Risk management for a bond using bond put options

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Abstract. This paper studies a strategy that minimizes the risk of a position in a zero coupon bond by buying a percentage of a put option, subject to a fixed budget available for hedging. We consider two popular risk measures: Value-at-Risk(VaR) and Tail Value-at-Risk (TVaR). We elaborate a formula for determining the optimal strike price for this put option in case of a Hull-White stochastic interest rate model. We calibrate the Hull-White model parameters to a set of cap prices, in order to provide a credible numerical illustration. We demonstrate the relevance of searching the optimal strike price, since moving away from the optimum implies a loss, due to an increased (T)VaR. In this way, we extend the results of [Ahn *et al.*, 1999], who minimize VaR for a position in a share.

Keywords: Value-at-Risk, Tail Value-at-Risk, bond hedging, Hull-White interest rate model.

1 Introduction

Many financial institutions and non-financial firms nowadays publicly report Value-at-Risk (VaR), a risk measure for potential losses. Internal uses of VaR and other sophisticated risk measures are on the rise in many financial institutions, where, for example, a bank risk committee may set VaR limits, both amounts and probabilities, for trading operations and fund management. At the industrial level, supervisors use VaR as a standard summary of market risk exposure. An advantage of the VaR measure, following from extreme value theory, is that it can be computed without full knowledge of the return distribution. Semi-parametric or fully non-parametric estimation methods are available for downside risk estimation. Furthermore, at a sufficiently low confidence level the VaR measure explicitly focuses risk managers and regulators attention on infrequent but potentially catastrophic extreme losses.

Value-at-Risk (VaR) has become the standard criterion for assessing risk in the financial industry. Given the widespread use of VaR, it becomes increasingly important to study the effects of options on the VaR-based risk management.

A drawback of the traditional Value-at-Risk measure is that it does not care about the tail behaviour of the losses. In other words, by focusing on the VaR at, let's say a 5% level, we ignore the potential severity of the losses below that 5% threshold. In other words, we have no information on how bad things can become in a real stress situation. Therefore, the important question of 'how bad is bad' is left unanswered. Tail Value-at-Risk (TVaR) is trying to capture this problem by considering the possible losses, once the VaR threshold is crossed.

The starting point of our analysis is the classical hedging example, where an institution has an exposure to the price risk of an underlying asset. This may be currency exchange rates in the case of a multinational corporation, oil prices in the case of an energy provider, gold prices in the case of a mining company, etc. The corporation chooses VaR as its measure of market risk. Faced with the unhedged VaR of the position, we assume that the institution chooses to use options and in particular put options to hedge a long position in the underlying.

[Ahn *et al.*, 1999] consider the problem of hedging the Value-at-Risk of a position in a single share by investing a fixed amount C in a put option.
The principal purpose of our study is to extend these results to the situation of a bond and to other risk measures. Although a (zero-coupon) bond is believed to face a lower risk than shares, we should still take into account the possibility of an unfavourable interest rate move. If the interest rate increases, the present value of future payments decreases, which thus decreases the price of a zero-coupon bond. We consider the well-known continuous-time stochastic interest rate model of [Hull and White, 1990] to form expectations concerning the movement in the instantaneous interest rate.

This model allows us to investigate the optimal speculative and hedging strategy by minimizing the Value-at-Risk and Tail Value-at-Risk of the bond, subject to the fixed amount C which is spent on put options, and which is to be considered as the hedging budget the company has.

The discussion is divided as follows: Section 2 presents the risk measures VaR and TVaR and the loss function, and introduces the Hull-White model and the pricing of bonds and options within this model. Afterwards, Section 3 discusses the optimal hedging policy for both risk measures and introduces comparative statics. Section 4 discusses the calibration procedure and results. Section 5 consists of a numerical illustration. Finally, Section 6 summarizes the paper, concludes and introduces further research possibilities.

2 The mathematical framework

2.1 Loss function and risk measures

Consider a portfolio with value W_t at time t. W_0 is then the value or price at which we buy the portfolio at time zero. W_T^d is the value of the portfolio at time T, discounted back until time zero by means of a zero-coupon bond with maturity T. The loss L_0 we make by buying at time zero and selling at time T is then given by L_0=W_0-W_T^d. The Value-at-Risk of this portfolio is defined as the $(1-\alpha)$-quantile of the loss distribution depending on a time interval with length T. The usual holding periods are one day or one month, but institutions can also operate on longer holding periods (e.g. one quarter or even one year), see [Dowd, 1998]. consideration are $T = 1$, 10, 20 days. A formal definition for the $\mathrm{VaR}_{\alpha,T}$ is

$$\Pr(L_0 \geq \mathrm{VaR}_{\alpha,T}) = \alpha.$$

In other words $\mathrm{VaR}_{\alpha,T}$ is the loss of the worst case scenario on the investment at a $(1-\alpha)$ confidence level during the period $[0,T]$. It is possible to define the $\mathrm{VaR}_{\alpha,T}$ in a more general way

$$\mathrm{VaR}_{\alpha,T}(L_0) = \inf\left\{Y \mid \Pr(L_0 \geq Y) < \alpha\right\}.$$

Our second measure, Tail Value-at-Risk, TVaR, is defined as follows:

$$\mathrm{TVaR}_{\alpha,T} = \frac{1}{\alpha}\int_{1-\alpha}^{1} \mathrm{VaR}_{1-\beta,T}\, d\beta.$$

This formula boils down to taking the arithmetic average of the quantiles of our loss, from $1-\alpha$ to 1 on, where we recall that $\mathrm{VaR}_{1-\beta,T}$ stands for the quantile at the level β.
If the cumulative distribution function of the loss is continuous, which is the case in our problem (cfr. infra), TVaR is equal to the Conditional Tail Expectation (CTE) which for the loss L_0 is calculated as:

$$\mathrm{CTE}_{\alpha,T}(L_0) = E[L_0 \mid L_0 > \mathrm{VaR}_{\alpha,T}(L_0)].$$

A closely related risk measure concerns Expected Shortfall (ESF). It is defined as:

$$\mathrm{ESF}(L_0) = E\left[(L_0 - \mathrm{VaR}_{\alpha,T}(L_0))_+\right].$$

In order to determine $\mathrm{TVaR}_{\alpha,T}(L_0)$, we can make use of the following equality:

$$\begin{aligned}\mathrm{TVaR}_{\alpha,T}(L_0) &= \mathrm{VaR}_{\alpha,T}(L_0) + \frac{1}{\alpha}\mathrm{ESF}(L_0)\\ &= \mathrm{VaR}_{\alpha,T}(L_0) + \frac{1}{\alpha}E\left[(L_0 - \mathrm{VaR}_{\alpha,T}(L_0))_+\right].\end{aligned}$$

This formula already makes clear that $\mathrm{TVaR}_{\alpha,T}(L_0)$ will always be larger than $\mathrm{VaR}_{\alpha,T}(L_0)$.

As mentioned in the introduction, we focus on the hedging problem of a zero-coupon bond. We suppose that at time zero a company buys a zero-coupon bond with maturity S, and sells this bond again at time T, with T being smaller than S. If we ignore the possibility of a change in the credit worthiness of the bond issuer, the most important impact on the bond price comes from changes in the interest rate curve. Therefore, we need to define a process that describes the evolution of the instantaneous interest rate, and enables us to value the zero-coupon bond and the hedging instruments. As term structure model, we consider the Hull-White model, which we describe in more detail in the next section.

2.2 The Hull-White model

There exists a whole literature concerning interest rate models. For a comprehensive overview, see [Brigo and Mercurio, 2001]. For our analysis, we focus on the Hull-White model, first discussed by Hull and White in 1990 (see [Hull and White, 1990]). We choose this model because it is still an often used model in financial institutions for risk management purposes, (see [Brigo and Mercurio, 2001]). Two main reasons explain this popularity. First of all, it is a model that allows closed form solutions for bond and plain vanilla European option pricing. So, since there are exact pricing formulas, there is no need to run time consuming simulations. But of course, if the model lacks credibility, fast but wrong price computations do not offer any benefit. But that is where the second big advantage of the Hull-White model comes from. It belongs to the class of so called no-arbitrage interest rate models. This means that, in contrast to equilibrium models (such as Vasicek, Cox-Ingersoll-Ross), no-arbitrage models succeed in fitting a given term structure, and thus can match today's bond prices perfectly. An often cited critique is that applying the model sometimes results in a negative interest rate.

[Hull and White, 1990] assume that the instantaneous interest rate follows a mean reverting process also known as an Ornstein-Uhlenbeck process:

$$dr(t) = (\theta(t) - \gamma(t)r(t))dt + \sigma(t)dZ(t) \tag{1}$$

for a standard Brownian motion $Z(t)$ under the risk-neutral measure Q, and with time dependent parameters $\theta(t)$, $\gamma(t)$, and $\sigma(t)$. The parameter $\theta(t)$ is the time dependent long-term average level of the spot interest rate around which $r(t)$ moves, $\gamma(t)$ controls the mean-reversion speed and $\sigma(t)$ is the volatility measure. By making the mean reversion level θ time dependent, a perfect fit with a given term structure can be achieved, and in this way arbitrage can be avoided. In our analysis, we will keep γ and σ constant, and thus time-independent. According to [Brigo and Mercurio, 2001], this is desirable when an exact calibration to an initial term structure is wanted. This perfect fit then occurs when $\theta(t)$ satisfies the following condition:

$$\theta(t) = F_t^M(0,t) + \gamma F^M(0,t) + \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t}),$$

where, $F^M(0,t)$ denotes the instantaneous forward rate observed in the market on time zero with maturity t.

It can be shown (see [Hull and White, 1990]) that the expectation and variance of the stochastic variable $r(t)$ are:

$$E\left[r(t)\right] = m = r(0)e^{-\gamma t} + a(t) - a(0)e^{-\gamma t} \tag{2}$$

$$\mathrm{Var}\left[r(t)\right] = s^2 = \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t}), \tag{3}$$

with the expression $a(t)$ calculated as follows:

$$a(t) = F^M(0,t) + \frac{\sigma^2}{2}\left(\frac{1 - e^{-\gamma(S-t)}}{\gamma}\right)^2.$$

Based on these results, Hull and White developed an analytical expression for the price of a zero-coupon bond with maturity date S

$$Y(t,S) = A(t,S)e^{-B(t,S)r(t)}, \tag{4}$$

where

$$B(t,S) = \frac{1 - e^{-\gamma(S-t)}}{\gamma}, \tag{5}$$

$$A(t,S) = \frac{Y^M(0,S)}{Y^M(0,t)}e^{B(t,S)F^M(0,t) - \frac{\sigma^2}{4\gamma}(1-e^{-2\gamma t})B^2(t,T)}, \tag{6}$$

with Y^M the bond price observed in the market. Since $A(t,S)$ and $B(t,S)$ are independent of $r(t)$, the distribution of a bond price at any given time must be lognormal with parameters Π and Σ^2:

$$\Pi(t,S) = \ln A(t,S) - B(t,S)m, \qquad \Sigma(t,S)^2 = B(t,S)^2 s^2, \tag{7}$$

with m and s^2 given by (2) and (3).

Hull and White also succeeded in developing bond option pricing formulas.
Consider a zero-coupon bond with principal K which matures at time S. Then the price of a European call option with this bond as the underlying security and with strike price X and exercise date T (with $T \leq S$) is at date zero given by:

$$C(0,T,S,X) = KY(0,S)\Phi(d_1) - XY(0,T)\Phi(d_2), \tag{8}$$

where

$$d_1 = \frac{1}{\sigma_p}\log\left(\frac{KY(0,S)}{XY(0,T)}\right) + \frac{\sigma_p}{2}, \qquad d_2 = d_1 - \sigma_p,$$

$$\sigma_p^2 = \frac{\sigma^2}{2\gamma^3}(1 - e^{-2\gamma T})(1 - e^{-\gamma(S-T)})^2$$

and $\Phi(z)$ is the cumulative distribution function of a standard normal random variable. The Put-Call parity model is used to determine the value of a put option from a corresponding call option and provides in this case the following European put option price corresponding to (8):

$$P(0, T, S, X) = -KY(0, S)\Phi(-d_1) + XY(0, T)\Phi(-d_2). \tag{9}$$

3 The bond hedging problem

Analogously to [Ahn *et al.*, 1999], we assume that we have bought, at time zero, one bond with maturity S and we will sell this bond at time T, which is prior to S. In case of an increase in the interest rate, not hedging can lead to severe losses. Therefore, the company decides to spend an amount C on hedging. This amount will be used to buy one or part of a bond put option, so that, in case of a substantial decrease in the bond price, the put option can be exercised in order to prevent large losses. The remaining question now is how to choose the strike price. We will find the optimal strike price which minimizes VaR and TVar for a given hedging cost.

3.1 VaR minimization

Let us assume that the institution has an exposure to a bond, $Y(0, S)$, with principal $K = 1$, which matures at time S, and that the company has decided to hedge the bond value by using a percentage h $(0 < h < 1)$ of one put option $P(0, T, S, X)$ with strike price X and exercise date T (with $T \leq S$). Analogously to the future value of the hedged portfolio on a stock, we can look at the future value of the hedged portfolio that is composed of the bond Y and the put option $P(0, T, S, X)$ at time T as a function of the form

$$H_T = \max(hX + (1 - h)Y(T, S), Y(T, S)).$$

If the put option finishes in-the-money (a case which is of interest to us), then the discounted value of the future value of the portfolio is

$$H_T^d = ((1 - h)Y(T, S) + hX)Y(0, T).$$

Taking into account the cost of setting up our hedged portfolio, which is given by the sum of the bond price $Y(0, S)$ and the cost C of the position in the put option, we get for the present value of the loss

$$L_0 = Y(0, S) + C - ((1 - h)Y(T, S) + hX)Y(0, T),$$

and this under the assumption that the put option finishes in-the-money.
From the formulae (4)-(6), we can see that, for $S \geq T$, $Y(T,S)$ has a lognormal distribution with parameters $\Pi(T,S)$ and $\Sigma^2(T,S)$, given by (7). Hence, the present value of the loss of the portfolio can be expressed as a strictly decreasing function function f of z:

$$L_0 = Y(0,S) + C - ((1-h)e^{\Pi(T,S)+\Sigma(T,S)z} + hX)Y(0,T) := f(z), \tag{10}$$

where z denotes a stochastic random variable with a standard normal distribution.
Then, the Value-at-Risk at an α percent level of a position $H = \{Y,\ h,\ P\}$ consisting of a bond Y and h put options P (which are assumed to be in-the-money) with a strike price X and an expiry date T is equal to[1]

$$\mathrm{VaR}_{\alpha,T}(L_0) = f(c(\alpha)) = Y(0,S) + C - ((1-h)e^{\theta_B(\alpha)} + hX)Y(0,T), \tag{11}$$

where

$$\theta_B(\alpha) = \Pi(T,S) + \Sigma(T,S)c(\alpha) \tag{12}$$

and $c(\alpha)$ is the percentile of the standard normal distribution, i.e. $\Pr(z \leq c(\alpha)) = \alpha$.

Similar to the Ahn et al. problem, we would like to minimize the risk of the discounted future value of the hedged bond H_T, given a maximum hedging expenditure C. More precisely, we consider the minimisation problem

$$\min_{X,h} Y(0,S) + C - ((1-h)e^{\theta_B(\alpha)} + hX)Y(0,T)$$

subject to the restrictions $C = hP(0,T,S,X)$ and $h \in (0,1)$.
Solving this constrained optimization problem, we find from the Kuhn-Tucker conditions that the optimal strike price X^* satisfies the following equation

$$P(0,T,S,X) - (X^* - e^{\theta_B(\alpha)})\frac{\partial P(0,T,X)}{\partial X} = 0,$$

or equivalently, when taking (9) into account, with $K = 1$,

$$e^{\theta_B(\alpha)} = \frac{Y(0,S)\Phi(-d_1)}{Y(0,T)\Phi(-d_2)}. \tag{13}$$

We note that the optimal strike price is independent of the hedging cost, meaning that a change in the cost only influences the percentage h invested in the put option.

[1] In case of an unhedged portfolio, take $C = h = 0$ in (10) and in (11) to obtain the loss function L_0 with corresponding $\mathrm{VaR}_{\alpha,T}(L_0)$.

3.2 Tail VaR minimization

In this section, we demonstrate the ease of extending our analysis to the alternative risk measure TVaR.

First we note that in view of (10)-(11) the ESF for the loss L_0 can be simplified to

$$\mathrm{ESF}(L_0) = (1-h)Y(0,T)e^{\Pi(T,S)}E[(e^{\theta_B(\alpha)} - e^{\Sigma(T,S)z})_+].$$

Because of the lognormality of the bond prices we easily obtain an analytical expression for the ESF:

$$\mathrm{ESF}(L_0) = (1-h)Y(0,T)e^{\Pi(T,S)}\Big[\alpha e^{\Sigma(T,S)c(\alpha)} - e^{\frac{1}{2}\Sigma^2(T,S)}\Phi(c(\alpha) - \Sigma(T,S))\Big].$$

This reduces the $\mathrm{TVaR}_{\alpha,T}(L_0)$ to:

$$\begin{aligned}\mathrm{TVaR}_{\alpha,T}(L_0) = {} & Y(0,S) + C - hXY(0,T) \\ & -\frac{1}{\alpha}(1-h)e^{\Pi(T,S)+\frac{1}{2}\Sigma^2(T,S)}\Phi(c(\alpha) - \Sigma(T,S))Y(0,T).\end{aligned}$$

We again seek to minimize this risk measure, in order to minimize potential losses. The procedure for minimizing this TVaR is analogue to the VaR minimization procedure. The resulting optimal strike price can thus be determined from the formula below:

$$\frac{1}{\alpha}e^{\Pi(T,S)+\frac{1}{2}\Sigma^2(T,S)}\Phi(c(\alpha) - \Sigma(T,S)) = \frac{Y(0,S)\Phi(-d_1)}{Y(0,T)\Phi(-d_2)}.$$

3.3 Comparative statics

We examine how changes in the parameters of the model influence the optimal strike price, by means of the derivatives of the optimal strike price with respect to these parameters.

For both $\mathrm{VaR}_{\alpha,T}$ and $\mathrm{TVaR}_{\alpha,T}$, the optimal strike price is implicitly defined by

$$F(X,\beta) = \mathrm{FAC}\cdot Y(0,T)\Phi(-d_2) - Y(0,S)\Phi(-d_1) = 0,$$

with β the vector including the Hull-White parameters, that is γ and σ, see Section 2.2, and with FAC representing $e^{\theta_B(\alpha)}$ in the case of $\mathrm{VaR}_{\alpha,T}$ and $\frac{1}{\alpha}e^{\Pi(T,S)+\frac{1}{2}\Sigma^2(T,S)}\Phi(c(\alpha) - \Sigma(T,S))$ in the case of $\mathrm{TVaR}_{\alpha,T}$.
Taking into account the implicit function theorem, we obtain the required derivatives as follows:

$$\frac{\partial F}{\partial X}dX + \frac{\partial F}{\partial \beta}d\beta = 0 \Longleftrightarrow \frac{dX}{d\beta} = -\frac{\frac{\partial F}{\partial \beta}}{\frac{\partial F}{\partial X}}. \tag{14}$$

The denominator of (14) is equal for the different derivatives, and is given by

$$\frac{\partial F}{\partial X} = \frac{\mathrm{FAC}\cdot Y(0,T)\varphi(d_2) - Y(0,S)\varphi(d_1)}{X\sigma_p}, \tag{15}$$

with φ being the density function of a standard normal random variable, while the numerator of (14) can be obtained by applying the following formula,

$$\frac{\partial F}{\partial \beta} = \frac{\partial \mathrm{FAC}}{\partial \beta} Y(0,T)\Phi(-d_2) + \mathrm{FAC} \cdot \frac{\partial Y(0,T)}{\partial \beta}\Phi(-d_2) \tag{16}$$
$$-\mathrm{FAC} \cdot Y(0,T)\varphi(d_2)\frac{\partial d_2}{\partial \beta} - \frac{\partial Y(0,S)}{\partial \beta}\Phi(-d_1) + Y(0,S)\varphi(d_1)\frac{\partial d_1}{\partial \beta}.$$

These derivatives are rather involved and do not lead to a straightforward interpretation of their sign and magnitude. Therefore, we will describe the influence of changes in the most important parameters by using a numerical illustration in a further section.

Further relevant derivatives are $\frac{dX}{dS}$ and $\frac{dX}{dT}$ to study the response of the optimal strike price to a change in the maturity of both the underlying bond and the maturity of the bond option used to hedge the exposure. They follow from formulae (14)-(16), after having replaced β by S and T respectively, and taking into account the simplification due to the fact that $Y(0,T)$ is independent of S, and $Y(0,S)$ is independent of T. Again, we leave the interpretation of these derivatives to the next section.

A last derivative of interest is the one with respect to α, formally $\frac{dX}{d\alpha}$:

$$\frac{dX}{d\alpha} = -\frac{1}{\frac{\partial F}{\partial X}} \cdot \frac{\partial \mathrm{FAC}}{\partial \alpha} Y(0,T)\Phi(-d_2),$$

where $\frac{\partial \mathrm{FAC}}{\partial \alpha}$ is respectively given by

$$\frac{e^{\theta_B(\alpha)}\Sigma(T,S)}{\varphi(c(\alpha))} \qquad \text{(VaR)}$$

$$\frac{e^{\Pi(T,S)+\frac{1}{2}\Sigma^2(T,S)}}{\alpha^2}\left[\frac{\alpha\varphi(c(\alpha)-\Sigma(T,S))}{\varphi(c(\alpha))} - \Phi(c(\alpha)-\Sigma(T,S))\right] \qquad \text{(TVaR).}$$

4 Calibration of the Hull-White model

Until now, we theoretically discussed the issue of minimizing the VaR and TVaR of our investment. A further step would be to provide a credible numerical illustration of this problem. The first problem we encounter when trying to implement this procedure, is the interest rate model we used. We need to have credible parameter values for γ and σ. The process to obtain these parameters is calibration. The most common way to calibrate the Hull-White model is by using interest rate options, such as swaptions or caps. The goal of the calibration is to find the model parameters that minimize the relative difference between the market prices of these interest rate options and the prices obtained by applying our model.

Suppose we have M market prices of swaptions or caps, then we search the γ and σ such that the sum of squared errors between the market and model prices are minimized. Formally,

$$\min_{\gamma,\sigma} \sqrt{\sum_{i=1}^{M} \left(\frac{\text{model}_i - \text{market}_i}{\text{market}_i} \right)^2}.$$

Interest rate caps are instruments that provide the holder of it protection against a specified interest rate (e.g. the three month LIBOR, R_L) rising above a specified level (the cap rate, R_C)). Suppose a company issued a floating rate note with as reference rate the three month LIBOR. When LIBOR rises above the cap rate, a payoff is generated such that the net payment of the holder only equals the cap rate. One cap consists of a series of caplets. These caplets can be seen as call options on the reference rate. The maturity of these call options equals the tenor, which is the time period between two resets of the reference rate. In our case, this is three months, or 0.25 year.

If in our case, at time t_k, the three month LIBOR rises above the cap rate, the call will be exercised, which leads to a payoff at time t_{k+1} (0.25 year later) that can be used to compensate the increased interest payment on the floating rate note. Formally, the payoff at time t_{k+1} equals (see [Hull, 2003]):

$$\max(0.25(R_L - R_C), 0).$$

This is equivalent to a payoff at time t_k of

$$\frac{\max(0.25(R_L - R_C), 0)}{1 + 0.25R_L}.$$

This can be restated as:

$$\max\left(1 - \frac{1 + 0.25R_C}{1 + 0.25R_L}, 0\right).$$

This is the payoff of a put option with strike 1, expiring at t_k, on a zero-coupon bond with principal $1 + 0.25R_C$, maturing at t_{k+1}. This means that each individual caplet corresponds to a put option on a zero-coupon bond. Thus, a cap can be valued as a sum of zero-coupon bond put options. Since these put options can be valued using the Hull-White model, this offers us a way to fit our model to the market data. The market data we have are to be found in table 1. In table 1, cap maturities are listed, along with the volatility quotes of these caps and the cap rate. Note that the volatility quotes have the traditional humped relation with respect to the maturity of the cap: the volatility reaches its peak at the 2 year cap and then decreases steadily as the maturity increases. Although the cap rate can be freely determined, it is most common to put it equal to the swap rate for a swap having the same payment dates as the cap. The volatility quotes that are provided are based on Black's model. This means that we first have to use Black's formula for valuing bond options in order to arrive at the prices of the caps.

Currency: EUR GOTTEX			
Cap maturity	**Volatility (in %)**	**Cap rate (in %)**	**Cap price**
6M	19.948029	2.200	0.00012
1Y	22.694659	2.310	0.00089
2Y	25.680017	2.542	0.00434
3Y	25.634871	2.741	0.00913
4Y	24.831329	2.911	0.01470
5Y	23.621903	3.065	0.02076
6Y	22.386273	3.206	0.02714
7Y	21.252854	3.334	0.03364
8Y	20.265208	3.448	0.04022
9Y	19.448656	3.548	0.04679
10Y	18.781271	3.634	0.05324
12Y	17.757097	3.775	0.06585
15Y	16.649579	3.934	0.08393
20Y	15.465745	4.093	0.11080

Table 1: Overview cap data

These prices are shown in the fourth column. Now we still have to calculate the model prices. Therefore, we use, for each caplet, formula (9). As strike price X we take 1. $Y(0,T)$ and $Y(0,S)$ can be read from the term structure. K equals $1 + 0.25R_C$. Taking the sum of all the caplets in a given cap, we get an expression for which we need to seek the parameters that, globally, make the best fit. The calibration procedure results in the following parameter values:

$$\gamma = 0.31621 \qquad \sigma = 0.011631.$$

The parameter γ is positive, which means that the process is mean-reverting.

5 Numerical results

Using the parameters obtained in the calibration section, we can illustrate the value of going through this minimisation procedure. We suppose the company has bought a bond with a remaining maturity of five years, and is planning to sell the bond after one year, at a time when the remaining maturity is four years. The price at which the bond is bought then is 0.8580. As initial instantaneous rate we take the overnight rate from the caps data, which is 0.0213. Then we still have to fix the hedging budget. The budget is set at 0.0043, or twenty percent of the one year rate.
Figure 1 shows the distribution of the bond price after one year, for 1.000.000 bond price calculations. Although theoretically this distribution is lognormal, it fits very well with

the normal distribution. This is mainly because of the small standard deviation we have. The mean of the distribution is 0.9032, but more importantly, the minimum is 0.8159, which is below the price at which we bought the bond. Note that the maximum bond price amounts to 1.0068. A zero-coupon bond price above 1 may sound peculiar, but can be explained by the possibility of negative interest rates in the Hull-White model. Concerning these negative interest rates,we note that in our sample, 0.6 % of the observations resulted in a negative interest rate.

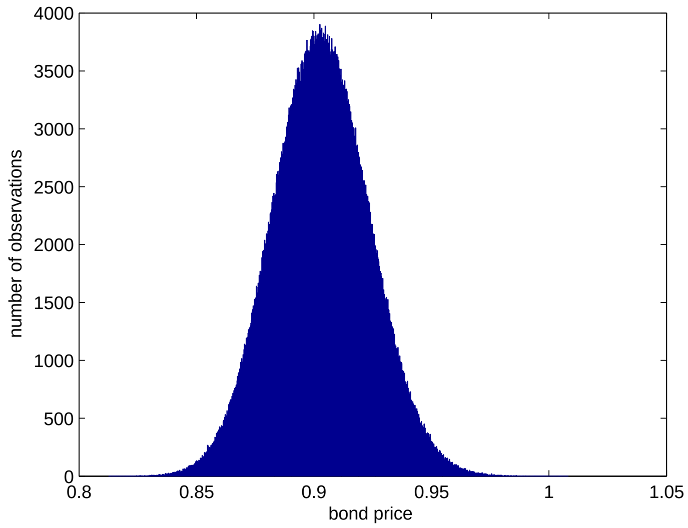
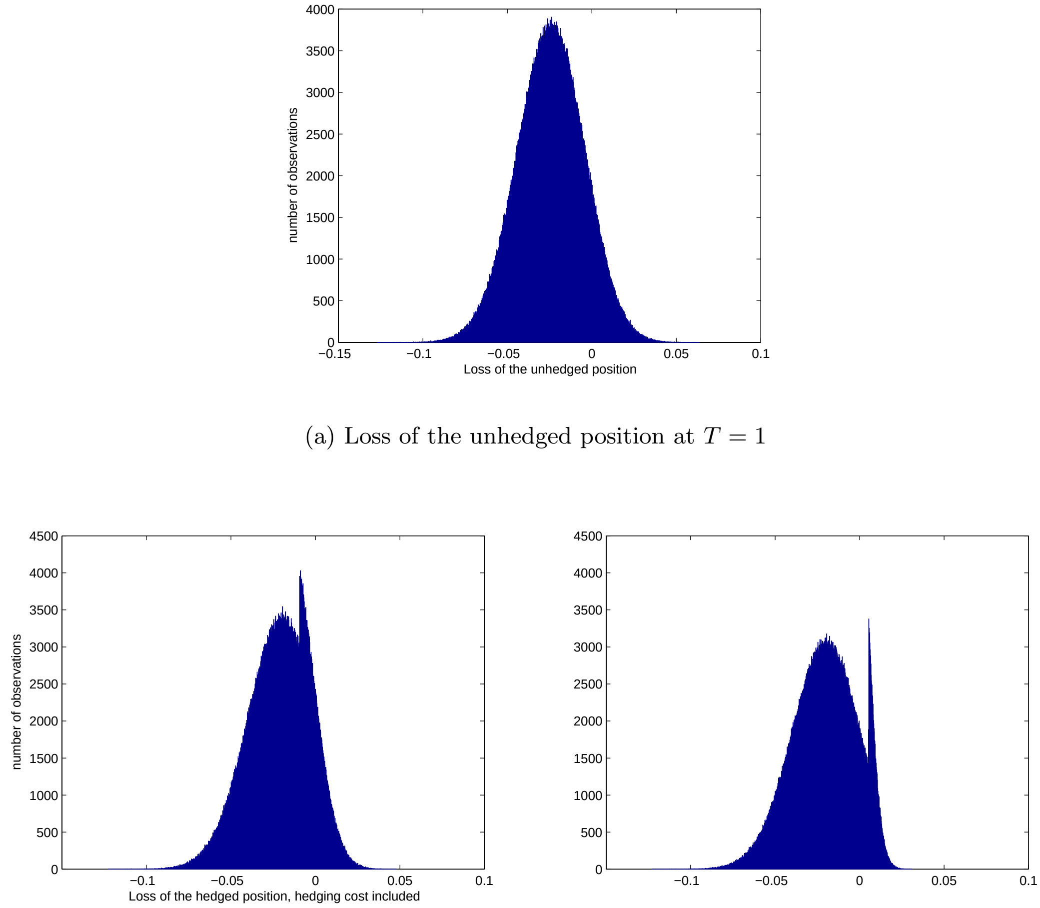


Fig. 1: Bond value at $T = 1$

Figure 2(a) shows the distribution of the unhedged position. Discounted losses can go up to 0.0608. In order to avoid that, the company has the hedging budget to buy bond put options. Following our VaR minimization procedure, the optimal strike price is given by 0.89195. This allows us to buy 26.23 percent of an option.

Figure 2(b) shows the loss distribution of the hedged position, in case the strike price is determined by minimizing VaR. This figure is fairly similar to figure 2(a), except for the right tail, where the large losses are to be situated. This right tail shows a peak, which was absent from the previous graph. This peak can be explained by the exercise of the put option once the bond price falls below 0.89195. If we had bought one entire put option ($h = 1$), we would be able to sell the whole bond at the strike price. In that case, we would be sure that our loss will be limited. This is shown in figure 3. Due to the fact that our budget doesn't allow us to buy one put option but only 26.23 percent, there is still some variation, because we have to sell 73.77 percent of the bond at the market price. Figure 2(c) shows the loss distribution of the hedged position, in case the strike price is determined by minimizing TVaR. Although the same shape is observed, we report two



(a) Loss of the unhedged position at $T = 1$

(b) Loss of the hedged position at $T = 1$, VaR minimisation

(c) Loss of the hedged position at $T = 1$, TVaR minimisation

Fig. 2: Loss distributions

differences. The peak is situated a bit more to the right, due to the lower strike price, which makes that the bond price should have dropped further in price before the put is exercised. Additionally, the right tail is smaller, because of the larger h, which makes sure that a larger part of the portfolio is hedged, and in this way the more extreme movements in the bond price do not influence the end value of the portfolio that much.

The VaR at the 5 percent level of the unhedged position is situated 0.0080248. The VaR of the hedged position, including hedging expenditure is 0.0066718. Figure 4(a) shows that indeed the strike price we calculated is optimal for VaR reduction and that it is valuable to go in search of the optimal strike price, since moving away from that price increases our risk. The full line in this figure shows the VaR for different strike prices,

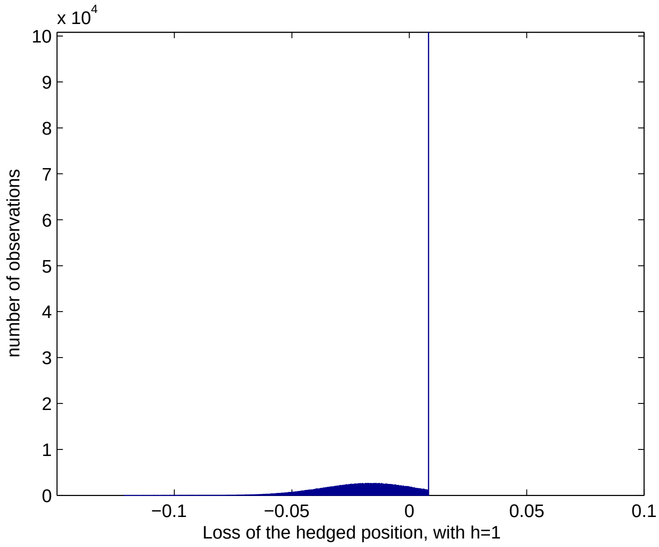
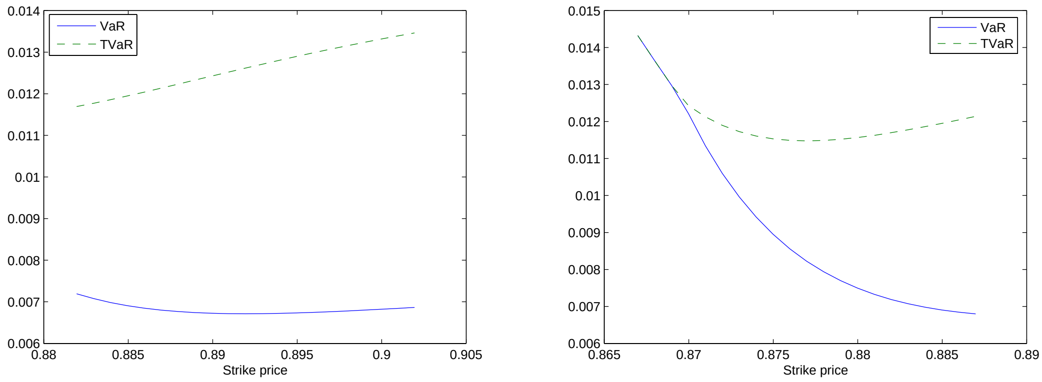


Fig. 3: Loss of the position, h=1

with the central strike price being the optimal. We observe a parabolic relationship, with the minimum VaR observed at 0.86932. Note that as the strike price changes in this graph, the percentage that we buy of the put option is also changed. The dashed line shows the TVaR. Two important remarks are to be made here. First of all, we indeed observe that TVaR always exceeds VaR, which was already clear from our theoretical discussion. Secondly, and more important, we observe a difference between the strike price that is optimal for VaR reduction and the one optimal for TVaR reduction. It seems that lowering the strike price decreases the TVaR. Further calculations show that, in order to minimize TVaR, the strike price should equal 0.87698. With this strike, 59.35 percent of a put option can be bought. Whereas TVaR for the unhedged position amounts to 0.01611, we are able to reduce it to 0.01148 by buying the put option at the optimal strike price. Again we graphically prove that our strike indeed minimizes the TVaR. This is done in figure 4(b) in which the full line shows TVaR and the dashed line VaR, both in the neighbourhood of the optimum. Whereas TVaR shows the expected parabolic behaviour, VaR constantly decreases, which is in line with what we expected, since we know that the minimum is situated further.

As already mentioned above, we are also interested in the effect of changes in the parameter estimates of the Hull-White model on the optimal strike price. An increase in one of these parameters always leads to a lower optimal strike price. The influence of a 1% increase in γ only marginally effects the strike price. The optimal strike price drops with 0.08%. The most influential parameter of the Hull-White model undoubtedly is the volatility. Increasing this parameter by 1% leads to a drop of 0.42% in the optimal strike price.



(a) VaR minimisation

(b) TVaR minimisation

Fig. 4: VaR and TVaR in function of distance from optimal strike price

Increasing the maturity of the bond decreases the strike price, while increasing the holding period (meaning that the holding period moves closer to the maturity of the bond) increases the strike price. Reducing the certainty with which a bank wishes to know the value it can lose, or in other words, increasing α leads to an increased strike price.

6 Conclusion

In this paper, we studied the optimal risk control for one bond using a percentage of a put option by means of Value-at-Risk and Tail Value-at-Risk, widespread concepts in the financial world. The interest model we use for valuation, is the Hull-White model. The optimal strategy corresponds to buying a percentage of a put option with optimal strike price in order to have a minimal VaR or TVaR given a fixed hedging cost. We did not obtain an explicit result, but numerical methods can be easily implemented to solve for the optimal strategy. In order to provide a credible numerical illustration, we calibrated the parameters of the Hull-White model to a set of cap prices. We then demonstrate the relevance of searching for the optimal strike price, since moving away from this optimum implies a loss because of an increased VaR or TVaR. We also showed that minimizing VaR results in a different strike price than minimising TVaR. It is up to the company to decide upon which risk measure to minimize and to then determine the appropriate strike price.

Further analysis can be oriented in a number of directions. First of all, we could examine the implications of assuming a different interest rate model than Hull-White. We could consider alternative means of hedging: swaps or swaptions could be used in order to prevent large losses to our portfolio.

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