Risk Management for a bond using bond put options

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Introduction: problem description and motivation

- Setup similar to Ahn et al.,Journal of Finance, 1999
- Classical hedging example: hedging exposure to price risk of an asset
- e.g. Currency, oil, gold
- Ahn et al. (1999): share as underlying asset
- Minimize VaR of position in share by using put options
- Optimal strike price of put option?
- Here:
  - underlying asset: bond
  - VaR and TVaR

Introduction: problem description and motivation

- Buy a zero-coupon bond with maturity S, $Y(0, S)$
- Sell it again at time T, with $T < S$, at price $Y(T, S)$
- No hedging: exposed to interest rate movements
- Hedging using bond put options
- Determine strike price such that risk is minimized, using budget C
- Risk measures: Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR)

Risk measurement

- $\mathrm{VaR}_{\alpha,T}$ satisfies:

$$\Pr(L_0 \geq \mathrm{VaR}_{\alpha,T}) = \alpha.$$

- $\mathrm{VaR}_{\alpha,T}$ is the loss of the worst case scenario on the investment at a $(1-\alpha)$ confidence level during the period $[0,T]$.

- $\mathrm{TVaR}_{\alpha,T}$, is defined as follows:

$$\mathrm{TVaR}_{\alpha,T} = \frac{1}{\alpha}\int_{1-\alpha}^{1} \mathrm{VaR}_{1-\beta,T}\, d\beta.$$

- $\mathrm{TVaR}_{\alpha,T}$ is the arithmetic average of the quantiles of our loss, from $1-\alpha$ to 1 on.

Hull-White model

- Advantages:
  - Perfect fit with an initial given term structure
  - Analytic solutions for zero-coupon bond and European options
- Disadvantage: Possibility of negative interest rates

$$\boxed{dr(t) = \; (\theta(t) - \gamma(t)r(t))dt + \sigma(t)dZ(t)}$$

- $Z(t)$ a standard Brownian motion under the risk-neutral measure Q
- $\theta(t)$: time dependent long-term average level of the spot interest rate
- $\gamma(t)$: the mean-reversion speed
- $\sigma(t)$: the volatility
- γ and σ constant

Hull-White model

$$\boxed{E\left[r(t)\right] = m = r(0)e^{-\gamma(t)} + a(t) - a(0)e^{-\gamma(t)}}$$

$$\boxed{Var\left[r(t)\right] = s^2 = \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t})}$$

- $a(t) = F^M(0,t) + \frac{\sigma^2}{2}\left(\frac{1 - e^{-\gamma(S-t)}}{\gamma}\right)^2$ with $F^M(0,t)$ the instantaneous forward rate observed in the market on time zero with maturity t

Bond valuation in the Hull-White model

Price of a zero-coupon bond at time t with maturity S:

$$\boxed{Y(t,S) = A(t,S)e^{-B(t,S)r(t)}}$$

- $B(t,S) = \dfrac{1 - e^{-\gamma(S-t)}}{\gamma}$
- $A(t,S) = \dfrac{Y^M(0,S)}{Y^M(0,t)} e^{B(t,S)F^M(0,t) - \frac{\sigma^2}{4\gamma}(1-e^{-2\gamma t})B^2(t,T)}$

  with $Y^M(0,t)$ the price of a zero coupon bond with maturity t observed in the market at time 0.
- Y(t,S) is lognormally distributed with

  mean $\Pi(t,S) = \ln A(t,S) - B(t,S)m$, and

  variance $\Sigma(t,S)^2 = B(t,S)^2 s^2$

Bond option valuation in the Hull-White model

Price of a bond put option:

$$\boxed{P(0,T,S,X) = -KY(0,S)\Phi(-d_1) + XY(0,T)\Phi(-d_2)}$$

with K the principal of the bond and X the strike price of the option

- $d_1 = \frac{1}{\sigma_p}\log(\frac{KY(0,S)}{XY(0,T)}) + \frac{\sigma_p}{2}$
- $d_2 = d_1 - \sigma_p$
- $\sigma_p^2 = \frac{\sigma^2}{2\gamma^3}(1-e^{-2\gamma T})(1-e^{-\gamma(S-T)})^2$

Risk minimization

- fixed budget C for hedging.
  - If C insufficient for buying entire option, buy portion h of option.
  - No overhedging
- Formally: $C = hP(0, T, S, X)$ and $h \in (0, 1)$
- Payoff of bond + put option at $T = \max(hX + (1 - h)Y(T, S), Y(T, S))$
- Discounted payoff $= ((1 - h)Y(T, S) + hX)Y(0, T)$
- $L_0 = Y(0, S) + C - ((1 - h)Y(T, S) + hX)Y(0, T)$
- Minimize $\mathrm{VaR}_{\alpha,T}$ and $\mathrm{TVaR}_{\alpha,T}$ of L_0, taking into account the budget constraint

Risk minimization

- $\mathrm{VaR}_{\alpha,T}$ minimization:

$$\boxed{e^{\Pi(T,S)+\Sigma(T,S)(c(\alpha))} = \frac{Y(0,S)\Phi(-d_1)}{Y(0,T)\Phi(-d_2)}}$$

with $c(\alpha)$ the percentile of the standard normal distribution, i.e.$\Pr(z \leq c(\alpha)) = \alpha$

- $\mathrm{TVaR}_{\alpha,T}$ minimization:

$$\boxed{\frac{1}{\alpha}e^{\Pi(T,S)+\frac{1}{2}\Sigma^2(T,S)}\Phi(c(\alpha) - \Sigma(T,S)) = \frac{Y(0,S)\Phi(-d_1)}{Y(0,T)\Phi(-d_2)}}$$

- Optimal strike price is implicitly defined (cfr. d_1)
- Optimal strike price is independent of hedging expenditure C !!

Hull-White model calibration

- Determine parameters of model in a credible way
- Suppose we have M market prices of traded instruments
- $\min_{\gamma,\sigma} \sqrt{\sum_{i=1}^{M} \left(\frac{\text{model}_i - \text{market}_i}{\text{market}_i} \right)^2}$
- Market instruments: caps
- provide protection against a specified interest rate (e.g. the three month LIBOR, R_L) rising above a specified level (the cap rate, R_C))
- Cap= series of caplets = series of zero-coupon bond put options
- Link: Zero-coupon bond put options can be valued by Hull-White model
- $\gamma = 0.31621 \qquad \sigma = 0.011631$

Numerical illustration

Parameters: $T = 1$, $S = 5$, $R(0) = 0.0213$, $C = 0.0043$, $\alpha = 0.05$

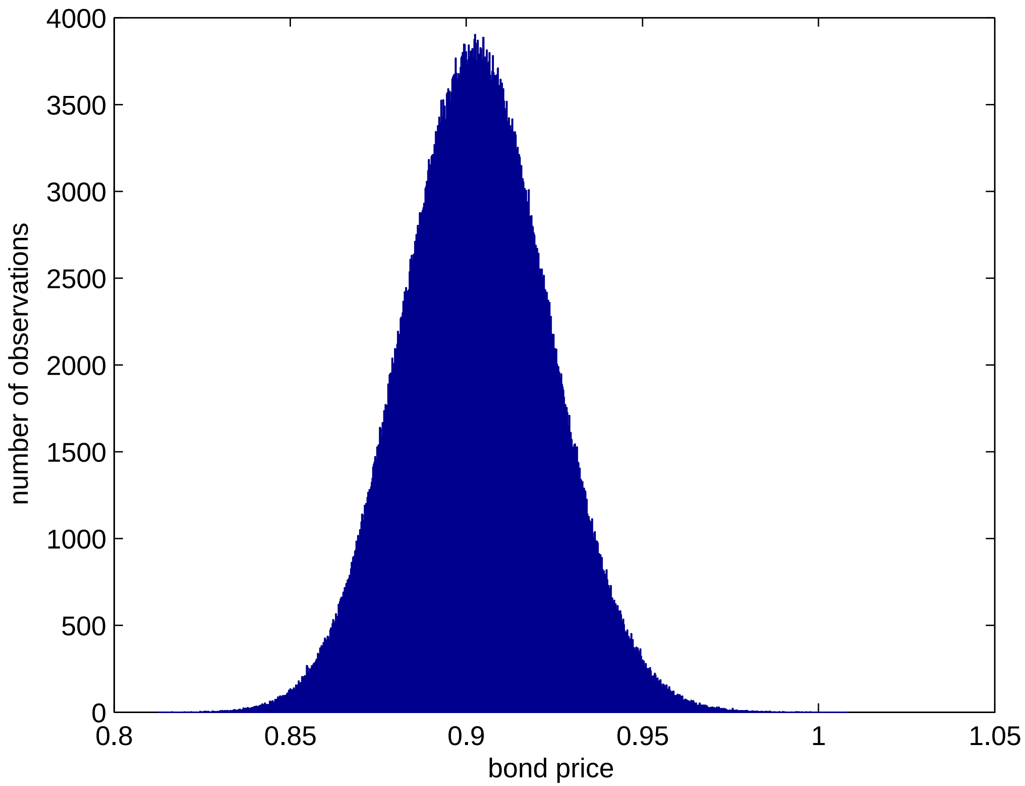

Figure 1: Bond value at $T = 1$

- Minimum bond price: 0.8159

Numerical illustration

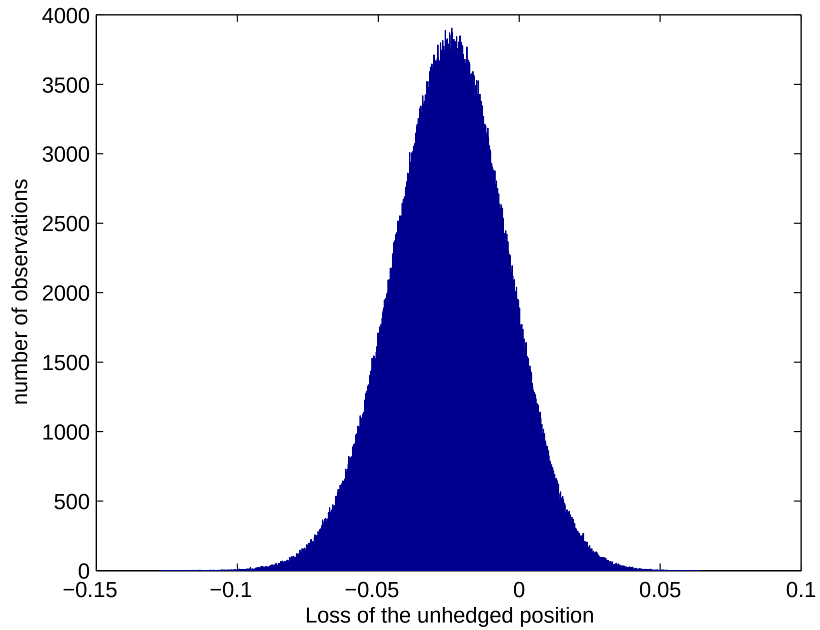

Figure 2: Loss of the unhedged position at $T = 1$

- No hedging: large potential losses
- Potential discounted loss: 0.0608
- Avoid this by hedging!

Numerical illustration

$\mathrm{VaR}_{\alpha,T}$ minimization:

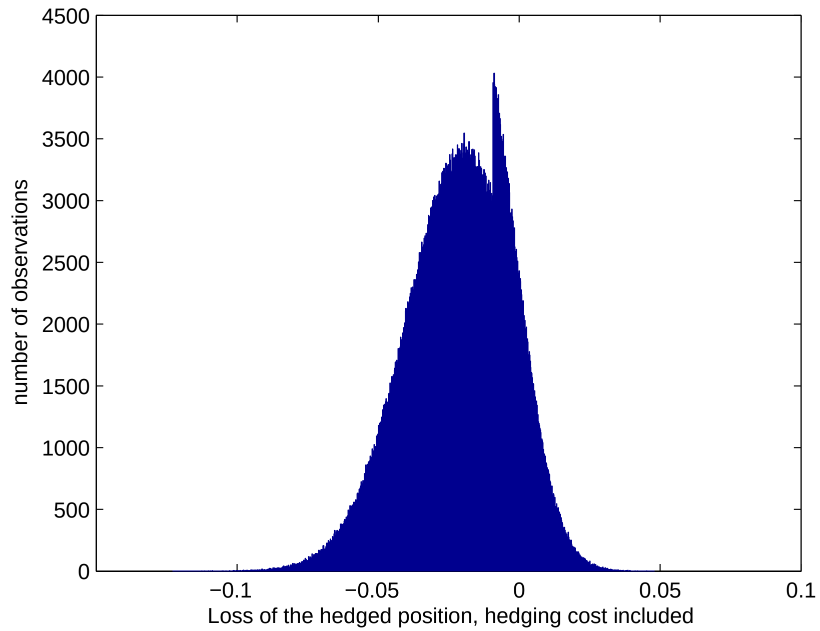

Figure 3: Loss of the hedged position at $T = 1$

- Optimal strike price: 0.8915
- buy 26.23% of an option
- Second peak.
- Shorter tail.

Numerical illustration

$\mathrm{TVaR}_{\alpha,T}$ minimization:

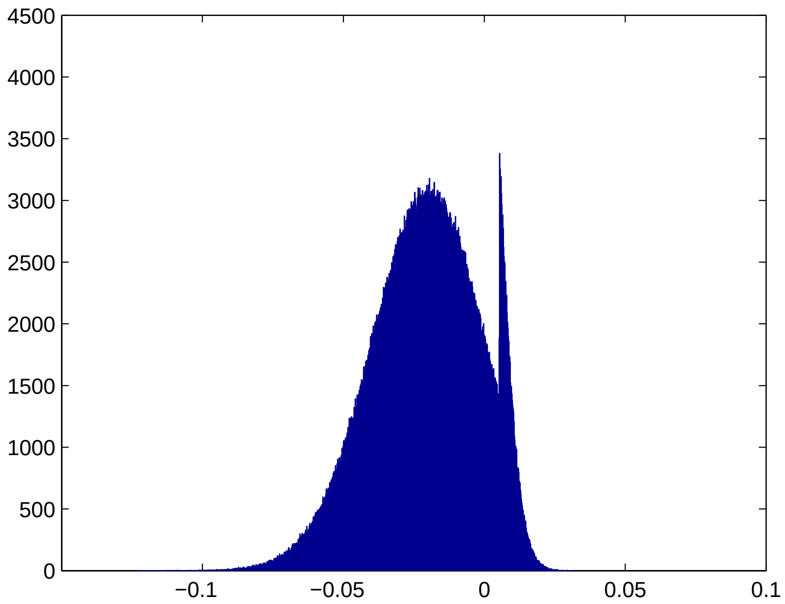

Figure 4: Loss of the hedged position at $T = 1$

- Lower optimal strike price: 0.87698
- Buy 59.35% of an option.
- second peak more to right
- shorter tail

Numerical illustration

$\mathrm{VaR}_{\alpha,T}$ minimization:

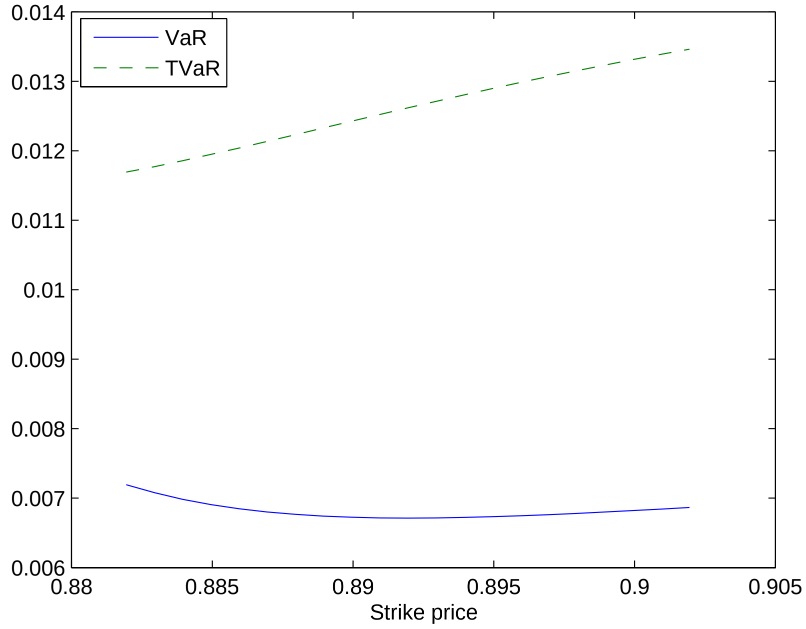

Figure 5: VaR and TVaR in function of distance from optimal strike price

Numerical illustration

$\mathrm{TVaR}_{\alpha,T}$ minimization:

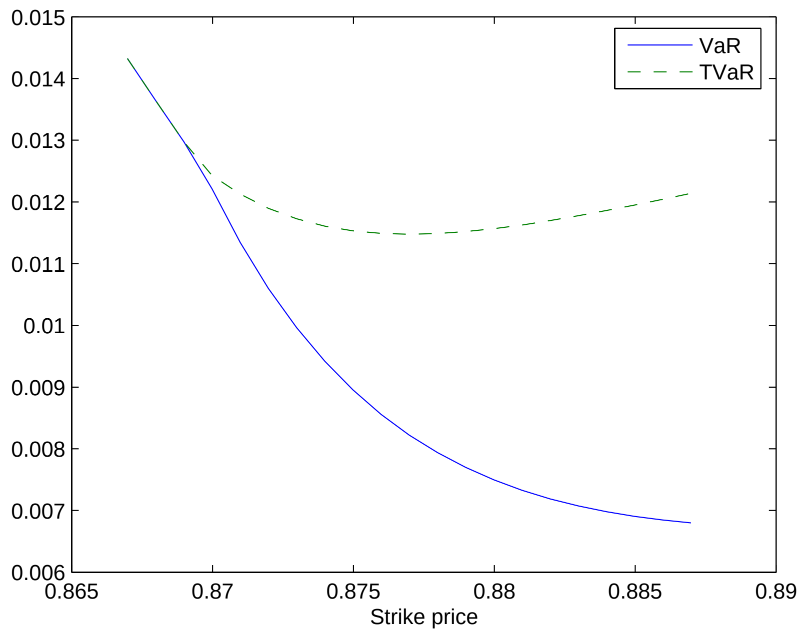

Figure 6: VaR and TVaR in function of distance from optimal strike price

Conclusions and further research

- VaR and TVaR Minimization of position in bond
- formula for determining optimal strike price for put option used for hedging
- model for instantaneous interest rate: Hull-White
- Extend to other interest rate models?
- Extend to other hedging instruments