A Continuous-Time Model for Reinvestment Risk in Bond Markets

Mikkel Dahl
Laboratory of Actuarial Mathematics
University of Copenhagen
Denmark
dahl@math.ku.dk

Outline of talk

- Motivation
- The model
- Hedging
- Conclusion

Motivation

Danish life insurance companies have to value liabilities at market value
$\rightsquigarrow$ Increased focus on correct valuation and risk management

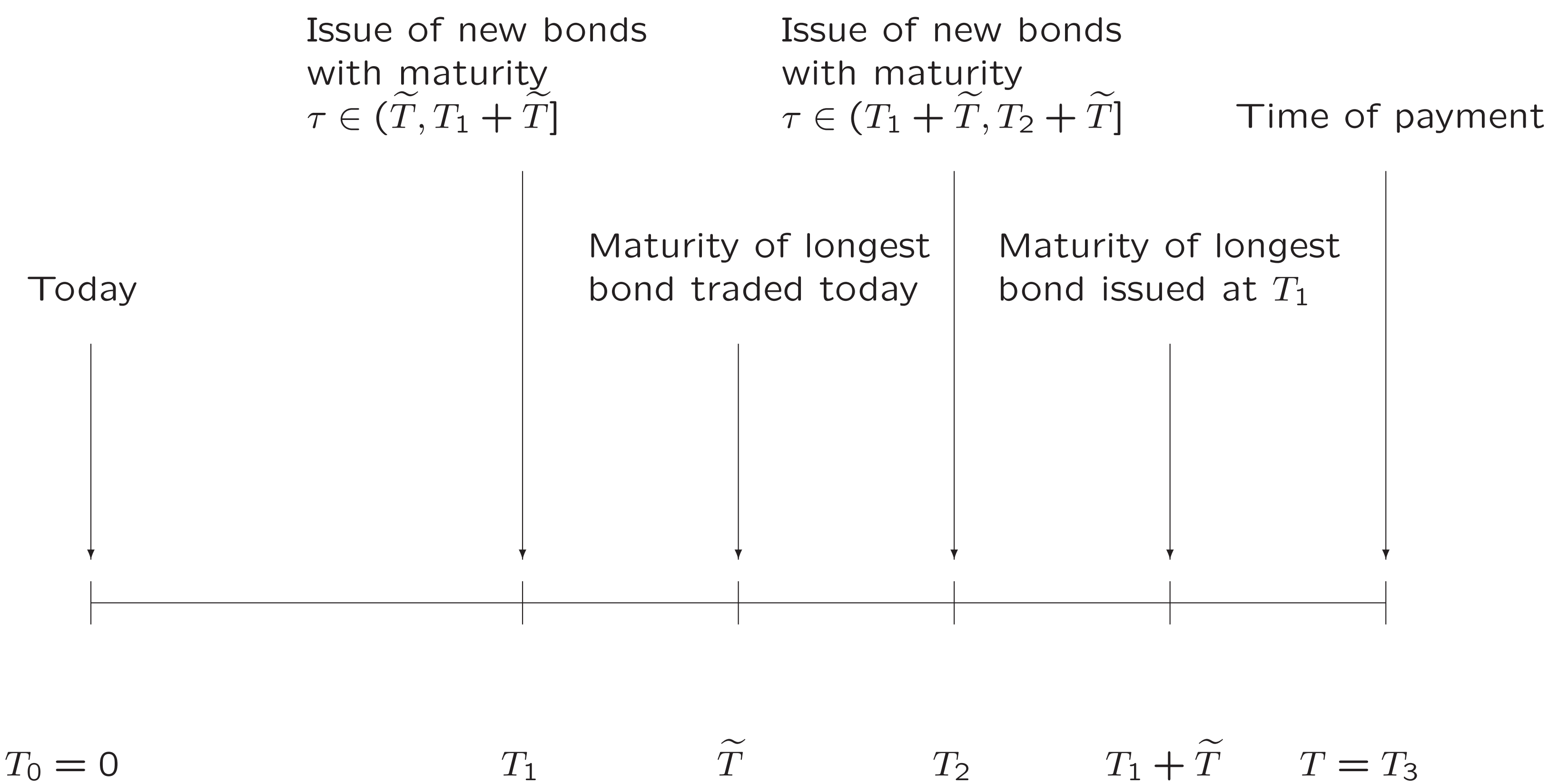
Special features concerning life insurance contracts

- Depends on death/survival of the insured
  - Unsystematic mortality risk (randomness of deaths in a portfolio with known mortality intensity)
  - Systematic mortality risk (stochastic mortality intensity)
- Very long term contracts $\rightsquigarrow$ no liquid market for sufficiently long bonds
  - Reinvestment risk (focus of this talk)

Example

- Assume the time to maturity of the longest bond traded is 10 years and new bonds are each year
- We have sold a claim of 1 with time of maturity 30 years
- What is the price? (Not unique)
- How to hedge? And what is the risk associated with the hedge?

The model


Issue of new bonds
with maturity
$\tau \in (\widetilde{T}, T_1 + \widetilde{T}]$
Issue of new bonds
with maturity
$\tau \in (T_1 + \widetilde{T}, T_2 + \widetilde{T}]$
Time of payment
Maturity of longest
bond traded today
Maturity of longest
bond issued at T_1
Today
$T_0 = 0$
T_1
$\widetilde{T}$
T_2
$T_1 + \widetilde{T}$
$T = T_3$

Notation

- Price at time t of a zero coupon bond maturing at time τ:

$$P(t, \tau)$$

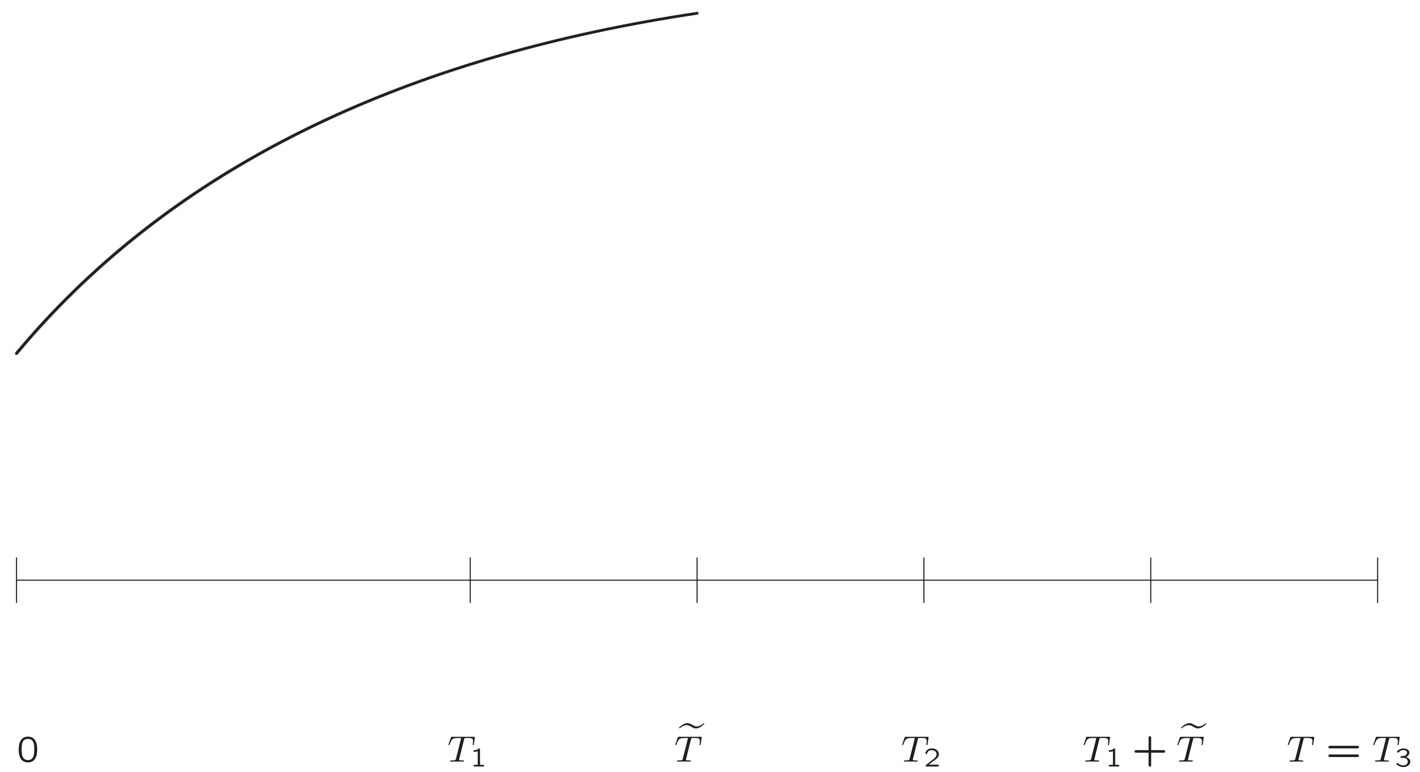
- Forward rates are given by

$$P(t, \tau) = e^{-\int_t^\tau f(t,u)du}$$

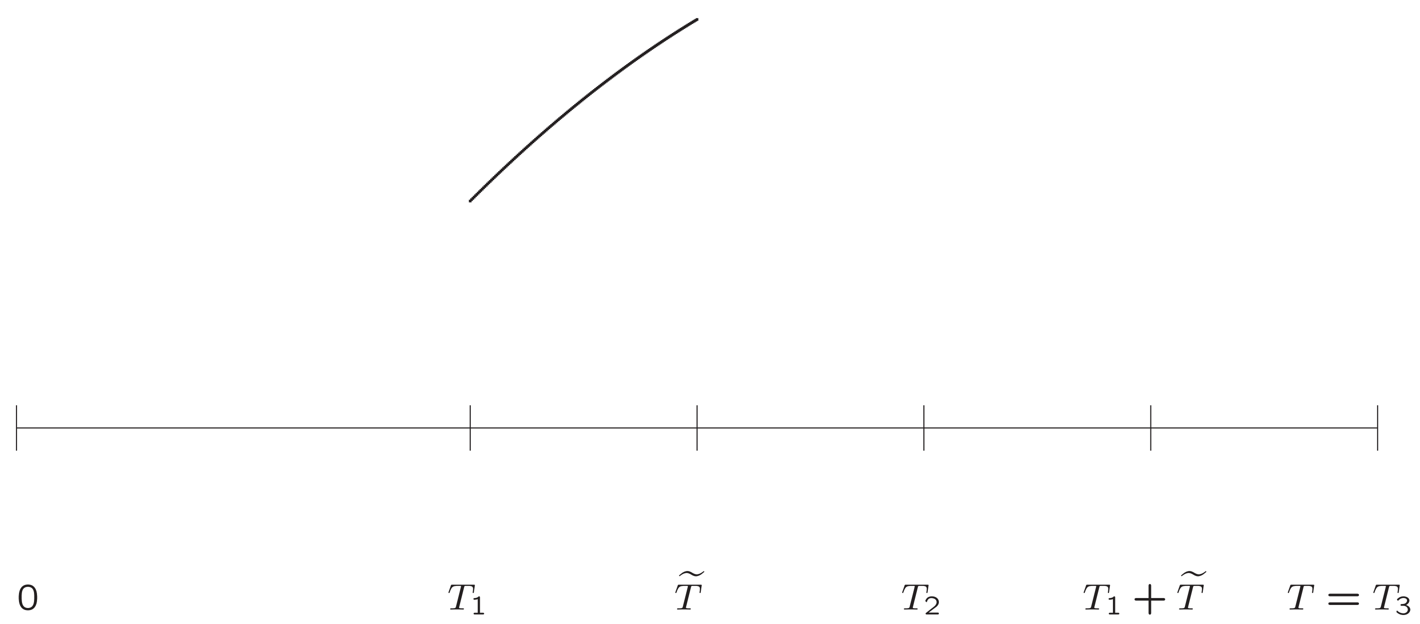
  The forward rate $f(t, \tau)$ can be interpreted as the riskfree rate of interest, contracted at time t over the infinitesimal interval $[\tau, \tau + d\tau)$.

- More convenient to model forward rates than bond prices

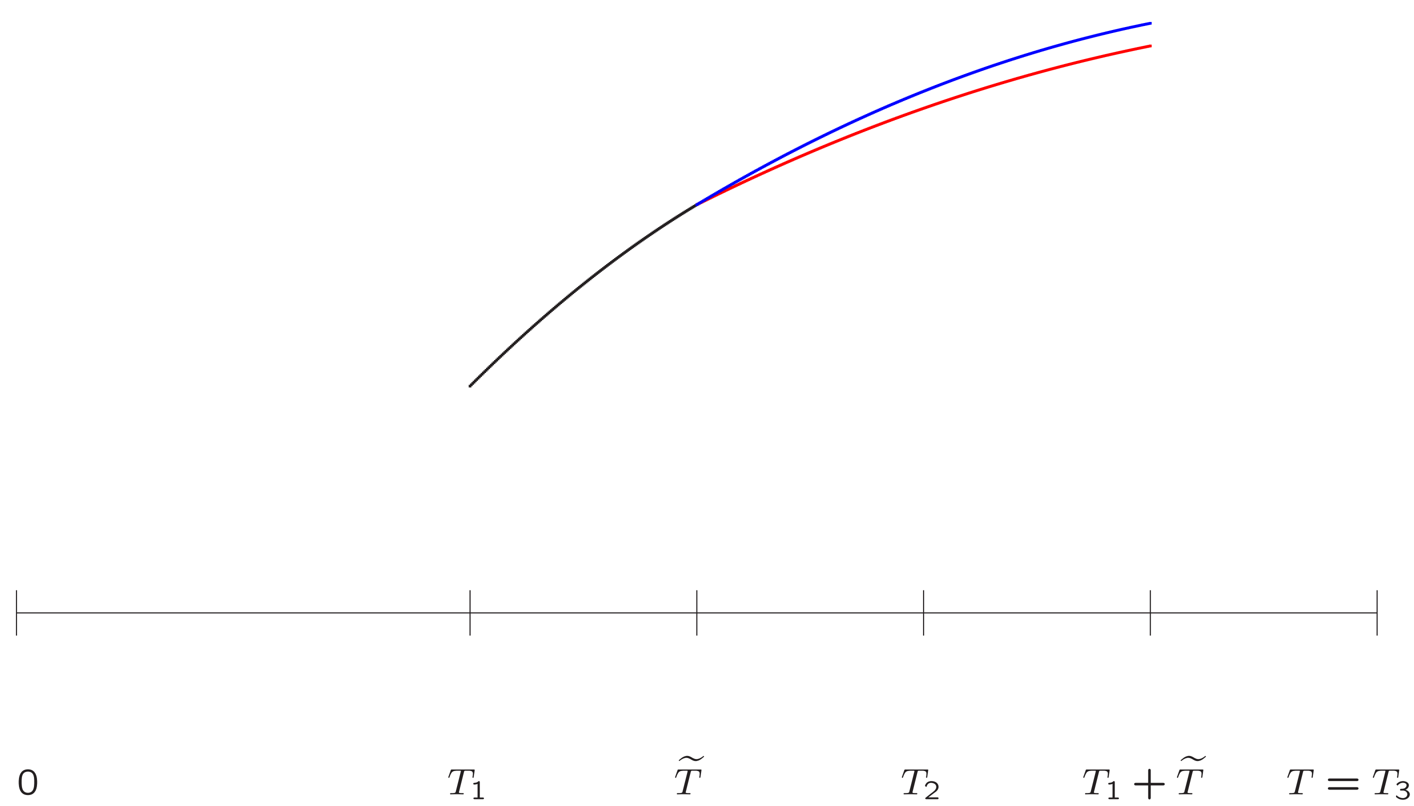
Forward rate curve at time 0



Forward rate curve at time T_1 prior to issue of new bonds



Examples of forward rate curves at time T_1 after issue of new bonds



- P-dynamics of forward rates between issue of new bonds

$$df(t,\tau) = \underbrace{\alpha(t,\tau)dt}_{\text{drift}} + \underbrace{\sigma(t,\tau)dW_t}_{\text{noise}}$$

- Initial value of new forward rates at time T_i, $i = 1, \ldots, n$

$$f(T_i,\tau) = \underbrace{f(T_i, T_{i-1} + \widetilde{T})}_{\text{Endpoint of known curve}} + \underbrace{\int_{T_{i-1}+\widetilde{T}}^{\tau} \gamma^i(u)du}_{\text{extension}}$$

γ^i depends on r.v. Y_i independent of W.
A possible choice of γ^i

$$\gamma^i(u) = \frac{1}{T_i - T_{i-1}} (k_1 Y_i + k_2(1 - Y_i))$$

$Y_i \in \{0,1\}$, k_1 and k_2 constants.
$\rightsquigarrow$ Extension is a straight line with slope $k_j/(T_i - T_{i-1})$, $j \in \{1,2\}$.

- Equivalent martingale measures (EMM's)
  - Unique dynamics of forward rates under EMM
  - No assets depending on $Y = (Y_1, \ldots, Y_n)$ are traded prior to observation
    $\rightsquigarrow$ No unique distribution of Y under EMM
    $\rightsquigarrow$ Infinitely many equivalent martingale measures
    $\rightsquigarrow$ Incomplete model

Hedging

Controlling the reinvestment risk

- Short term contracts (depends on bonds with maturity before time $\tilde{T}$)
  $\rightsquigarrow$ Attainable
  $\rightsquigarrow$ Perfect hedge for short term claims
  $\rightsquigarrow$ No risk

- Long term contracts (depends on bonds with maturity after time $\tilde{T}$)
  $\rightsquigarrow$ In general: Unattainable
  $\rightsquigarrow$ In general: No perfect hedge
  $\rightsquigarrow$ Minimize risk associated with hedge
  $\rightsquigarrow$ Choose criterion measuring risk
  $\rightsquigarrow$ Considered risk-minimization (in the paper)

Conclusion

- Main contribution (and focus of this talk): Proposed a bond market model including reinvestment risk

- Applied criterion of risk-minimization
  - Risk-minimizing strategy
  - Quantify reinvestment risk

- A possible implementation

- Currently working on applications in insurance