reform of the social security and the shift from pure pay as you go systems to funding methods. Another motive is the huge volatility of alternative investments, such those available on financial markets: a life annuity is coupled to an interest rate guarantee which is a long term protection against financial losses. However, in practice, few retirees choose this solution. In this paper, we develop a model that answers to the question: "Why and how much should an agent purchase a life annuity?".

Literature counts many articles studying this problem known as the "annuity puzzle". Yaari (1965) was one of the first to prove that a consumer, without bequest motive should prefer a full annuitization in a market without risky assets. The underlying intuition is that the rate of return of an annuity is always higher than the risk free rate because it includes a mortality risk premium. Richard (1974) has solved the problem of optimal consumption/investment for an uncertain lifetime, in a financial markets composed of risky/non risky assets, and of death/life insurances. Kapur and Orszag (1999) have determined the optimal investment/consumption policy for an individual who invests in equities and endowments. Milevsky (1998, 2001) and Young (2000) have developed a model in which the individual defers the purchase of a life annuity until it is not possible to beat the rate of return from an annuity. Pirical (2003) has calibrated the Richard's model to the Japanese market in order to explain the demand of life annuity. The Yaari's model was discretized by Petrova (2004) and calibrated on the US market.

This paper proposes a methodology to find the optimal consumption for an individual who invests a part of his richness in a life annuity and the surplus in a cash account with a deterministic return, without bequest motives and without any possibility of borrowing. To our knowledge, the pattern of consumption has not yet been developed in this framework. This apparently easy problem may not be solved without taking into account the constraint that the individual doesn't borrow to consume because he doesn't want to leave a negative inheritance to his relatives. This constraint is incorporated in the Bellman's equation with Lagrangian multipliers. The optimal scheme of consumption may be decomposed in two periods. During the first one, the constraint on the wealth is inactive. In the second period, the constraint is active and the optimal consumption is equal to the life annuity. The time of transition between periods is unknown and is computed numerically. It is then possible to calculate the optimal fraction of the initial wealth that should be invested in a life annuity to maximize the utility arising from the consumption.

2. The classical approach.

We consider an agent who retires at age x and who decides to invest his wealth, noted W_0, in a life annuity and in a cash account. $P_0 = \alpha.W_0$ is the insurance premium paid to purchase a life insurance, which provides a continuous annuity B. $F_0 = (1-\alpha).W_0$ is the initial amount deposited on the cash account. The life annuity B is determined by:

$$B = \frac{\alpha.W_0}{\overline{a}_{x,x+T}}.\frac{1}{1-\epsilon} \tag{2.1}$$

Where :

- ϵ: commercial loading
- $\overline{a}_{x,x+T} = \int_x^{x+T} e^{-r.(s-x)}.{}_sp_x^{tf}.ds$: is a continuous life annuity from age x to age x+T. r is the interest rate granted to the costumer and ${}_sp_x^{tf}$ is the probability that an individual of age x is still alive at age x+s. T is the maximum age that a human being could reach.
- ${}_sp_x^{tf} = e^{-\int_x^s \mu^{tf}(x+z)dz}$ where $\mu^{tf}(t)$ is the mortality rate of the tariff. The real probability of decease and the associated mortality rate are noted respectively ${}_sp_x$ and $\mu(t)$ and will be used later in the formulation of the problem.

We assume that the cash account has a constant return equal to r, the same guarantee as the life insurance. This assumption will be relaxed in paragraph 3.3. The dynamic of the agent's asset at time t, F_t, is therefore:

$$dF_t = (r.F_t + B - c_t).dt$$

With c_t , the consumption of the individual. If the agent hasn't any bequest motive, he seeks to maximize the discounted utility of consumption till his decease. Let $U(c_t)$ be the utility of the consumption at time t. The problem is formulated as :

$$v(F_0, 0) = \max_{c_t \in \mathbb{R}} \mathbb{E}\left(\int_0^{\tau} e^{-\rho.s}.U(c_s).ds\right)$$

Where τ is the random instant of decease, ρ is the discount factor and $v(F_t, t)$ is the value function at time t. For the sake of simplicity, we choose $\rho = r$, this assumption will be also relaxed in paragraph 3.3. Under the additional assumption that the utility function is C.A.R.A. with a risk aversion parameter noted γ, the objective at time t is :

$$v(F_t,t) = \max_{c_t\in\mathbb{R}} \mathbb{E}\left(\int_t^{\tau} e^{-r.(s-t)}.\left(\frac{c_s}{\gamma}\right)^{\gamma} ds\,|\mathbb{F}_t\right) \tag{2.2}$$

$\mathbb{F}_t$ is the information at time t. It may be proved (see Yaari 1965, Richard 1974) that the objective (2.2) is equivalent to :

$$v(F_t,t) = \max_{c_t\in\mathbb{R}} \int_t^{T} e^{-r.(s-t)}.{}_sp_{x+t}.\left(\frac{c_t}{\gamma}\right)^{\gamma} ds \tag{2.3}$$

As mentioned before, ${}_sp_x$ is the real probability of decease of the individual and the related mortality rate is noted $\mu(x+s)$. The value function $v(F_t,t)$ is the $\mathbb{C}^{2,1}$ function, solution of the Hamilton Jacobi Bellman equation :

$$\frac{\delta v(F_t,t)}{\delta t} - (r+\mu(x+t)).v(F_t,t) + sup_{c_t}\left(\underbrace{L^{c_t}v(F_t,t) + \frac{c_t^{\gamma}}{\gamma}}_{Hamiltonian}\right) = 0 \tag{2.4}$$

Where $L^{c_t}v(F_t,t)$ is the infinitesimal generator of $v(F_t,t)$:

$$L^{c_t}v(F_t,t) = (r.F_t + B - c_t)\,.\frac{\delta v(F_t,t)}{\delta F_t}$$

And with the terminal condition:

$$v(F_T,T) = 0$$

We will see that this formulation isn't sufficient to find an acceptable solution, excepted if the agent doesn't purchase a life annuity ($\alpha = 0$). Before any further developments, we redevelop the solution found by Merton (1969-1971) for the scenario of full investment in a cash account.

2.1. **No annuitization.** If the agent doesn't purchase an annuity ($\alpha = 0$), the HJB equation takes the form :

$$v_t - (r+\mu(x+t)).v + sup_{c_t}\left(\underbrace{(r.F_t - c_t)\,.v_F + \frac{c_t^{\gamma}}{\gamma}}_{H}\right) = 0 \tag{2.5}$$

Where v_t and v_F designate respectively the derivative of $v(F_t,t)$ with regards to t and F_t. Deriving the Hamiltonian leads to an expression for c_t^*, the optimal consumption:

$$c_t^* = (v_F)^{\frac{1}{\gamma-1}}$$

If we try a solution of the form:

$$v(F_t,t) = b(t).\frac{F_t^{\gamma}}{\gamma}$$

We obtain the next ODE for $b(t)$:

$$b(t)' + b(t).\left(r.(\gamma-1) - \mu(x+t)\right) + b(t)^{\frac{\gamma}{\gamma-1}}.(1-\gamma) = 0$$

With the boundary condition that $b(T) = 0$. This is a Bernouilli's equation and its solution is:

$$b(t)^{\frac{1}{1-\gamma}} = \int_t^T e^{-\int_t^s r + \frac{1}{1-\gamma}.\mu(x+z).dz} ds \tag{2.6}$$

It is easily checked that $b(t) < 1$ (it's a kind of life annuity) and that the fund remains positive during the individual's life. In the Merton's problem, it isn't necessary to add a constraint which specifies that the wealth of the individual has to remain positive. This constraint is known as the wealth constraint (or budget constraint) in the literature and corresponds to the fact that an individual doesn't want to leave a negative heritage to his descendants.

2.2. **Partial annuitization.** In this case ($\alpha > 0$), we will see that it's absolutely necessary to integrate the wealth constraint in the problem. If we ignore it, we solve the problem:

$$v_t - (r + \mu(x+t)).v + sup_{c_t} \left(\underbrace{(r.F_t + B - c_t)\,.v_F + \frac{c_t^\gamma}{\gamma}}_{H} \right) = 0 \tag{2.7}$$

The optimal consumption is again obtained by deriving H:

$$c_t^* = (v_F)^{\frac{1}{\gamma-1}}$$

And we try a solution of the form:

$$v(F_t, t) = b(t).\frac{(F_t + a(t))^\gamma}{\gamma}$$

After calculations, we get two ODE that characterize $b(t)$ and $a(t)$:

$$b(t)' + b(t).\left(r.(\gamma - 1) - \mu(x+t)\right) + b(t)^{\frac{\gamma}{\gamma-1}}.(1-\gamma) = 0$$

$$a(t)' - r.a(t) + B = 0$$

With the boundary condition that $b(T) = 0$, $b(t)$ is identical to (2.6):

$$b(t)^{\frac{1}{1-\gamma}} = \int_t^T e^{-\int_t^s r + \frac{1}{1-\gamma}.\mu(x+z).dz} ds \tag{2.8}$$

The function $a(t)$ is of the form:

$$a(t) = -\int_0^t B.e^{-r.s}.ds + K$$

Where K is a constant. The boundary condition on $a(t)$ may not be derived from the constraint $v(T, F_T) = 0$, which is achieved by the terminal condition $b(T) = 0$. If we choose the natural boundary condition $a(T) = 0$, we get that:

$$a(t) = \int_t^T B.e^{-r.(s-t)}.ds$$

It means that the consumption anticipates future incomes resulting from the life annuity:

$$c_t = \int_t^T e^{-\int_t^s r+\frac{1}{1-\gamma}.\mu(x+z).dz} ds. \left(F_t + \int_t^T B.e^{-r.(s-t)}.ds \right)$$

It's interesting to develop the value function at time $t = 0$ with $F(0) = (1-\alpha).W_0$:

$$v\left((1-\alpha).W_0, 0\right) = \frac{b(0)}{\gamma}. \left((1-\alpha).W_0 + \int_0^T B.e^{-r.s}.ds \right)^{\gamma}$$

If we combine this last expression with the definition of the life annuity (2.1), we get that:

$$v\left((1-\alpha).W_0, 0\right) = \frac{b(0)}{\gamma}. \left((1-\alpha).W_0 + \frac{\alpha.W_0}{1-\epsilon}. \frac{\int_0^T e^{-r.s}.ds}{\int_0^T e^{-r.s-\int_0^s \mu^{tf}(x+z).dz} ds} \right)^{\gamma}$$

If the individual seeks to maximize at time $t = 0$ the utility arising from his future consumption, the level of annuitization α will be chosen in function of the ratio ψ:

$$\psi = \frac{1}{1-\epsilon}. \frac{\int_0^T e^{-r.s}.ds}{\int_0^T e^{-r.s-\int_0^s \mu^{tf}(x+z).dz} ds}$$

The numerator of ψ is a financial annuity while the denominator is a life annuity times one less the commercial loading. If the commercial loading ϵ is relatively small, the ratio ψ is bigger than one because the financial annuity is always bigger than the life annuity. In that case, the individual has advantage to invest 100% of his wealth in a life annuity. On the contrary, if the ratio ψ is lower than one, the agent has to invest his wealth in cash. We have an “all or nothing” policy.

Unfortunately, the solution developed in this paragraph isn't admissible. Indeed, after analysis of the consumption pattern on the example detailled hereafter, we observe that the optimal consumption is too high and that the fund F_t becomes negative after a few years (see graphs 2.1 and 2.2, curves "$a(T) = 0$").

As the consumption is function of $a(t)$ and that the constraint $v(T, F_T) = 0$ is achieved by the terminal condition $b(T) = 0$, we may modify the boundary condition on $a(t)$. If we try an initial boundary condition of the form $a(0) = \frac{\alpha.W_0}{1-\epsilon} = \frac{P_0}{1-\epsilon}$, the fund F_t remains positive but the consumption is clearly suboptimal. The consumption falls to zero while the funds is still positive (see example, graphs 2.1 and 2.2, curves "$a(0) = P_0$").

The figures 2.1 and 2.2 illustrate the influence of the $a(t)$ boundary conditions on the evolution of the fund F_t and of the consumption c_t, when a man, 60 years old, chooses to invest 75% of his wealth W_0=1000, in a annuity. The risk free rate r worths 3.25% and the mortality rates (real and in the tariff) are given by a Gompertz-Makeham distribution (see appendix for details). There isn't any commercial loading $\epsilon = 0$. The utility factor, γ, is equal to 60% .

Figure 2.1. Evolution of the fund.

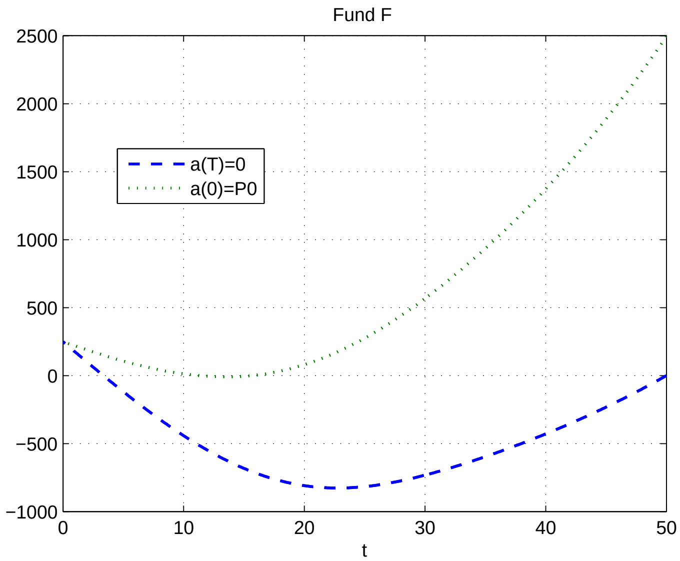
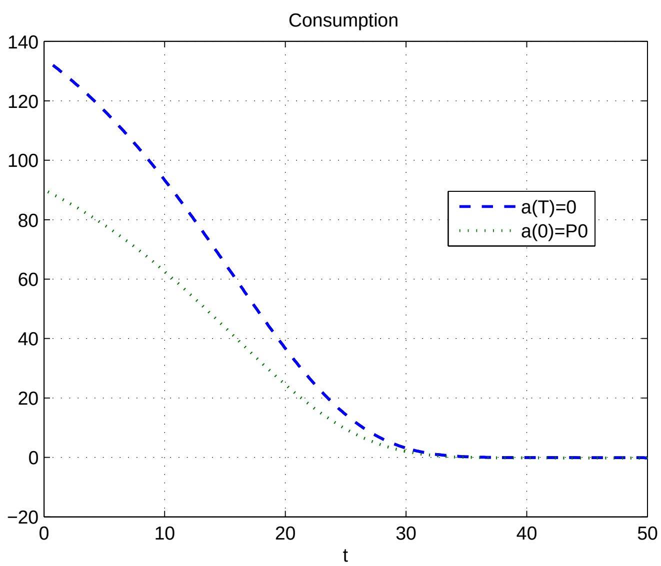


Figure 2.2. Evolution of the consumption.



Therefore, this apparently simple problem, doesn't accept a realizable solution. It's absolutely necessary to integrate the wealth constraint in our research of a solution.

3. Optimal consumption with wealth constraint.

In this section, we try to solve the problem of the maximization of the utility of the consumption under the constraint that the fund F_t has to remain positive during the whole life of the agent:

$$v(F_t,t)=\max_{c_t\in\mathbb{R}}\int_t^T e^{-r.(s-t)}.{}_sp_{x+t}.\left(\frac{c_t}{\gamma}\right)^{\gamma}ds \tag{3.1}$$

Subject to

$$F_t\geq 0 \tag{3.2}$$

Let $\Lambda(F_t,t)$ be the positive Lagrangian multiplier associated to the constraint (3.2). $\Lambda(F_t,t)$ is strictly positive when the constraint is active ($F_t=0$) while $\Lambda(F_t,t)$ is null when the constraint is not active ($F_t>0$). The HJB equation of the constrained problem is therefore:

$$v_t-(r+\mu(x+t)).v+sup_{c_t}\left(\underbrace{\left(r.F_t+B-c_t\right).v_F+\frac{c_t^{\gamma}}{\gamma}}_{H}\right)+\Lambda(F_t,t).F=0 \tag{3.3}$$

Again, we try a solution of the type:

$$v(F_t,t)=b(t).\frac{\left(F_t+a(t)\right)^{\gamma}}{\gamma} \tag{3.4}$$

And if we try a Lagrangian multiplier that looks like:

$$\Lambda(F_t,t)=\lambda(t).b(t).\left(F_t+a(t)\right)^{\gamma-1} \tag{3.5}$$

The optimal consumption c_t^* is still:

$$c_t^*=b(t)^{\frac{1}{\gamma-1}}.(F_t+a(t)) \tag{3.6}$$

If we inject expressions (3.4) (3.5) (3.6) in the HJB equation (3.3), we get that:

$$\begin{aligned}0 \;=\;& b(t)'.\frac{\left(F_t+a(t)\right)^{\gamma}}{\gamma}+a(t)'.b(t).\left(F_t+a(t)\right)^{\gamma-1}-(r+\mu(x+t)).b(t).\frac{\left(F_t+a(t)\right)^{\gamma}}{\gamma}\\ &+\left(r.F_t+r.a(t)-r.a(t)+B\right).b(t).\left(F_t+a(t)\right)^{\gamma-1}+(1-\gamma).b(t)^{\frac{\gamma}{\gamma-1}}.\frac{\left(F_t+a(t)\right)^{\gamma}}{\gamma}\\ &+\lambda(t).b(t).\left(F_t+a(t)\right)^{\gamma-1}.\left(F_t+a(t)-a(t)\right).\frac{\gamma}{\gamma}=0\end{aligned}$$

Regrouping the terms in $\frac{(F_t+a(t))^{\gamma}}{\gamma}$ and in $(F_t+a(t))^{\gamma-1}$ leads to two ODE :

$$b(t)'-(r.(1-\gamma)+\mu(x+t)-\lambda(t).\gamma).b(t)-b^{\frac{\gamma}{\gamma-1}}.(\gamma-1)=0 \tag{3.7}$$

$$a(t)'-r.a(t)+B-\lambda(t).a(t)=0 \tag{3.8}$$

When the constraint is inactive, the Lagrangian multiplier is null, $\lambda(t)=0$. When the constraint is active, the multiplier is positive and non null, $\lambda(t)>0$, but the product $\lambda(t).F_t$ is null. The form of $\lambda(t)$ may be deduced when the agent doesn't invest in a cash ($\alpha=1$). Next, we build numerically the optimal solution for the case of an initial investment in cash ($\alpha<1$).

3.1. **Full annuitization.** If the agent invests 100% of its initial wealth in a life annuity, the optimal consumption is obviously the annuity, $c_t^* = B$, because the discount rate of the objective, $r+\mu(x+t)$ is higher than the rate of return of the cash account, r. The Lagrangian function $\lambda(t)$ must therefore be the mortality rate, $\mu(x+t)$. Indeed, equations (3.7) and (3.8) become:

$$b(t)' - (r + \mu(x+t)).(1-\gamma).b(t) - b^{\frac{\gamma}{\gamma-1}}.(\gamma - 1) = 0 \tag{3.9}$$

$$a(t)' - (r + \mu(x+t)).a(t) + B = 0 \tag{3.10}$$

With the terminal conditions $b(T) = 0$ and $a(T) = 0$, we have that:

$$b(t)^{\frac{1}{1-\gamma}} = \int_t^T e^{-\int_t^s r+\mu(x+z).dz} ds = \bar{a}_{x+t} \tag{3.11}$$

$$a(t) = \int_t^T B.e^{-\int_t^s r+\mu(x+z).dz}.ds = \bar{a}_{x+t} \tag{3.12}$$

$a(t)$ and $b(t)^{\frac{1}{1-\gamma}}$ are both equal to the mathematical reserve at time t. As foreseen, the optimal consumption is well equal to B:

$$c_t^* = b(t)^{\frac{1}{\gamma-1}}.(0 + a(t)) = B$$

And the value function is:

$$v(F_t, t) = \frac{B^\gamma}{\gamma} \int_t^T e^{-\int_t^s r+\mu(x+z).dz}.ds = \frac{B^\gamma}{\gamma}.\bar{a}_{x+t} \tag{3.13}$$

3.2. **Partial annuitization.** It seems wise to assume that the agent will totally consume its fund before the time t^*, which is unknown. At this instant, the fund F_{t^*} is null and as showed in the previous paragraph, the optimal consumption will be equal to the annuity B and therefore $\lambda(t) = \mu(x+t)$. Before t^*, the fund F_t is positive and therefore the Lagrangian function $\lambda(t)$ must be null (the constraint $F_t \geq 0$ is inactive). The solution is then described by a system of four ODE, two equations for instant $t \leq t^*$ and two for $t \geq t^*$:

$$\begin{cases} b_1(t)' - (r.(1-\gamma) + \mu(x+t)).b_1(t) - b_1^{\frac{\gamma}{\gamma-1}}.(\gamma-1) = 0 & \forall t \leq t^* \\ a_1(t)' - r.a_1(t) + B = 0 & \forall t \leq t^* \\ b_2(t)' - (r + \mu(x+t)).(1-\gamma).b_2(t) - b_2^{\frac{\gamma}{\gamma-1}}.(\gamma-1) = 0 & \forall t \geq t^* \\ a_2(t)' - (r + \mu(x+t)).a_2(t) + B = 0 & \forall t \geq t^* \end{cases}$$

With the following terminal conditions:

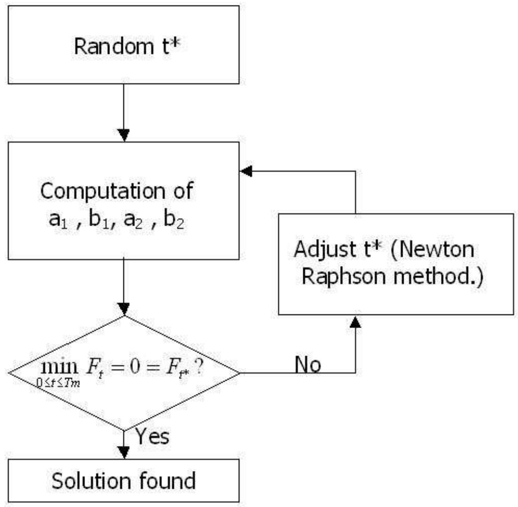
$$\begin{cases} b_1(t^*) = b_2(t^*) & a_1(t^*) = a_2(t^*) \\ b_2(T) = 0 & a_1(T) = 0 \end{cases}$$

The value function is equal to:

$$\begin{cases} v(F_t, t) = b_1(t)\frac{(F_t + a_1(t))^\gamma}{\gamma} & \forall t \leq t^* \\ v(F_t, t) = b_2(t)\frac{(a_2(t))^\gamma}{\gamma} & \forall t \geq t^* \end{cases}$$

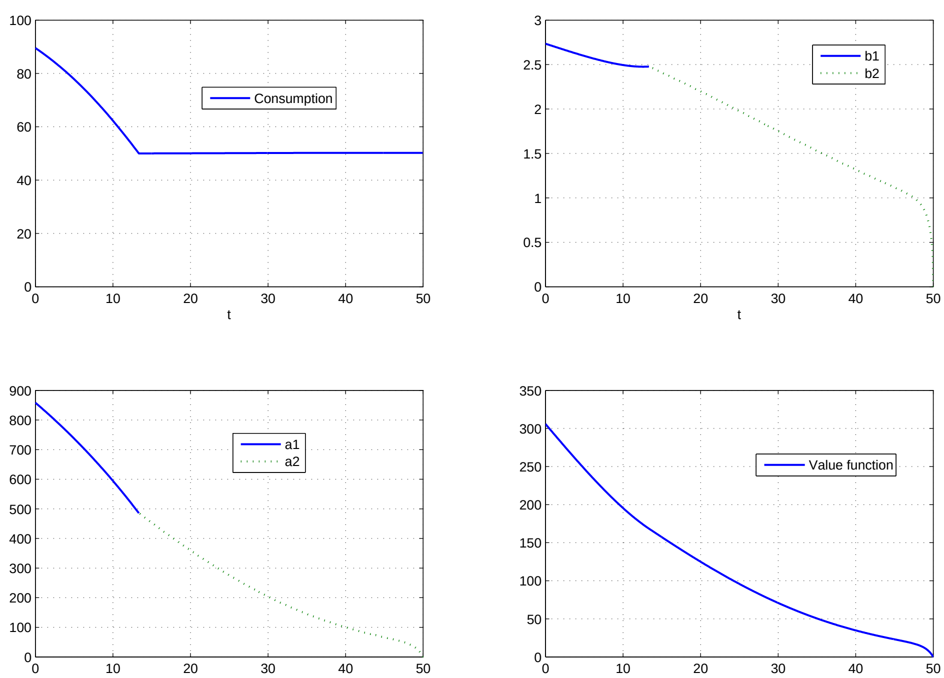
It is not possible to obtain an analytical expression for t^*. We compute it with an iterative algorithm depicted in figure (3.1).

FIGURE 3.1. Algorithm.



If we apply this algorithm to the example developed in the subsection 2.2 (an agent of age 60 chooses to invest 75% of his wealth W_0=1000, in a annuity with r =3.25% and $\gamma = 0.6$), we get that $t^* = 13,35$. The individual will therefore consume the totality of his fund of cash before the age of 73 years. The next graph presents the pattern of consumption.

FIGURE 3.2. Consumption, a and b functions, value function.



The figure 3.2 points out that the value function is $\mathbb{C}^1$ with regard to the time, even if functions $a(t)$ and $b(t)$ aren't differentiable at $t = t^*$.

It is also possible to derive an analytical expression of the consumption c_t. We know that:

$$\begin{cases} c_t^* = b_1(t)^{\frac{1}{\gamma-1}}.(F_t + a_1(t)) & \forall t \leq t^* \\ c_t^* = B & \forall t \geq t^* \end{cases}$$

If we derive c_t^*, for $t < t^*$, we obtain that:

$$\begin{aligned} dc_t^* &= \frac{1}{\gamma-1}.b_1(t)^{\frac{2-\gamma}{\gamma-1}}.b_1(t)'.(F_t + a_1(t)).dt + b_1(t)^{\frac{1}{\gamma-1}}.(dF_t + a_1(t)'.dt) \\ &= \frac{1}{\gamma-1}.b_1(t)^{\frac{2-\gamma}{\gamma-1}}.\left(-\left(-r.(1-\gamma) - \mu(x+t)\right).b_1(t) - b_1(t)^{\frac{\gamma}{\gamma-1}}.(1-\gamma)\right). \\ &\quad (F_t + a_1(t)).dt + b_1(t)^{\frac{1}{\gamma-1}}.(r.(F_t + a_1(t)) - c_t^*).dt \\ &= \left(-c_t^*.r + \frac{1}{\gamma-1}.\mu(x+t).c_t^* + b_1(t)^{\frac{1}{\gamma-1}}.c_t^* + r.c_t^* - b_1(t)^{\frac{1}{\gamma-1}}.c_t^*\right).dt \\ &= -\frac{1}{1-\gamma}.\mu(x+t).c_t^*.dt \end{aligned}$$

The consumption is then ruled by the following dynamic:

$$c_t^* = c_0.e^{-\int_0^t \frac{\mu(x+z)}{1-\gamma}.dz} \qquad c_0 = b_1(0)^{\frac{1}{\gamma-1}}.(F_{t=0} + a_1(0))$$

The optimal consumption is therefore a decreasing function of the initial consumption $c_{t=0}$. The consumption is reduced by a factor which is similar to a probability of decease corrected by $\frac{1}{1-\gamma}$:

$$c_t^* = c_0.\left(e^{-\int_0^t \mu(x+z).dz}\right)^{\frac{1}{1-\gamma}} = c_0.\left({}_tp_x\right)^{\frac{1}{1-\gamma}}$$

The evolution of the fund, F_t is then easily calculated:

$$\begin{cases} F_t = F_{t=0}.e^{r.t} + B.\int_0^t e^{r.(t-s)}ds - c_0.\int_0^t \left({}_sp_x\right)^{\frac{1}{1-\gamma}}.e^{r.(t-s)}ds & \forall t \leq t^* \\ F_t = 0 & \forall t \geq t^* \end{cases}$$

3.3. **Influence of the cash return on the consumption policy.** We consider now that the cash return is different from the guarantee of the life insurance. Let r_F be the rate of return of the cash account. The difference between the insurance guarantee, r, and the financial return r_F, is noted $\theta = r_F - r \geq 0$. The dynamic of the cash account is then described by the next ODE:

$$dF_t = ((r+\theta).F_t + B - c_t).dt$$

The discount factor of the value function is equal to $\rho = r + \phi$ with $\phi \geq 0$. And the Bellman's equation becomes:

$$v_t - (r + \phi + \mu(x+t)).v + sup_{c_t}\left(\underbrace{((r+\theta).F_t + B - c_t).v_F + \frac{c_t^\gamma}{\gamma}}_{H}\right) + \Lambda(F_t, t).F = 0 \tag{3.14}$$

Again, we try a value function and a Lagrangian multiplier given by (3.4) and by (3.5). After calculations, the components $a(t)$ and $b(t)$ of the value function are solutions of:

$$b(t)' - (r.(1-\gamma) + \phi + \mu(x+t) - \theta.\gamma - \lambda(t).\gamma).b(t) - b^{\frac{\gamma}{\gamma-1}}.(\gamma - 1) = 0 \tag{3.15}$$

$$a(t)' - (r+\theta).a(t) + B - \lambda(t).a(t) = 0 \tag{3.16}$$

With the terminal conditions $a(T) = 0$ and $b(T) = 0$. As in the previous section, we assume that there exists a instant t^* such that the optimal consumption c_t^* for $t \geq t^*$, is equal to the life annuity B. The wealth constraint imposes then that:

$$\lambda(t) = \mu(x+t) - \theta + \phi \qquad \forall t \in [t^*, T]$$

Note that the Lagrangian multiplier must remain positive. Consequence : we must have $t^* \geq \bar{t}$ with $\bar{t}$ such that $\lambda(\bar{t}) = \mu(x+\bar{t}) - \theta + \phi = 0$. There is no easy way to prove that $t^* \geq \bar{t}$ but it may be checked after computations. Before t^*, the funds F_t is positive and therefore the Lagrangian function $\lambda(t)$ is null (the constraint $F_t \geq 0$ is inactive). The solution is again described by a system of four ODE:

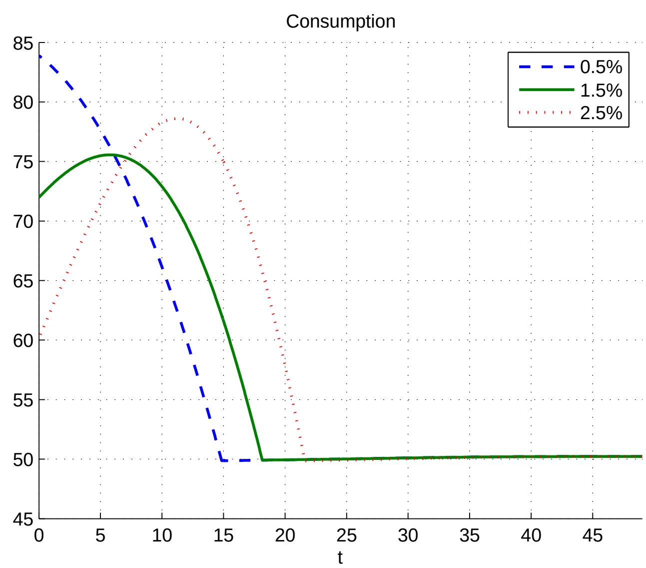
$$\begin{cases} b_1(t)' - (r.(1-\gamma) + \mu(x+t) + \phi - \theta.\gamma).b_1(t) - b_1^{\frac{\gamma}{\gamma-1}}.(\gamma-1) = 0 & \forall t \leq t^* \\ a_1(t)' - (r+\theta).a_1(t) + B = 0 & \forall t \leq t^* \\ b_2(t)' - (r + \mu(x+t) + \phi).(1-\gamma).b_2(t) - b_2^{\frac{\gamma}{\gamma-1}}.(\gamma-1) = 0 & \forall t \geq t^* \\ a_2(t)' - (r + \mu(x+t) + \phi).a_2(t) + B = 0 & \forall t \geq t^* \end{cases}$$

With the following terminal conditions:

$$\begin{cases} b_1(t^*) = b_2(t^*) & a_1(t^*) = a_2(t^*) \\ b_2(T) = 0 & a_1(T) = 0 \end{cases}$$

We solve numerically the ODE, applied to the example developed in the subsection 2.2 (an agent of age 60 chooses to invest 75% of his wealth in a annuity with r =3.25% and $\gamma = 0.6$, W_0=1000), for different cash returns. In order to facilitate the interpretation of results, we let the discount rate ρ equal to r ($\phi = 0$).

FIGURE 3.3. Consumption in function of θ.



Higher is the cash return, lower is the consumption during the first years. Indeed, the individual has an advantage to delay his consumption to profit of the cash return which is higher than the rate included in the discount factor of the utility of the consumption $(r + \mu(x + t))$. This policy of consumption also influences the evolution of the fund and delays the instant t^* when the fund becomes null.

FIGURE 3.4. Fund in function of θ.

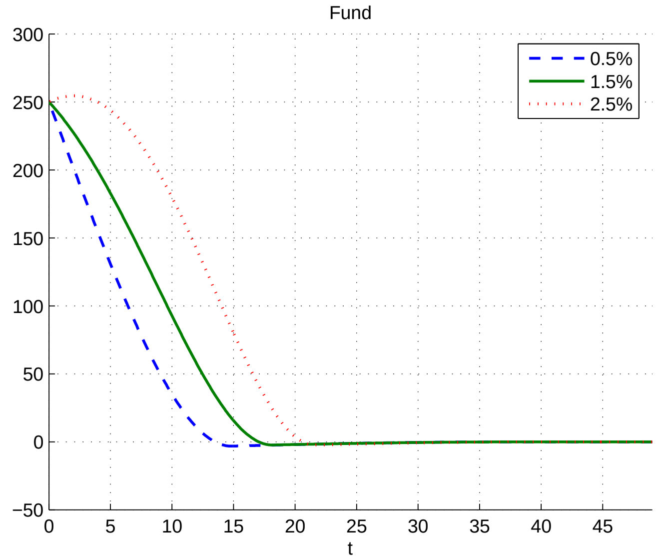


Table 1 shows the instants t^* for different θ. Clearly, we have $t^* \geq \bar{t}$ with $\bar{t}$ solution to the equation: $\lambda(\bar{t}) = \mu(x + \bar{t}) - \theta + \phi = 0$. It confirms that the Lagrangian multipliers are positive for each tests.

TABLE 1. Time t^* and $\mu(x + t^*)$

$\theta - \phi$	t^*	$\mu(60 + t^*)$	$\bar{t}$
0.5%	14.85	3.52%	-6.45
1.0%	16.45	4.10%	1.30
1.5%	18.15	4.81%	5.75
2.0%	19.90	5.69%	8.85
2.5%	21.60	6.69%	11.25
3.0%	23.25	7.83%	13.15

As in section 3, it is possible to obtain an expression of the optimal consumption c_t^* in function of c_0, for time $t \leq t^*$. The dynamic of c_t^* is:

$$dc_t^* = -\frac{1}{1-\gamma}.\left(\mu(x+t) + \phi - \theta.\gamma\right).c_t^*.dt$$

And by integration we get that :

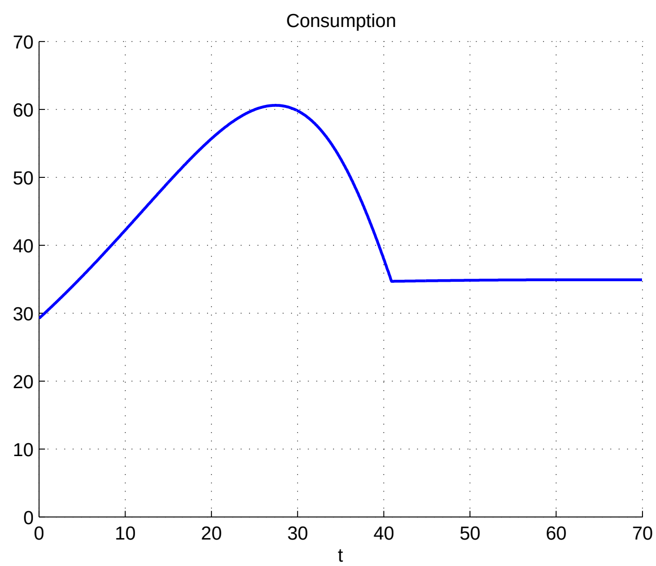
$$c_t^* = c_0.\left(e^{-\int_0^t \mu(x+z)+\phi-\theta.\gamma dz}\right)^{\frac{1}{1-\gamma}}$$

As we have observed on an example, the consumption pattern can count two phases : one of growing and one of decreasing consumption. Let $\tilde{t}$ be the instant such that:

$$\int_0^{\tilde{t}} (\mu(x+z) + \phi - \theta.\gamma) \, dz = 0$$

The growing and decreasing phases extend respectively from $t = 0$ to $\tilde{t}$ and from $\tilde{t}$ to t^*. It's interesting to observe on the example that the consumption can be lower than the annuity if the agent purchases an annuity earlier than 60 years old. The next figure depicts this for an individual who buy a life annuity at the age of 40 years (75% of his wealth in a annuity, r =3.25%, $\gamma = 0.6$, W_0=1000 and $\theta = 2\%$). From 30 years until 34.65 years, the agent reinvests a part of the life annuity on the cash account. The consumption increases till $\tilde{t} = 27.5$ and $t^* = 40.9$.

FIGURE 3.5. Consumption, purchase of the annuity when 40 years old.



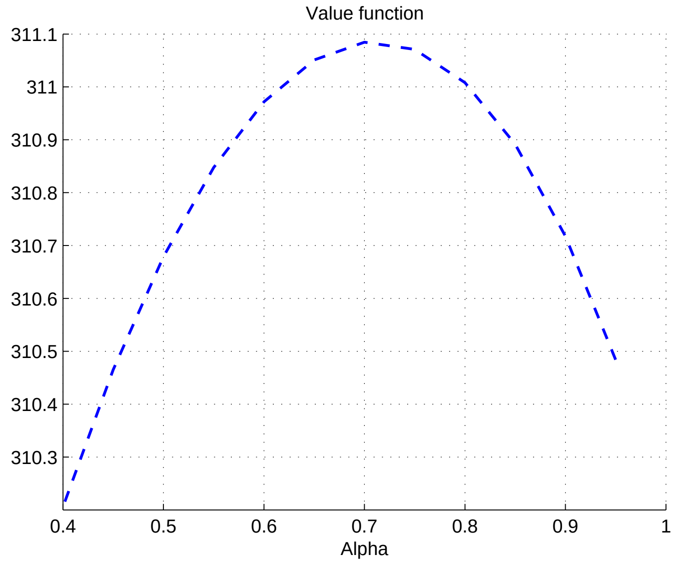
4. OPTIMAL LEVEL OF ANNUITIZATION.

In the previous sections, the agent invests a fixed fraction α of his initial wealth in a annuity. If the cash account provides a higher return than the insurance contract ($r_F > r$), does there exist an optimal ratio α^* which maximizes the discounted value of the utility arising from the consumption?

$$\alpha^* = \arg\sup_{\alpha} v\,(\ (1-\alpha).W_0 \ , \ 0)$$

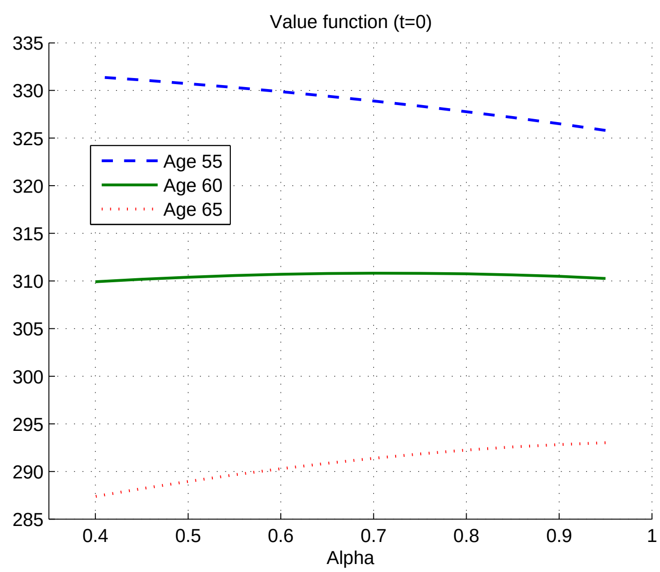
Contrary to the first observations done in paragraph 2.2, the solution doesn't consist in an "all or nothing" policy (by "all or nothing" we means $\alpha = 1$ or $\alpha = 0$). we will see on an example that the optimal ratio α^* is function of the age of the agent, of the spread between the cash return and of the guarantee of the annuity. Again, we consider the example of an individual, 60 years old, who invests $\alpha.W_0$ in a annuity (r =3.25%, $\gamma = 0.6$, W_0=1000, $\rho = r$) and $(1-\alpha).W_0$ on a cash account with a return of $r_F = r + 2\%$=5.25% ($\theta = 2\%$) .

FIGURE 4.1. $v((1-\alpha).W_0, 0)$ in function of α, age : 60 years.



The value function is maximized for $\alpha = 70\%$ but we clearly see that the influence of the ratio α on the value function at time $t = 0$ is relatively small: ($v((1-\alpha).W_0, 0)$ remains in the interval $[310, 311]$). In fact, the optimum is a function of the time of annuity purchase (see graphic 4.2). At the age of 55 years, the agent doesn't has advantage to buy a life annuity if he can get a return of 5.25% on a cash account. At the age of 65 years, it is more interesting to invest the totality of the capital in a life annuity because the average return of the annuity is always higher than 5.25%.

FIGURE 4.2. $V((1-\alpha).W_0, 0)$ in function of the age of annuity purchase.



The parameter α influences widely the consumption pattern (see figure 4.3) : to privilege the annuity leads to a higher consumption when the individual reaches the end of his life.

FIGURE 4.3. Consumption in function of α, 60 years old.

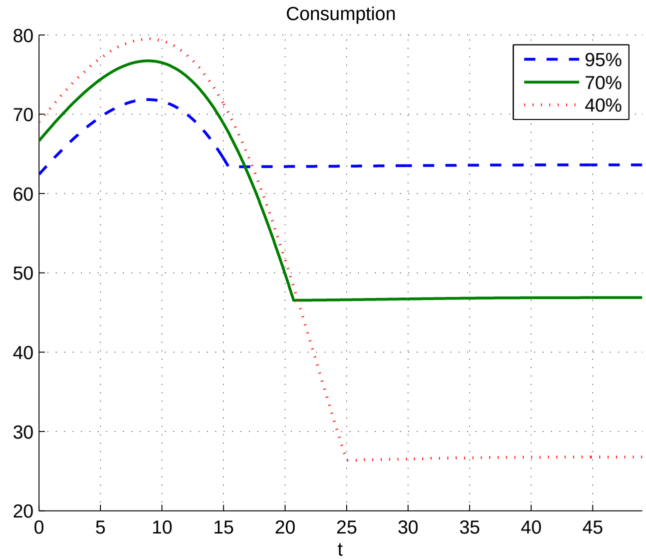


Table 2 confirms that the solutions found are optimal : all Lagrangian multipliers are positive since $\mu(x+t)$ are always bigger than the cash account premium θ=2%.

TABLE 2. Time t^* and $\mu(x+t^*)$

	55 y		60 y		65 y	
α	t^*	$\mu(55+t^*)$	t^*	$\mu(60+t^*)$	t^*	$\mu(65+t^*)$
95%	21.45	4.10%	15.30	3.67%	9.95	3.55%
90%	22.60	4.57%	16.80	4.23%	11.80	4.23%
85%	23.60	5.03%	17.95	4.72%	13.40	4.93%
80%	24.50	5.48%	18.95	5.20%	14.75	5.61%
75%	25.30	5.91%	19.85	5.66%	14.90	5.69%
70%	26.05	6.35%	20.65	6.11%	15.75	6.17%
65%	26.80	6.82%	21.40	6.56%	16.50	6.63%
60%	27.45	7.26%	22.15	7.05%	17.20	7.08%
55%	28.35	7.91%	22.85	7.54%	17.90	7.57%
50%	28.85	8.29%	23.55	8.06%	18.60	8.10%
45%	30.15	9.39%	24.25	8.62%	19.30	8.66%
40%	30.20	9.44%	24.95	9.22%	20.00	9.26%

5. CONCLUSIONS.

In this paper, we derive the optimal consumption of an individual, without bequest motives and without possibility of borrowing, who decides at retirement to allocate his wealth between a cash account and a life annuity. Contrary to the classical consumption/investment problem solved by Merton, we have to incorporate in the Bellman's equation the wealth constraint, which specifies that the agent doesn't borrow money to consume. This constraint is inserted in the framework of resolution with Lagrangian multipliers. We have found that the consumption planning is subdivided in two periods. During the first one, the wealth constraint is inactive and the consumption is proportional to the initial consumption. In the second period, the constraint is active and the optimal consumption is equal to the annuity. The instant of transition between those periods is calculated numerically.

Then, we determine the impact of the cash return on the policy of consumption and observe that higher is the cash return, lower is the consumption during the first years. Moreover, when the cash return is high and if the agent is young enough , the consumption may be lower than the annuity. The consumption policy is explicitly dependent on the spread between the cash return and the guarantee coupled to the annuity.

In the last part of our work, we show the influence of the cash return on the optimal allocation of the initial richness between a cash account and a life annuity. We qualify the results of Yaari in the sense that an individual old enough and without bequest motives should privilege the annuitization. Before a certain age, which is function of the individual's mortality rate and of the cash return, the individual has advantage to invest, at least partially, his richness in a cash account.

There are still unanswered issues that motives further researches. what is the optimal consumption if the individual invests a fraction of his wealth in a risky asset and in an annuity? In a stochastic environment, the problem becomes quite complex and the instant of activation of the wealth constraint becomes random.

6. Appendix.

In the examples presented in this paper, we assume that the real mortality rates and the mortality rates used for pricing, $\mu(x+t)$ are given by a Gompertz-Makeham distribution. The parameters are those defined by the Belgian regulator for the pricing of a life insurance purchased by a man. For an individual of age x, the mortality rate is :

$$\mu(x) = \mu_1(x) = a_\mu + b_\mu.c^x \qquad a_\mu = -\ln(s_\mu) \qquad b_\mu = \ln(g_\mu).\ln(c_\mu)$$

Where the parameters s_μ, g_μ, c_μ take the values showed in the table 3. Table 4 presents the evolution of mortality rates in function of the age of the individual.

Table 3. Belgian legal mortality, for life insurance products, and for a male population.

s_μ:	0.999441703848
g_μ:	0.999733441115
c_μ :	1.116792453830

Table 4. Mortality rates.

Age x	$\mu(x)$
30	0.10%
40	0.18%
50	0.37%
60	0.88%
70	2.23%
80	5.74%

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