Development and Pricing of a New Participating Contract

Carole Bernard *

Olivier Le Courtois †

François Quittard-Pinon ‡

18th July 2005

Abstract: The purpose of this article is to design and price a new type of participating life insurance contract. Participating contracts are popular in France, Denmark, the UK (where they are called with-profit contracts)... They satisfy various covenants and obey various regulations depending on the country where they are issued. The standard literature is Briys and de Varenne [1994, 1997a] and Grosen and Jørgensen [2000], to quote only a few. Bernard, Le Courtois and Quittard-Pinon [2005] study a particular type of participating contract, where an HJM term structure of interest rates is assumed and where the potential default of the issuing company is taken into account. In this paper, we design a new participating contract very similar to the one considered in the above-mentioned study, but where the guaranteed rate corresponds to the return of a portfolio of Government bonds. We show that this new type of contract can be valued in closed-form even when interest rates are stochastic and default of the company is taken into account. Also, this new type of contract is likely to reduce the default risk of the issuing companies, and is less sensitive to interest risk than more standard contracts.

Keywords: Participating Life Insurance Policies, Contingent Claims Valuation, Default Risk, Stochastic Interest Rates, Dubins-Schwarz theorem.

Subject and Insurance Branch Codes: IM10, IE01

Journal of Economic Literature Classification: G13, G22

* C. Bernard is PhD Student and Teaching Assistant at ISFA Graduate School of Actuarial Studies (University of Lyon 1), France.

† O. Le Courtois is Associate Professor of Finance at E.M. Lyon Graduate School of Management, France.

‡ F. Quittard-Pinon is Professor of Finance at ISFA Graduate School of Actuarial Studies (University of Lyon 1), France. Corresponding Author: quittard@univ − lyon1.fr, Address: 110, Rue Sully, 69006 Lyon, France, Phone and Fax: 33-4-78-93-96-05.

Introduction

Life insurance contracts and in particular participating contracts are at the origin of a huge literature. Because these contracts bear many covenants (surrender options, bonus options...) and are subject to potentially many risks (interest rate, default, legal, mortality risks...), the need for valuation tools and methods, to price and manage them, is an important issue. The arising of new accounting standards, with the implementation by the International Accounting Standard Board (IASB) of the International Accounting Standards (IAS) 32 and 39 and of the International Financial Reporting Standard (IFRS) 4, calls for a stronger integration of financial and actuarial methods, when fair value is becoming the key concept of corporate finance and insurance theory.

Briys and de Varenne [1994, 1997a] built a simple framework for the pricing of life insurance contracts that has been used extensively in the following literature (see for instance Grosen and Jørgensen [2002]). They value life insurance contracts in a stochastic interest rates environment and take into account the default risk of the issuing company. Nevertheless, default can only occur at the maturity of the contract, similarly as in a Merton model of the capital structure of the firm, and this is clearly a limitation to their framework. In an other branch of the literature, Briys and de Varenne [1997b] valued risky debt in a stochastic interest rates environment when the default barrier is stochastic - say proportional to a zero-coupon bond. This approach, which is an extension of the Black and Cox model where default can occur at any time, is particularly interesting from a financial viewpoint, but has not, to the best of our knowledge, been used in the insurance domain.

Since the works of Briys and de Varenne, lots of articles have appeared on participating contracts in the insurance literature. Amongst these contributions, we can quote the ones of Miltersen and Persson [2003]. They provide closed form formulae for guaranteed investment contracts with participating covenants. Bacinello [2001] values life insurance contracts with binomial trees. This method proves very useful in dealing with various actuarial features: mortality, surrender options, minimum guarantees, annual participating bonuses, periodic premia... Tanskanen and Lukkarinen [2003] study participating life insurance contracts in a Black and Scholes framework. The insured has the right to surrender at some prespecified dates, he can also change the reference portfolio. The pricing is done thanks to numerical schemes. One can also cite Ballotta [2005] who evaluates with-profit contracts written on an asset modeled by a jump diffusion process. As a conclusion, there is a huge diversity of products, models and numerical methods to tackle the challenge of valuing in fair value life-insurance contracts.

Bernard, Le Courtois and Quittard-Pinon [2005] priced some participating contracts under default risk and a stochastic term structure of interest rates. They model the firm assets by a standard geometric Brownian motion that is correlated with the driving factor of the interest rates. Their framework is the one of Heath, Jarrow and Morton [1992], and they use a Hull and White specification for the risk-free zero-coupon bonds volatility structure. To solve their problem and avoid heavy numerical schemes, they rely on the extended Fortet equations [1943] that were developed by Collin-Dufresne and Goldstein [2001] (this paper is an outgrowth of the well-known Longstaff and Schwartz [1995] approach to risky fixed and floating-rate debt). More specifically, this resumes to implementing recurrence equations. It should be noted that even though the use of extended Fortet's equations allows them to price these contracts quite quickly, this numerical scheme is not instantaneous either.

In this article, we construct a new type of contract that is close to the one valued in Bernard, Le Courtois and Quittard-Pinon [2005] but where the minimum guaranteed rate is of a different nature. We assume that, under this new contract, the life-insurance company guarantees a Government rate to the insured, in other words a stochastic interest rate: if and when the company defaults, the covenant guarantees a sum proportional to the value of a basket of Government Bonds - whose value is unknown at present by definition. We shall denote these new contracts by GP-LICs, meaning "Government rate guarantee Participating Life Insurance Contracts". In a similar fashion, the more classical contracts studied by Bernard, Le Courtois and Quittard-Pinon [2005] will be denoted by CP-LICs, where it is meant "Constant rate guarantee Participating Life Insurance Contracts".

The first contribution of this article is to show that the introduced new contracts, or GP-LICs, can be priced in closed-form for any HJM specification of the term structure of interest rates (*idem est* for any specification of the risk-free zero-coupon bond volatility). Our second contribution is a comparison of GP-LICs with CP-LICs, from which stems that the new contracts are less sensitive to interest risk and less akin to induce bankruptcy of the issuing company. Obviously, these features should be of interest not only to academics, but also to practitioners.

1 Design of a New Contract

We shall study in this article a particular type of contract (GP-LIC) that is a generalization of the participating contract (CP-LIC) studied in Bernard, Le Courtois and Quittard-Pinon [2005]. The originality of this new contract lies in its minimum guarantee which is proportional to the value of a basket of Government bonds and hence stochastic.

1.1 Contract's Payoff

We classically assume that the funds raised by the insurance company constitute its assets and are modeled by a lognormal diffusion. From the proceeds made by the company, a part is distributed as a minimum rate and an other part as a bonus or participating interest on the financial success of the firm. The originality of a GP-LIC lies in its minimum guaranteed rate. We assume that the company guarantees a rate proportional to the one of a set of Government zero-coupon bonds maturing, as a first step, at the same time T as the contract. Indeed, if the company defaults, the insured recovers the amount he would have obtained by investing initially in Government bonds, times a proportionality coefficient.

We suppose that the insured invests the initial capital L_0 in the participating policy. Leaving aside the participating bonus temporarily, the minimum guarantee is equivalent to buying $\frac{\beta L_0}{P(0,T)}$ Government zero-coupon bonds with initial value equal to $P(0,T)$ at time 0. Here β is chosen inferior to one, so as to allow the company to guarantee a minimum rate inferior to the Government rate. Indeed, at maturity, this position is worth $\frac{\beta L_0}{P(0,T)}$ contracts whose value is equal to $P(T,T) = 1$, in other words, it is worth $l_T^g = \frac{\beta L_0}{P(0,T)}$. Provided $\beta > P(0,T)$, the insured has indeed a minimum amount guaranteed at time T that is superior to his initial investment L_0. Now, if the company defaults at time $t < T$, a minimum amount $l_t^g = \frac{\beta L_0 P(t,T)}{P(0,T)}$ is guaranteed (value at time t of $\frac{\beta L_0}{P(0,T)}$ zero-coupon bonds maturing at time T) from the covenant, and of course this is, for the moment, neglecting the potential bankruptcy costs occurring at that time.

Let A be the assets of the firm, modeled as a lognormal process. The initial contribution of the insured, L_0, satisfies the following relationship: $L_0 = \alpha A_0$ where A_0 is the initial value of the assets. As concerns the initial amount of equity, it readily verifies: $E_0 = (1-\alpha)A_0$. The insured receive at maturity T of their contracts, provided the company did not bankrupt in the meanwhile:

$$\Theta_L(T) = \begin{cases} A_T & \text{if } A_T < l_T^g \\ \\ l_T^g & \text{if } l_T^g \leq A_T \leq \frac{l_T^g}{\alpha} \\ \\ l_T^g + \delta\left(\alpha A_T - l_T^g\right) & \text{if } A_T > \frac{l_T^g}{\alpha} \end{cases}$$

where, in the first situation, the company has defaulted at maturity and the assets are returned to the insured, whereas in the second situation, the company performs correctly and the insured receive the guaranteed rate defined above. Finally, in the third situation, an additional participating rate is redistributed. Remark that in the discriminating value $\frac{l_T^g}{\alpha}$, where the participating rate starts being distributed, intervenes the coefficient α, putting forward the equitable distribution of benefits between the insured and the equityholders. Indeed, initially, the insured possess the amount αA_0, or L_0. At maturity, one simply has to compare αA_T with l_T^g, to decide whether a participation rate is added up to the guaranteed rate.

To sum up, the final payoff to the insured can be written as:

$$\Theta_L(T) = l_T^g + \delta\left(\alpha A_T - l_T^g\right)^+ - \left(l_T^g - A_T\right)^+ \tag{1}$$

where here the interpretation is that we add up, to a promised sum to the insured, a bonus option corresponding to a participation in the benefits of the company, and a put option directly related to the terminal default risk of the issuing company.

A first idea would be to value these contracts by taking the risk-neutral expectation of the above payoff. This would resume essentially to computing a linear combination of standard European options whose closed-form formulae could be easily obtained. For more details on such types of calculations, we refer the reader to Briys and De Varenne [1994]. In the next paragraph, we shall extend this contract to a setting where default can happen at any time (put differently we shall move from a Merton like model to a Black and Cox like model).

1.2 A More Refined Default Model

A life-insurance company should be able to honor its commitments to the insured. In this context, it should be solvent at any time, and not only at the maturity of the issued contracts (this corresponds to assuming classically that a covenant forces the company to be solvent at any time even if it pays back the insured only at a fixed date in the future). In the case we consider, the insured has a minimum amount guaranteed l_t^g at any time t if the company defaults early. Let us recall that this is the value at time t of $\frac{\beta L_0}{P(0,T)}$ zero-coupon bonds maturing at time T. Of course, in the real world, this is a basis for the calculation of what is really going to be distributed to the insured upon default, because bankruptcy costs will have to be taken into account.

Default occurs when the level of the assets is not sufficient to reimburse the insured. Let τ be the company's early default time. It may be written as:

$$\tau = \inf \{ \; s < T \; / \; A_s < \lambda_1 \, l_s^g \; \} \tag{2}$$

where λ_1 is a proportionality coefficient whose meaning is that the managers cannot fully anticipate bankruptcy and declare it when the assets attain a level $\lambda_1 \, l_s^g$ (which is inferior to l_s^g at time s).

Now, let λ_2 be the bankruptcy costs parameter: upon default, a fraction of the assets are wasted to cover various costs such as court fees. The amount of assets remaining at default $\lambda_1 \, l_\tau^g$ has to be diminished using λ_2 to obtain the residual amount $\lambda_2 \, \lambda_1 \, l_\tau^g$ that will be redistributed to the insured. In full generality, λ_1 and λ_2 should be estimated from past records on the assets values and recovery rates of defaulting life insurance companies.

The payoff to the insured at default time $\tau < T$, assuming $\lambda_1 < 1$ and $\lambda_2 < 1$, writes:

$$\Theta_L(\tau) \; = \; \lambda_1 \, \lambda_2 \, l_\tau^g \; \mathbb{1}_{\tau < T} \tag{3}$$

The general pricing formula for our contract can then be established using the above expressions $\Theta_L(T)$ and $\Theta_L(\tau)$, given in (1) and (3). Denoting by r the risk-free interest rate process, one readily has under the risk-neutral measure Q the formula allowing to compute V_1, the value of a GP-LIC:

$$V_1(0) \; = \; \mathbb{E}_Q \left[\left(e^{-\int_0^T r_s ds} \; \Theta_L(T) \right) \; \mathbb{1}_{\tau \geq T} + \; \left(e^{-\int_0^\tau r_s ds} \; \Theta_L(\tau) \right) \; \mathbb{1}_{\tau < T} \right]$$

This last equation writes more explicitly as:

$$
\begin{aligned}
V_1(0) \;=\; \mathbb{E}_Q \Big[& e^{-\int_0^T r_s ds} \left(l_T^g + \delta \left(\alpha A_T - l_T^g \right)^+ - \left(l_T^g - A_T \right)^+ \right) \mathbb{1}_{\tau \geq T} \\
& + e^{-\int_0^\tau r_s ds} \, \lambda_1 \, \lambda_2 \, l_\tau^g \, \mathbb{1}_{\tau < T} \Big] \qquad (4)
\end{aligned}
$$

To do the valuation of this guarantee, one has to postulate some dynamics for the interest rates and assets. This is precisely what we are doing in the following subsection.

1.3 Assets and Interest Rate Dynamics

We set ourselves in a general HJM [1992] framework where we need to know for our study the forward-neutral dynamics of the assets A_t and the zero-coupon bonds $P(t,T)$. We assume that the assets follow a lognormal dynamics correlated to the interest rates, which themselves possess an exponential volatility structure σ_P (standard Hull and White specification). The interest rate model considered here is driven by a unique factor, correlated to the one of the assets, as mentioned before.

Setting $\nu > 0$ and $a > 0$, the volatility structure expresses simply as:

$$\sigma_P(t,T) = \frac{\nu}{a} \left(1 - e^{-a(T-t)} \right)$$

Under the risk-neutral measure Q, the dynamics of the assets A_t and of the zero-coupon bond $P(t,T)$ express as:

$$\frac{dA_t}{A_t} = r_t dt + \sigma dZ^{\mathrm{Q}}(t) \qquad (5)$$

and:

$$\frac{dP(t,T)}{P(t,T)} = r_t dt - \sigma_P(t,T) dZ_1^{\mathrm{Q}}(t)$$

where $Z^{\mathrm{Q}}(t)$ and $Z_1^{\mathrm{Q}}(t)$ are standard Q-Brownian motions with correlation coefficient equal to ρ.

Let us now construct a Brownian motion Z_2^{Q} independent from Z_1^{Q}, that is, such as $dZ_1^{\mathrm{Q}}.dZ_2^{\mathrm{Q}} = 0$. It is possible to split up Z^{Q} into the two following components:

$$dZ^{\mathrm{Q}}(t) = \rho dZ_1^{\mathrm{Q}}(t) + \sqrt{1-\rho^2} dZ_2^{\mathrm{Q}}(t)$$

We have therefore decorrelated the pure interest rate risk from the other sources of risk. The dynamics of the assets given in (5) can now be reexpressed as:

$$\frac{dA_t}{A_t} = r_t dt + \sigma \left(\rho dZ_1^{\mathrm{Q}}(t) + \sqrt{1-\rho^2} dZ_2^{\mathrm{Q}}(t) \right)$$

Recall that the Radon-Nikodym density allowing to build the forward-neutral measure Q_T is defined by:

$$\frac{d\mathrm{Q}_T}{dQ} = e^{-\int_0^T \sigma_P(s,T) dZ_1^{\mathrm{Q}}(s) - \frac{1}{2} \int_0^T \sigma_P^2(s,T) ds}$$

In this case, the short-term interest rate dynamics obey the following relationship:

$$dr_t = a(\theta_t - r_t)dt + \nu dZ_1^{\mathrm{Q}_T}(t)$$

where $\theta_t = \theta - \frac{\nu^2}{a^2}\left(1 - e^{-a(T-t)}\right)$ and where we have defined a new Brownian motion $Z_1^{\mathrm{Q}_T}$ satisfying under Q_T the relationship: $dZ_1^{\mathrm{Q}_T} = dZ_1^{\mathrm{Q}} + \sigma_P(t,T)dt$.

We also define $Z_2^{\mathrm{Q}_T}$ such as $Z_1^{\mathrm{Q}_T}$ and $Z_2^{\mathrm{Q}_T}$ be non-correlated Q_T-Brownian motions. The dynamics of A_t and $P(t,T)$ under Q_T finally write as:

$$\frac{dA_t}{A_t} = (r_t - \sigma\rho\sigma_P(t,T))dt + \sigma\left(\rho dZ_1^{\mathrm{Q}_T} + \sqrt{1-\rho^2}dZ_2^{\mathrm{Q}_T}\right)$$

and:

$$\frac{dP(t,T)}{P(t,T)} = (r_t + \sigma_P^2(t,T))dt - \sigma_P(t,T)dZ_1^{\mathrm{Q}_T}$$

Upon integration of these two dynamics, one obtains:

$$A_t = \frac{A_0}{P(0,t)}\ \exp\left(\int_0^t (\sigma_P(u,t) + \rho\sigma\ )dZ_1^{\mathrm{Q}_T}(u) + \int_0^t \sigma\sqrt{1-\rho^2}\ dZ_2^{\mathrm{Q}_T}(u)\right.$$
$$\left. + \int_0^t \left(-\sigma_P(u,T)(\sigma_P(u,t) + \rho\sigma) + \frac{\sigma_P^2(u,t) - \sigma^2}{2}\right) du\right)$$

and:

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left(-\int_0^t (\sigma_P(u,T) - \sigma_P(u,t))\, dZ_1^{\mathrm{Q}_T}\right.$$
$$\left. + \frac{1}{2}\int_0^t (\sigma_P(u,T) - \sigma_P(u,t))^2\, du\right)$$

Finally, note that the following dynamics will also be useful in the coming developments:

$$\frac{A_t}{P(t,T)} = \frac{A_0}{P(0,T)} \exp\left(\int_0^t (\sigma_P(u,T) + \rho\sigma)\, dZ_1^{\mathrm{Q}_T}(u) + \int_0^t \sigma\sqrt{1-\rho^2}dZ_2^{\mathrm{Q}_T}(u)\right.$$
$$\left. - \frac{1}{2}\int_0^t \left((\sigma_P(u,T) + \rho\sigma)^2 + \sigma^2(1-\rho^2)\right) du\right) \quad (6)$$

Let us now come to the exhibition of the main formulae giving the price of a GP-LIC in the setting just defined.

1.4 Main Formulae

We are now able to give a general valuation formula for our guarantee. Starting from equation (4), and moving towards the forward-neutral world, one obtains (see the Appendix for more details):

$$V_1(0) = P(0,T)\ \mathbb{E}_{\mathrm{Q}_T}\left[\left(l_T^g + \delta\,(\alpha A_T - l_T^g)^+ - (l_T^g - A_T)^+\right)\mathbb{1}_{\tau\geq T} + \lambda_1\,\lambda_2\, l_T^g\, \mathbb{1}_{\tau<T}\right] \quad (7)$$

We wish to write the above formula in a simplified form:

$$V_1(0) = P(0,T)\,[GF + BO - PO + LR] \tag{8}$$

and we define for this purpose:

$$\begin{cases} GF &= l^g_T\ (1 - E_1) \\ BO &= \alpha\delta(E_6 - E_2) - \delta l^g_T\ (E_7 - E_3) \\ PO &= l^g_T\ (E_8 - E_4) - E_9 + E_5 \\ LR &= \lambda_1\,\lambda_2\,l^g_T\ E_1 \end{cases}$$

where the fundamental contributions to the contract's value express as:

$$E_1 = \mathbb{Q}_T\,[\tau < T]$$

$$\begin{array}{l|l} E_2 = \mathbb{E}_{\mathbb{Q}_T}\Big[A_T \mathbb{1}_{A_T > \frac{l^g_T}{\alpha}} \mathbb{1}_{\tau<T}\Big] & E_6 = \mathbb{E}_{\mathbb{Q}_T}\Big[A_T \mathbb{1}_{A_T > \frac{l^g_T}{\alpha}}\Big] \\ E_3 = \mathbb{Q}_T\Big[A_T > \frac{l^g_T}{\alpha}\ ,\ \tau < T\Big] & E_7 = \mathbb{Q}_T\Big[A_T > \frac{l^g_T}{\alpha}\Big] \\ E_4 = \mathbb{Q}_T\,[A_T < l^g_T\ ,\ \tau < T] & E_8 = \mathbb{Q}_T\,[A_T < l^g_T] \\ E_5 = \mathbb{E}_{\mathbb{Q}_T}\Big[A_T \mathbb{1}_{A_T < l^g_T} \mathbb{1}_{\tau<T}\Big] & E_9 = \mathbb{E}_{\mathbb{Q}_T}\Big[A_T \mathbb{1}_{A_T < l^g_T}\Big] \end{array}$$

The goal of the next section will be to compute explicitly the nine contributions defined just above.

2 Contract Valuation

We start by detailing the mechanics allowing to compute the first subcontract term, E_1:

Computation of E_1

E_1 is the probability that bankruptcy occurs before T. Using (2), this is the probability that the process $\frac{A_u}{P(u,T)}$ crosses the barrier $\lambda_1\,\beta\,\frac{L_0}{P(0,T)}$ before T, or written more simply:

$$E_1 = \mathbb{Q}_T\left(\inf_{u\in[0,T[}\left(\frac{A_u}{P(u,T)}\right) < \lambda_1\,l^g_T\right)$$

Start noting that equation (6) writes:

$$\frac{A_u}{P(u,T)} = \frac{A_0}{P(0,T)} e^{N_u - \frac{1}{2}\xi(u)}$$

where the differential of N is defined by:

$$dN_s = (\sigma_P(s,T) + \rho\sigma)\, dZ_1^{\mathrm{Q}_T}(s) + \sigma\sqrt{1-\rho^2}\, dZ_2^{\mathrm{Q}_T}(s)$$

and the quadratic variation of N is:

$$\xi(u) \ = < N >_u = \int_0^u [(\sigma_P(s,T) + \rho\sigma)^2 + \sigma^2(1-\rho^2)]ds \tag{9}$$

The key of the computation of E_1 is the Dubins-Schwarz theorem (see for instance Karatzas and Shreve [1991]) which states that there exists a unique Q_T-Brownian motion B such that:

$$\forall u \in [0,T]\ ,\ N_u = N_0 + B_{\xi(u)}$$

Using this representation theorem, the searched probability develops according as:

$$\begin{aligned}
E_1 &= \mathrm{Q}_T\left(\inf_{u\in[0,T[}\left(\frac{A_u}{P(u,T)}\right) < \lambda_1\, l_T^g\right) \\
&= \mathrm{Q}_T\left\{\min_{u\in[0,T]}\left(\frac{A_0}{P(0,T)}e^{N_u-\frac{1}{2}\xi(u)}\right) < \lambda_1\, l_T^g\right\} \\
&= \mathrm{Q}_T\left\{\min_{u\in[0,T]}\left(e^{B_{\xi(u)}-\frac{1}{2}\xi(u)}\right) < \frac{P(0,T)\,\lambda_1\, l_T^g}{A_0}\right\} \\
&= \mathrm{Q}_T\left\{\min_{s\in[0,\xi(T)]}\left(B_s - \frac{1}{2}s\right) < \ln(\lambda_1\,\beta\alpha)\right\}
\end{aligned}$$

It appears from this formula that we only need to know the law of the minimum of an arithmetic Brownian motion to compute E_1 (it can be found for example in Jeanblanc, Yor and Chesney [2005]). One then obtains:

$$E_1 = \mathcal{N}\left(\frac{\ln(\lambda_1\alpha\beta) + \frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right) + \frac{1}{\lambda_1\alpha\beta}\,\mathcal{N}\left(\frac{\ln(\lambda_1\alpha\beta) - \frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right)$$

To simplify notations, we use the auxiliary functions η^+ and η^- defined by the following expressions:

$$\eta^+(x) = \mathcal{N}\left(\frac{\ln(x) + \frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right) \quad \text{and} \quad \eta^-(x) = \mathcal{N}\left(\frac{\ln(x) - \frac{1}{2}\xi(T)}{\sqrt{\xi(T)}}\right)$$

where $\mathcal{N}$ denotes the cumulative standard normal distribution function.

In this setting, one finally obtains for E_1:

$$E_1 = \eta^+(\lambda_1\alpha\beta) + \frac{1}{\lambda_1\alpha\beta}\,\eta^-(\lambda_1\alpha\beta)$$

We shall now see that the computation of E_2, though different from the one of E_1, follows from the same tools and principles.

Computation of E_2

Let us first recall the expression of E_2:

$$\begin{aligned} E_2 = & \ \mathbb{E}_{\mathrm{Q}_T}\left[A_T \mathbb{1}_{\{A_T > \frac{l_T^g}{\alpha}\}} \mathbb{1}_{\tau < T}\right] \\ = & \ \mathbb{E}_{\mathrm{Q}_T}\left[A_T \mathbb{1}_{\{A_T > \frac{l_T^g}{\alpha}\}} \mathbb{1}_{\inf_{u \in [0,T[}\left(\frac{A_u}{P(u,T)}\right) < \lambda_1 l_T^g}\right] \end{aligned}$$

To compute this formula, two steps are in order. First, one has to apply Girsanov's theorem to move from the probability Q_T to a new probability $\widetilde{Q}$ allowing to greatly simplify the above expression (by getting rid of A_T). Then, another application of the Dubins-Schwarz theorem under the new probability $\widetilde{Q}$ will help concluding on E_2.

Define the following Radon-Nikodym measure:

$$\frac{d\widetilde{Q}}{d\mathrm{Q}_T} = \exp\left(\int_0^T (\sigma_P(u,T) + \rho\sigma)\, dZ_1^{\mathrm{Q}_T}(u) + \int_0^T \sigma\sqrt{1-\rho^2}\, dZ_2^{\mathrm{Q}_T} - \frac{1}{2}\xi(T)\right)$$

where $\xi(T)$ is defined as in (9). Thanks to Girsanov's theorem, it is possible to construct under $\widetilde{Q}$ the two standard Brownian motions $\widetilde{Z}_1$ and $\widetilde{Z}_2$ defined by:

$$d\widetilde{Z}_1(s) = dZ_1^{\mathrm{Q}_T}(s) - \int_0^s (\sigma_P(u,T) + \rho\sigma)\, du$$

and:

$$d\widetilde{Z}_2(s) = dZ_2^{\mathrm{Q}_T}(s) - \int_0^s \sigma\sqrt{1-\rho^2}\, du$$

Let us express A_T under $\widetilde{Q}$:

$$A_T = \frac{A_0}{P(0,T)} \exp\left(\int_0^T (\sigma_P(u,T) + \rho\sigma) d\widetilde{Z}_1(u) + \int_0^T \sigma\sqrt{1-\rho^2} d\widetilde{Z}_2 + \frac{1}{2}\xi(T)\right)$$

Then the expression of E_2 becomes:

$$E_2 = \frac{A_0}{P(0,T)} \widetilde{Q}\left(A_T > \frac{l_T^g}{\alpha} \ ; \ \inf_{u \in [0,T[}\left(\frac{A_u}{P(u,T)}\right) < \lambda_1 l_T^g\right)$$

Define the martingale H as:

$$H_s \ = \ \int_0^s (\sigma_P(u,T) + \rho\sigma) d\widetilde{Z}_1(u) + \int_0^s \sigma\sqrt{1-\rho^2} d\widetilde{Z}_2$$

Note that the quadratic variation of H is equal to the one of N; therefore, we denote it by ξ. Thanks to the Dubins-Schwarz theorem, there exists a $\widetilde{Q}$-Brownian motion $\widetilde{B}$ which satisfies:

$$H_s \ = \ \widetilde{B}_{\xi(s)}$$

This allows simplifying E_2 according as:

$$E_2 = \frac{A_0}{P(0,T)} \widetilde{Q}\left[\widetilde{B}_{\xi(T)} + \frac{1}{2}\xi(T) > \ln\left(\frac{l_T^g P(0,T)}{\alpha\, A_0}\right) \ , \ \inf_{s \in [0,\xi(T)[}\left(\widetilde{B}_s + \frac{1}{2}s\right) < \ln\left(\frac{\lambda_1 l_T^g P(0,T)}{A_0}\right)\right]$$

Finally, using the joint law of an arithmetic Brownian motion and its infimum, one obtains:

$$E_2 = \frac{A_0\,\lambda_1\alpha\beta}{P(0,T)}\;\eta^+(\lambda_1^2\alpha^2\beta)$$

Applying the same method, we are able to compute the expressions of E_3, E_4 and E_5.

Final Results

As far as the five first terms are concerned, they are obtained following the above methodology, and write as:

$$\begin{cases} E_1 &= \eta^+(\lambda_1\alpha\beta) + \frac{1}{\lambda_1\alpha\beta}\;\eta^-(\lambda_1\alpha\beta) \\ E_2 &= \frac{A_0\,\lambda_1\alpha\beta}{P(0,T)}\;\eta^+(\lambda_1^2\alpha^2\beta) \\ E_3 &= \frac{1}{\lambda_1\alpha\beta}\;\eta^-(\lambda_1^2\alpha^2\beta) \\ E_4 &= \eta^+(\lambda_1\alpha\beta) + \frac{1}{\lambda_1\alpha\beta}\left(\eta^-(\lambda_1\alpha\beta) - \eta^-(\lambda_1^2\alpha\beta)\right) \\ E_5 &= \frac{A_0}{P(0,T)}\;\left[\eta^-(\lambda_1\alpha\beta) + \lambda_1\alpha\beta\left(\eta^+(\lambda_1\alpha\beta) - \eta^+(\lambda_1^2\alpha\beta)\right)\right] \end{cases} \tag{10}$$

The four last terms, where τ does not appear explicitly, express readily as simple Gaussian functions:

$$\begin{array}{c|c} E_6 = \;\Phi_1\Big(\mathrm{M}_T;\;\sqrt{\mathrm{V}_T};\;\frac{l_T^g}{\alpha}\Big) & E_7 = \mathcal{N}\left(\frac{\mathrm{M}_T - \ln\left(\frac{l_T^g}{\alpha}\right)}{\sqrt{\mathrm{V}_T}}\right) \\ E_8 = \mathcal{N}\left(\frac{\ln\left(l_T^g\right) - \mathrm{M}_T}{\sqrt{\mathrm{V}_T}}\right) & E_9 = \;\Phi_2\big(\mathrm{M}_T;\;\sqrt{\mathrm{V}_T};\;l_T^g\big) \end{array} \tag{11}$$

where A_T follows a lognormal law with moments M_T and V_T, and where Φ_1 and Φ_2 are defined by:

$$\Phi_1(m;\sigma;a) \;=\; \mathbb{E}[e^X \mathbb{1}_{e^X > a}] \;=\; \exp\left(m + \frac{\sigma^2}{2}\right)\mathcal{N}\left(\frac{m + \sigma^2 - \ln(a)}{\sigma}\right)$$

and:

$$\Phi_2(m;\sigma;a) \;=\; \mathbb{E}[e^X \mathbb{1}_{e^X < a}] \;=\; \exp\left(m + \frac{\sigma^2}{2}\right)\mathcal{N}\left(\frac{\ln(a) - m - \sigma^2}{\sigma}\right)$$

when X is a random variable distributed as $\mathcal{N}(m, \sigma^2)$.

Formulae (10) and (11), are truly closed-form formulae of the market value of our contract, and they can be computed instantaneously once all the parameters are specified. In the last section, we shall provide some interesting numerical results obtained using these formulae.

3 Numerical Analysis

Our numerical analysis is organized as follows. In a first part, we give some numerical results related to GP-LICs, and explain how to set the different parameters in order to obtain fair priced contracts. A second part is devoted to the comparison of GP-LICs and CP-LICs (whose characteristics are recalled in subsection 3.3). The only difference between these contracts is their guaranteed part. Indeed, the guaranteed interest rate of a GP-LIC is proportional to the yield of a zero-coupon bond $P(0,T)$, whilst a CP-LIC guarantees a constant interest rate r_g. We will first describe this former contract, before comparing it to GP-LICs.

3.1 Data

We give in table 1 our chosen parameter values. Some of them will change during the numerical study, and we shall precise in due time whether we take the following values or not.

A_0	α	a	ν	ρ	σ	T	P(0,T)	λ_1	λ_2
100	0.85	0.4	0.008	0.2	0.1	10	0.6703	0.6	0.4

Table 1: Data

Let us first recall the meaning of the above coefficients. A_0 refers to the initial assets value of the company, and α is a coefficient yielding the part invested by the insured (that is: $L_0 = \alpha\, A_0 = 85$). The two parameters a and ν define the volatility σ_P of the instantaneous interest rate process. The correlation coefficient between the assets process A and the instantaneous interest rate process r is ρ. The volatility σ of the assets is set to 10%, which is quite low and is due to the presence of investment grade bonds in the portfolio of the insurance company. T stands for the maturity of the contract; we suppose it is equal to 10 years. $P(0,T)$ is a Government zero-coupon bond maturing at T. Finally, λ_1 is the scale factor on the threshold triggering bankruptcy; the coefficient defining the rebate on the assets in case of prior default is given by λ_2 and is set to 40% (this could be estimated by studying recovery rates).

Note that some parameters have not been given yet: the participating bonus δ and the proportional coefficient β that determines the minimum guarantee. They will be specified when needed. Let us now present some numerical results on the fair valuation of GP-LICs and their components.

3.2 The New Contract

We first recall how the fairness of a contract can be assessed. One has set the contract parameters in such a way as to establish the equality between the initial policyholders' investment L_0 and the initial market value $V_1(0)$ of their contracts. That is, formally:

$$L_0 = V_1(0)$$

$V_1(0)$ can be computed instantaneously thanks to formulae (8), (10) and (11). Assigning a value to all the parameters except one, one obtains the remaining parameter's value by means of a simple root search algorithm. Note that the participation coefficient is extremely easy to obtain in terms of the other parameters (for this, use equation (8)):

$$\delta = \frac{\frac{L_0}{P(0,T)} - GF + PO - LR}{\alpha(E_6 - E_2) - l_T^g(E_7 - E_3)} \tag{12}$$

In such contracts, two parameters play a key role: the guaranteed rate and the participation coefficient. We first concentrate on the guaranteed rate, which is proportional to the yield of the zero-coupon bond $P(0,T)$. Indeed, our contract guarantees the amount $\beta L_0/P(0,T)$ at time T.

Now, let us define y_0 according as:

$$L_0\, e^{y_0 T} \;=\; L_0\, \frac{\beta}{P(0,T)},$$

this is the yield that can be anticipated at time zero by the insured, assuming subsequent bankruptcy will not occur. Of course, this is only an indication at time zero of the true yield of a GP-LIC; note also that in this case, the yield at time t is unknown due to the stochastic nature of the guarantee.

Keeping the values of table 1 (except for the volatility σ that is allowed to change), we compute and graph in figure 1 the fair participation coefficient δ with respect to y_0, at some given fixed values of the volatility ($\sigma = 6\%$, $\sigma = 10\%$, $\sigma = 13\%$ and $\sigma = 15\%$).

The interpretation of the two first curves ($\sigma = 6\%$ and $\sigma = 10\%$) is very standard: they are negatively sloped because - in a general context - a higher yield y_0 should be compensated by a lower participation coefficient. When the volatility of the underlying assets portfolio is higher ($\sigma = 13\%$ or $\sigma = 15\%$), one can observe different curves - where δ tends to be an increasing function of y_0. The interpretation is such: when σ is high, the insured are facing a quite important default risk; because of this and the fact that extracting more value from the assets (by distributing y_0) increases the ruin probability, policyholders will require higher participating rates δ under higher yields y_0. The horizontal dashed line gives a limit beyond which fair contracts cannot be built without seriously harming shareholders.

It is obvious that not every choice of parameters will yield acceptable fair contracts. In particular, some parameters obey regulatory constraints and cannot be fixed arbitrarily. For instance, in France participation coefficients should be higher than 85%, guaranteed interest rates (corresponding here to the yield y_0) should be less than 75% of the average Governement yield and α should necessarily be inferior to 96% (existence of minimum solvency margin of 4%).

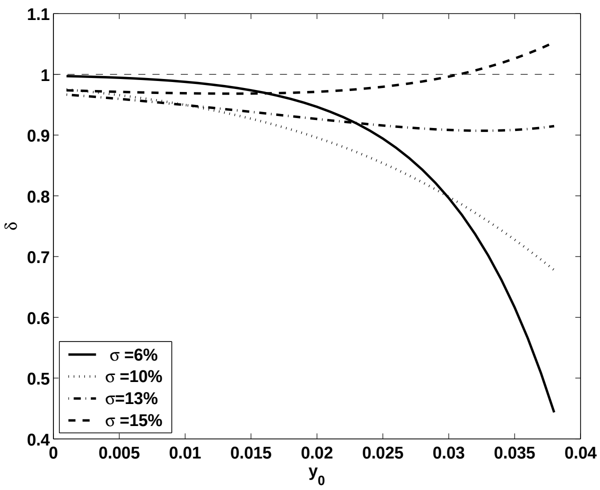

Figure 1: Participation coefficient δ with respect to y_0

Parameters should range between realistic values: the participation coefficient should preferably be less than 100% and guaranteed interest rates must of course be positive. These constraints are illustrated in figure 2 where plots of the participation coefficient δ with respect to α can be observed. In this graph, the first coordinate ranges between 80% and 100%, and y_0 is set to 0.5%, 2% and 3%.

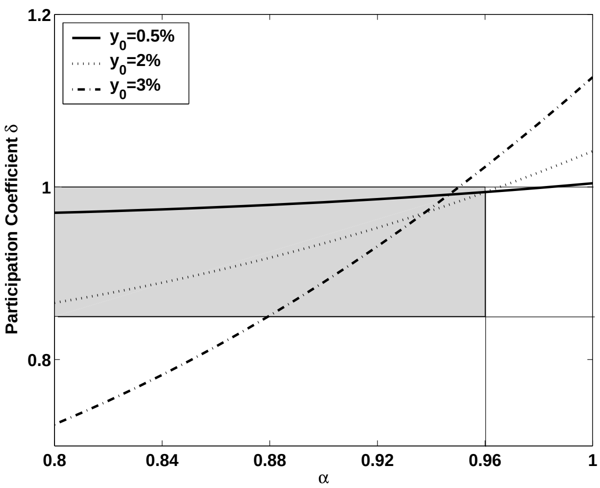

Figure 2: Participation coefficient δ with respect to α

Under the existing constraints some contracts cannot exist; for instance it is impossible to set $\alpha = 84\%$, $y_0 = 3\%$ and give a participation rate of $\delta = 80\%$. The range of possibilities is located in the grey zone (graphing the constraints $0.85 < \delta < 1$ and $\alpha < 0.96$). An important remark is that when solvency is in danger (when equity decreases or α increases), then for the contract to be fair, its associated participation coefficient should increase.

The above numerical study reveals that the contracts we introduced, or GP-LICs, display very standard features. Our next goal will be to show in what respects they defer from other existing contracts (CP-LICs), and what are the specific advantages of creating and using them.

3.3 Comparison with existing contracts

We first recall briefly the design of CP-LICs, based on the description done in Bernard *et alii* [2005]. Then, we study the dependence GP-LICs and CP-LICs on the guaranteed interest rate, the interest rate volatility and the correlation between the interest rate and the assets processes.

Description

A CP-LIC is a participating contract with minimum and constant (instantaneously compounded) interest rate guaranteed r_g, and a participation coefficient equal to δ. Do remark that in the case of CP-LICs r_g holds between 0 and T and is contractual (whereas in the case of GP-LICs, y_0 is just an equivalent yield representing the rate guaranteed from time 0, only, and up to time T).

The initial investment of the insured is L_0; at maturity T, in case of no prior default, he will receive his investment put up by the guaranteed rate, that is $L_T^g = L_0 e^{r_g T}$. At that time, he should also get the participating part of the contract, $\delta\left(\alpha A_T - l_T^g\right)^+$, provided the company performed well. Default risk is taken into account by introducing a regulatory barrier in the valuation model of the contract. The level of the assets of the company has to be above $\lambda_1 L_0 e^{r_g t}$ at any given time t. τ is the default or the first passage time of the assets process at the barrier $B_t = \lambda_1 L_0 e^{r_g t} = \lambda_1 L_t^g$. A CP-LIC therefore admits, under the risk-neutral probability Q, the valuation formula:

$$V_2(0) = \mathbb{E}_Q\left[e^{-\int_0^T r_s ds}\left(L_T^g + \delta\left(\alpha A_T - L_T^g\right)^+ - \left(L_T^g - A_T\right)^+\right)\mathbb{1}_{\tau \geq T} \right.$$
$$\left. + \quad \lambda_1\ \lambda_2\ e^{-\int_0^\tau r_s ds}\ L_\tau^g\ \mathbb{1}_{\tau < T}\right]$$

Under a stochastic interest rate environment, this formula cannot be developed in closed-form because the minimum guaranteed interest rate (and hence the default barrier) is deterministic and not proportional to risk-free zero-coupon bonds - as is the case for the new contracts introduced in this article. Instead, this formula can be developed in semi closed-form, as shown by Bernard *et alii* [2005], and based on the Collin-Dufresne and Goldstein [2001] approach. We used their methodology to price CP-LICs - upon a slight adaptation because a clear distinction between λ_1 and λ_2 had not been made in their paper.

In fact, in a simple model where interest rates are constant and not stochastic, this standard contract can be priced in closed-form as shown by Grosen and Jørgensen [2002]. We will use the notation V_3 for the value of a CP-LIC evaluated in a constant interest rate model. We recall the notation V_1 (resp. V_2) for the price of a GP-LIC (resp. a CP-LIC) evaluated under a stochastic interest rate assumption.

Recall also that the yield of a GP-LIC is proportional to the yield of a Government zero-coupon bond maturing at time T. Before comparing the two contracts, we want them to guarantee approximately the same yield. In fact, a GP-LIC's yield can be known only at time 0 (because the contractual guarantee at time t is proportional to $P(t,T)$, which is stochastic). Denoting by r_g the minimum guaranteed rate of a CP-LIC, we set β for the GP-LIC in such a way as to satisfy the following equality:

$$L_0 e^{r_g T} \;=\; L_0 \;\beta \frac{1}{P(0,T)}$$

where it is meant that both contracts should start offering the same initial yield (put differently, $r_g = y_0$ should hold).

Participation Coefficient of a Fair Contract

We choose the parameters as in table 1 and graph in figure 3 the value of the participation coefficient δ with respect to r_g (where r_g ranges between 1% and 3% and is equal to y_0) for both contracts. Recall that the participation coefficient δ admits a closed-form expression. Indeed, equation (12) gives the corresponding formula for the case of GP-LICs (a similar expression can be found in Bernard *et alii* [2005] for the case of CP-LICs).

We need to affect a value to the constant risk-free interest rate to compute V_3, that is, the value of a CP-LIC in a constant interest rate framework. We chose $r = 3.9\%$ for the pricing of V_3: this is this particular value which makes the third curve in figure 3 close enough to the two first ones.

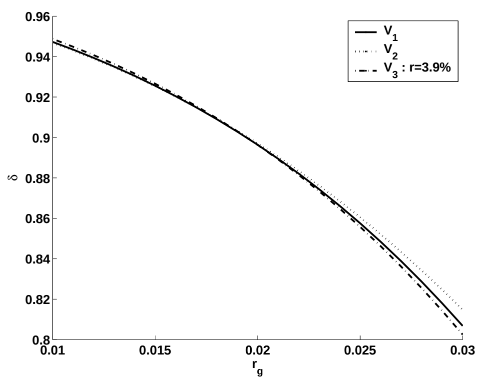

Figure 3: δ as a function of r_g

Again, graph 3 is very typical and its shape can be explained simply: to compensate for a low guaranteed rate, the insurance company has to provide a high level of participation on the assets' performance. In the remainder of this study, we shall assume $r_g = 2\%$, and keep the fair value of δ for each of the existing three situations. This means that we will assume $\delta_1 = 89.70\%$ as far as GP-LICs will be concerned, $\delta_2 = 90.25\%$ for CP-LICs in a stochastic interest rate context, and $\delta_3 = 89.63\%$ for CP-LICs in a constant interest rate context. The sensitivity analysis developed in the next paragraphs will show that GP-LICs are contracts safer than CP-LICS.

Default Probability

We denote by E_1 the default probability, and display its dependence with regards to the minimum guaranteed rate r_g in figure 4.

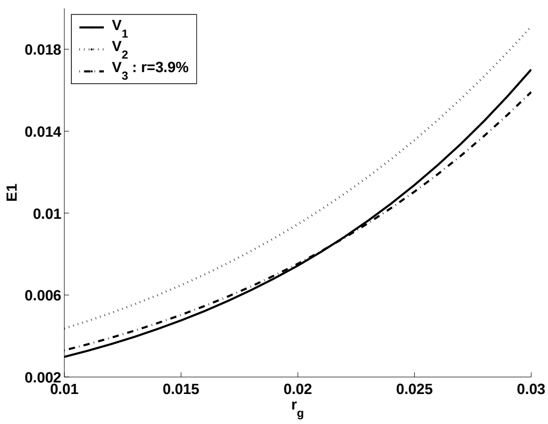

Figure 4: E_1 with respect to r_g

It is very interesting to construe how the default probability varies. First of all, and this is commonsense, it increases with r_g. Indeed, when all the other parameters are kept constant, an increase of r_g means an increase of the payout rate withdrawn from the assets, and hence a higher default probability.

Secondly, and more interestingly, for a very analogous design and similar parameters, a GP-LIC is less likely to induce bankruptcy of the issuing company than a CP-LIC. Because E_1 is smaller for a GP-LIC than for a CP-LIC, this is a clear advantage of GP-LICs that is displayed here.

It can also be noticed that E_1 is small for CP-LICs computed with V_3, in other words with a constant interest rate model. Of course, these values (that would be obtained as simplifications by actuaries, and would lead them to underestimate the risk of CP-LICs) are wrong. A constant interest rate model is not sufficient to price efficiently such contracts (for instance, and as will be shown in the following, it cannot take into account the correlation between the assets and interest rates), and this is another conclusion of our study.

Sensitivity to the Assets Volatility σ

We graph in figure 5 the contract values as a function of the underlying assets' volatility σ. The first element that can be noted is that all contracts have values that start increasing with the level of volatility and then decrease (after an optimum at about $\sigma = 10\%$) when the volatility continues increasing. Indeed, the optimum corresponds to the fair coefficients computed based on table 1. Decreasing the volatility from this point corresponds to decreasing the appeal of the product to the investors. Increasing it yields to an increase of the default probability and therefore to a decrease of the policy value.

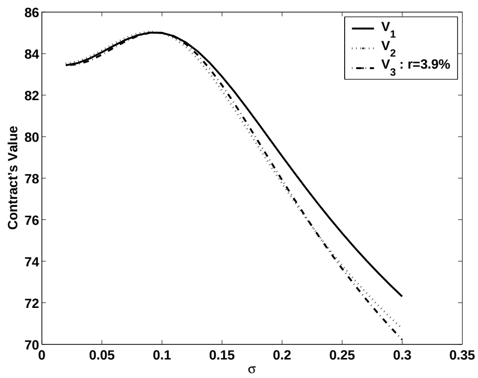

Figure 5: V with respect to σ

Comparing both contracts, it appears that V_1 always remains higher than V_2 and has a smaller tendency to decrease with respect to the volatility σ. Again, this is in advantage of GP-LICs. Less sensitive to the volatility of the assets, it should be of greater interest to the investors. This is of course a consequence of the use of a floating-rate guarantee (reducing in particular the default probability of the issuing company) which is the only distinction between the two contracts. Less likely to induce default, this type of contract is compatible with a higher level of σ, and this is a sufficient reason for it to be worth more than a CP-LIC.

Sensitivity to the Correlation

In this paragraph, we study the sensitivity of GP and CP-LICs to the correlation between the assets and interest rate processes. To start with, we plot in figure 6 the values of these participating contracts with respect to the correlation.

It can be observed from figure 6 that, again, V_1 admits a more stable behavior than V_2 - this time with respect to the correlation. GP-LICs seem to be more appealing than CP-LICs in this context. Yet, it could be noticed that CP-LICs would be privileged as a speculative instrument by some insured anticipating a sharp decrease in the correlation, because in that case the contract's value

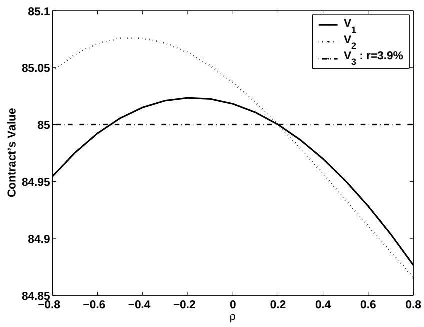

Figure 6: V w.r.t. Correlation ρ

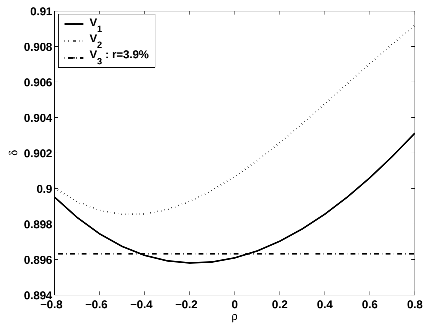

Figure 7: δ w.r.t. Correlation ρ

would increase more.

V_3, corresponding to the value of a CP-LIC computed with constant interest rates (and therefore a null correlation), is necessarily constant, and does not provide any information on the type of interest rate risk policyholders are facing. Definitely, and despite the fact that we needed to do it to arrive at this point, constant interest rate models should be avoided. To conclude on this graph, one can also remark that the same value of a contract can in many situations be attained with two very distinct values of the correlation.

In figure 7, the participation coefficient δ is plotted as a function of ρ (with $r_g = 2\%$), where ρ varies between -0.8 and 0.8. One should first note that the participation coefficient is less sensitive to the correlation for GP-LICs. Then, it should also be remarked that the fair participation coefficient is always smaller in absolute value in the case of these GP-LICs. This feature should be particularly relevant to the companies issuing or willing to issue such contracts.

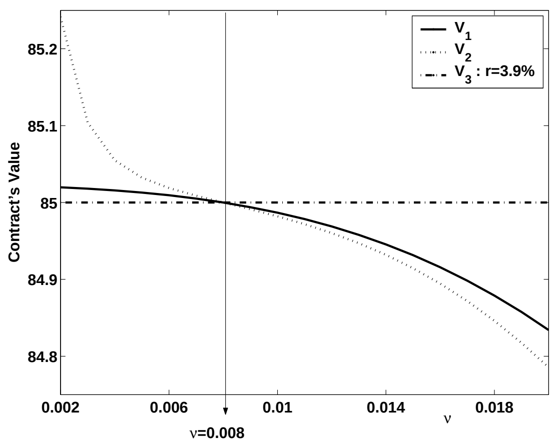

Figure 8: V w.r.t. ν

Finally, figure 8 clearly exhibits the higher sensitivity to interest rates of a CP-LIC compared to a GP-LIC. ν is here the parameter driving the size of the volatility of the interest rates, and could be interpreted as a proxy of this volatility itself. Again, our conclusion is that a GP-LIC is more akin to resist environment changes than a standard participating contract.

Conclusion

In a general framework taking into account many actual features such as stochastic interest rates and default probability, we suggest to study a new contract bearing many characteristics of usual participating life insurance contracts. This new contract is designed in such a way as to lead to an easy understanding of its behavior. The technical reason why it is so is because we obtain its valuation in a closed-form formula. Our new contract reacts to economic and financial parameters qualitatively in a similar way as ordinary participating LICs, but more efficiently. It is less risky: the bankruptcy probability of the issuing company is lower. As far as the insurer's investment is concerned, the contract value rises with the asset volatility as long as it can be perceived as not too much risky and then decreases but always a higher value than that of a standard contract. Furthermore GP-LICs react more nicely to changes in the correlation or interest rate parameters than CP-LICs, and therefore the former contracts would appear to be more interesting in terms of risk management. The building of GP-LICs seems very easy to achieve but it rests on Government bonds which are a very liquid instrument.

Appendix
Going Forward-Neutral

We explain briefly how to go from formula (4):

$$V(0) \;=\; \mathbb{E}_Q\left[e^{-\int_0^T r_s ds}\left(l_T^g + \delta\,(\alpha A_T - l_T^g)^+ - (l_T^g - A_T)^+\right)\mathbb{1}_{\tau\geq T}\right.$$
$$\left. + \; e^{-\int_0^\tau r_s ds}\;\lambda_1\,\lambda_2\,l_\tau^g\,\mathbb{1}_{\tau<T}\right]$$

to formula (7):

$$V(0) = P(0,T)\;\mathbb{E}_{\mathrm{Q}_T}\left[\left(l_T^g + \delta\,(\alpha A_T - l_T^g)^+ - (l_T^g - A_T)^+\right)\mathbb{1}_{\tau\geq T} + \;\lambda_1\,\lambda_2\,l_T^g\,\mathbb{1}_{\tau<T}\right]$$

The main difficulty here is to show that:

$$\mathbb{E}_Q\left[e^{-\int_0^\tau r_s ds}\,\lambda_1\,\lambda_2\,l_\tau^g\,\mathbb{1}_{\tau<T}\right] = P(0,T)\,\mathbb{E}_{\mathrm{Q}_T}\left[\lambda_1\,\lambda_2\,l_T^g\,\mathbb{1}_{\tau<T}\right]$$

because the passage from the risk-neutral probability to the forward-neutral one is direct in the first part of formula (4) and simply stems from the definition of these two worlds.

Indeed, one can write:

$$\mathbb{E}_Q\left[e^{-\int_0^\tau r_s ds}\,\lambda_1\,\lambda_2\,l_\tau^g\,\mathbb{1}_{\tau<T}\right] = \lambda_1\,\lambda_2\,l_T^g\,\mathbb{E}_Q\left[e^{-\int_0^\tau r_s ds}\,P(\tau,T)\,\mathbb{1}_{\tau<T}\right]$$

where we are discounting a payoff of $P(\tau,T)\,\mathbb{1}_{\tau<T}$ from τ to 0.

Taking as new numéraire $P(.\,,T)$, one can write under Q_T:

$$\mathbb{E}_Q\left[e^{-\int_0^\tau r_s ds}\,\lambda_1\,\lambda_2\,l_\tau^g\,\mathbb{1}_{\tau<T}\right] = \lambda_1\,\lambda_2\,l_T^g\,P(0,T)\,\mathbb{E}_{\mathrm{Q}_T}\left[\frac{P(\tau,T)\,\mathbb{1}_{\tau<T}}{P(\tau,T)}\right]$$

which immediately simplifies as:

$$\mathbb{E}_Q\left[e^{-\int_0^\tau r_s ds}\,\lambda_1\,\lambda_2\,l_\tau^g\,\mathbb{1}_{\tau<T}\right] = \lambda_1\,\lambda_2\,l_T^g\,P(0,T)\,\mathbb{E}_{\mathrm{Q}_T}\left[\mathbb{1}_{\tau<T}\right]$$

and then the result obtains.

References

BACINELLO, A. (2001): "Fair Pricing of Life Insurance Participating Policies With a Minimum Interest Rate Guaranteed," *Astin Bulletin*, 31(2), 275–297.

BACINELLO, A., AND S. PERSSON (2000): "Design and Pricing of Equity-linked Life Insurance under Stochastic Interest Rates," *Working Paper*.

BALLOTTA, L. (2005): "A Lévy Process-Based Framework for the Fair Valuation of Participating Life Insurance Contracts," *Special Issue of Insurance: Mathematics and Economics, to appear*.

BERNARD, C., O. LE COURTOIS, AND F. QUITTARD-PINON (2005): "Market Value of Life Insurance Contracts under Stochastic Interest Rates and Default Risk," *Insurance: Mathematics and Economics, to appear*.

BRIGO, AND MERCURIO (2001): *Interest Rate Models: Theory and Practice*. Springer Finance, Heidelberg.

BRIYS, E., AND F. DE VARENNE (1994): "Life Insurance in a Contingent Claim Framework: Pricing and Regulatory Implications," *The Geneva Papers on Risk and Insurance Theory*, 19(1), 53–72.

——— (1997a): "On The Risk of Life Insurance Liabilities: Debunking Some Common Pitfalls," *Journal of Risk and Insurance*, 64(4), 673–694.

——— (1997b): "Valuing Risky Fixed Rate Debt: An Extension," *Journal of Financial and Quantitative Analysis*, 32(2), 239–248.

CHRÉTIEN, L., AND F. QUITTARD-PINON (2000): "Pricing Shark Options in a Stochastic Interest Rate Environment," *First World Bachelier Congress, Paris*.

COLLIN-DUFRESNE, P., AND R. GOLDSTEIN (2001): "Do Credit Spreads Reflect Stationary Leverage Ratios?," *The Journal of Finance*, 56(5), 1929–1957.

FORTET, R. (1943): "Les Fonctions Aléatoires du Type de Markov Associées à Certaines Equations Linéaires aux Dérivées Partielles du Type Parabolique," *Journal Math. Pures Appl.*, 22, 177–243.

GROSEN, A., AND P. JØRGENSEN (2000): "Fair Valuation of Life Insurance Liabilities: The Impact of Interest Rate Guarantees, Surrender Options, and Bonus Policies," *Insurance: Mathematics and Economics*, 26(1), 37–57.

——— (2002): "Life Insurance Liabilities at Market Value: An Analysis of Insolvency Risk, Bonus Policy, and Regulatory Intervention Rules in a Barrier Option Framework," *Journal of Risk and Insurance*, 69(1), 63–91.

HEATH, D., R. JARROW, AND A. MORTON (1992): "Bond Pricing and the Term Structure of Interest Rates: a New Methodology for Contingent Claims Valuation," *Econometrica*, 60, 77–105.

JEANBLANC, M., M. YOR, AND M. CHESNEY (2005): *Mathematical Methods for Financial Markets*. Springer-Verlag, to appear.

JØRGENSEN, P. (2001): "Life Insurance Contracts with Embedded Options," *Journal of Risk Finance*, 3(1), 19–30.

KARATZAS, I., AND S. SHREVE (1991): *Brownian Motion and Stochastic Calculus*. Springer.

LONGSTAFF, F., AND E. SCHWARTZ (1995): "A Simple Approach to Valuing Risky Fixed and Floating Rate Debt," *The Journal of Finance*, 50(3), 789–820.

MILTERSEN, K., AND S. PERSSON (2003): "Guaranteed Investment Contracts: Distributed and Undistributed Excess Return," *Scandinavian Actuarial Journal*, 4, 257–279.

NIELSEN, J., AND K. SANDMANN (1995): "Equity-linked Life Insurance: A Model With Stochastic Interest Rates," *Insurance: Mathematics and Economics*, 16, 225–253.

TANSKANEN, A., AND J. LUKKARINEN (2003): "Fair Valuation of Path-dependent Participating Life Insurance Contracts," *Insurance: Mathematics and Economics*, 33, 595–609.