

RISK MEASURES AND EFFICIENT USE OF CAPITAL

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ABSTRACT

This note investigates risk measures constructed from acceptance sets by deriving the cost of making a given position acceptable by adding to it an optimal position containing a combination in multiple admissible assets. These risk measures are shown to be both coherent and “numéraire invariant”.

KEYWORDS

Acceptance set, Admissible assets, Currency invariance, Distribution invariance, Numéraire invariance.

INTRODUCTON

Measuring the risk of a portfolio of assets and liabilities by determining the minimum amount of supporting capital that needs to be added to the portfolio to make the risk “acceptable” has now become a standard in the financial services industry. The theoretical framework is that of coherent risk measures at first described in [ADEHb].

In that paper the basic approach consisted in defining a set of portfolios that are deemed acceptable (the acceptance set), specifying a traded asset in which the supporting capital may be invested (the “admissible” asset, e.g. risk free asset in the given currency), and determining the minimum investment in the admissible asset that needs to be added to the original portfolio to make it acceptable (the “required” capital). This minimum investment defines a risk measure which is coherent.

In general the minimum required capital will of course depend on the definition of acceptability but also on the choice of the admissible asset. The limitation to a single admissible asset does not necessarily yield the least amount of required capital. In this note we investigate allowing multiple admissible (traded/financial) assets. We look at the minimum investment in a portfolio of these admissible assets

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that needs to be added to the original portfolio to make it acceptable. It is not clear from the outset that defining a risk measure in this way will yield a coherent risk measure. However this can be shown by defining a new coherent risk measure with a single admissible asset but a different acceptance set. This new risk measure is then shown to coincide with the multiple-admissible-asset measure.

This result can be applied to a variety of problems. In particular this arises naturally when dealing with portfolios where assets and liabilities are denominated in both domestic and foreign currency. While foreign exchange risk certainly needs to be taken into consideration as a risk factor, acceptability of a portfolio should be independent of the particular currency one chooses to account in and, for the case of an unacceptable portfolio, the minimum amount of capital required should not depend on the unit of account. This is generally not satisfied if one allows investments only in one admissible asset as is illustrated by the example in the appendix.

The lesson of this example is that in the case of two currencies it is not clear which of the “risk free” assets to choose, the domestic or the foreign one. The solution to this problem relies on extending the set of admissible assets to allow supporting the original portfolio by injecting a mixture of the domestic and the foreign asset. This brings us back to the problem of enlarging the set of admissible assets. The result is moreover a “currency invariant” measure.

This note aims therefore also at defining and constructing numéraire and currency invariant risk measure (in the setting of one period of uncertainty). Along the way it provides some appreciation of distribution invariant risk measures as they are shown not to allow for currency invariance.

1. REVIEW OF ONE PERIOD COHERENT ACCEPTABILITY

Coherent one-period risk measurement at date 0 is best approached (see [ADEHa], p. 69, [ADEHb], Section 2.2) by taking the primitive objects to be

- (i) a probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$ supposed finite, for simplicity,
- (ii) a random variable on $(\Omega, \mathcal{F}, \mathbb{P}_0)$ representing an admissible asset r (chosen as “numéraire”) i.e. a traded instrument exchanging one unit of date 0 money for a (possibly random) strictly positive number r of date 1 money,
- (iii) an “acceptance set” \mathcal{A} , a set of random variables on $(\Omega, \mathcal{F}, \mathbb{P}_0)$.

The random variables related to random amounts of date 1 money are mathematically described by the ratio obtained in each state of the world: “net worth at date 1, in date 1 money, divided by value at date 1 of the admissible asset, in date 1 money”.

Remark. The set $r\mathcal{A}$ describes the acceptable net worths expressed in date 1 money. Therefore another numéraire r' would introduce an acceptance set \mathcal{B} such that $r\mathcal{A} = r'\mathcal{B}$ as we shall see in Section 3.

Given the numéraire r , an acceptance set \mathcal{A} defines the *risk measure* $\rho_{\mathcal{A},r}$, a functional on the space $L^0 = L(\Omega, \mathcal{F})$ of measurable functions on (Ω, \mathcal{F}) , as follows:

$$\rho_{\mathcal{A},r}(X) = \inf\{m \mid X + m \in \mathcal{A}\}.$$

The risk measure is the smallest amount of units of date 0 money which invested in the admissible asset, must be added at date 0 to the planned net worth X to make it acceptable.

When the acceptance set satisfies some “coherence” requirements, namely to be a convex cone \mathcal{A} of \mathbb{R}^Ω with vertex at the origin, containing the positive orthant and not intersecting the interior of the negative orthant, the associated risk measure is *coherent*, i.e. satisfies the four conditions:

- monotonicity: for each X and each Y , if $X \geq Y$ then $\rho_{\mathcal{A},r}(X) \leq \rho_{\mathcal{A},r}(Y)$,
- translation invariance: for each constant a and each X , $\rho_{\mathcal{A},r}(X + a) = \rho_{\mathcal{A},r}(X) - a$, (note the simplification vis-à-vis Def.2.2 in [ADEHb], due to the division by the “numéraire”, also called “discounting”),
- positive homogeneity: if $\lambda \geq 0$ then for each X , $\rho_{\mathcal{A},r}(\lambda X) = \lambda \rho_{\mathcal{A},r}(X)$,
- subadditivity: for each X and each Y , $\rho_{\mathcal{A},r}(X + Y) \leq \rho_{\mathcal{A},r}(X) + \rho_{\mathcal{A},r}(Y)$.

Remark. We do not require here acceptance cones to be closed. As a consequence, $X + \rho(X)$ which has zero risk measure is not necessarily in \mathcal{A} .

2. REACHING THE ACCEPTANCE SET WHEN SEVERAL TRADED ASSETS ARE ADMISSIBLE

Keeping the numéraire r and the acceptance set \mathcal{A} as in Section 1 we suppose now that there is a family $(S_i)_{1 \leq i \leq m}$ of securities traded at date 0 at market prices S_i^0 (in date 0 money) and worth S_i^1 at date 1 in date 1 money. We note \mathcal{M} the linear subspace of L^0 generated by the $\frac{1}{r}S_i$ and \mathcal{M}_0 the subspace of the $H \in \mathcal{M}$ such that the portfolio rH has initial price 0. The securities S_i are called admissible to describe the fact that some risk management/measurement is done by measuring the risk of a given future value in terms of an acceptance set larger than \mathcal{A} , because trading admissible assets amounts to “augment” the acceptance sets.

Definition 1. *Given the numéraire r , the coherent acceptance set \mathcal{A} and the space \mathcal{M} of admissible assets, the risk measure $\rho^{\mathcal{A},\mathcal{M},r}$ is defined by the relation*

$$\begin{aligned} \rho^{\mathcal{A},\mathcal{M},r}(X) &= \inf\{m \mid X + \sum_i \delta_i \frac{S_i^1}{r} \in \mathcal{A}, \text{ for some } \delta_i \text{ with } \sum_i \delta_i S_i^0 = m\} \\ &= \inf\{m \mid X + m \in \mathcal{A} + \mathcal{M}_0\}, \end{aligned}$$

Proposition 1. *We have the equality $\rho^{\mathcal{A},\mathcal{M},r} = \rho_{\mathcal{A} + \mathcal{M}_0, r}$.*

Remark. Even if we had required acceptance cones to be closed, this would not guarantee that $\mathcal{A} + \mathcal{M}_0$ would be closed

The authors thank Jean-Marc Eber for insisting on the necessity of an absence of “acceptability arbitrage”, to avoid that *any* unacceptable position becomes acceptable by a mere cost free addition of long and short positions in the tradable admissible assets:

Assumption NAA. *No acceptability arbitrage: the space \mathcal{M}_0 does not intersect the interior of \mathcal{A} .*

If assumption NAA is not satisfied there exists indeed a cost-free portfolio of the admissible assets with a (strictly) negative measure of risk. Such a portfolio would allow to change costlessly any position to an acceptable one.

Proposition 2. *Under assumption NAA the set $\mathcal{A} + \mathcal{M}_0$ is a coherent acceptance set.*

Proof. This set is clearly a convex homogeneous cone containing \mathbb{R}_{++}^Ω . If for $A \in \mathcal{A}, M_0 \in \mathcal{M}_0, A + M_0$ is also in \mathbb{R}_{--}^Ω , we would find that $-M_0$ is in the interior of \mathcal{A} , and Assumption NAA would be contradicted.

We remark that the measure $\rho^{\mathcal{A}, \mathcal{M}, r}$ is built by a minimization procedure depending on the space \mathcal{M} . It also depends a priori on the initially chosen numéraire r . We shall examine now the effect of a change to another numéraire r' . Let us remark that the relevant coherent acceptance cone \mathcal{A}' is $\frac{r}{r'}\mathcal{A}$ and that the spaces $\mathcal{M}', \mathcal{M}'_0$ deduced from r' and from the S_i satisfy $\mathcal{M}' = \frac{r}{r'}\mathcal{M}, \mathcal{M}'_0 = \frac{r}{r'}\mathcal{M}_0$.

Proposition 3. *Numéraire invariance: given the numéraires r and r' , the coherent acceptance set \mathcal{A} related to the numéraire r and the space \mathcal{M} of admissible assets, the two risk measures $\rho_{\mathcal{A}, \mathcal{M}_0, r}$ and $\rho_{\mathcal{A}', \mathcal{M}'_0, r'}$ are linked by the relation*

$$\text{for each } X, \rho_{\mathcal{A} + \mathcal{M}_0, r}(X) = \rho_{\mathcal{A}' + \mathcal{M}'_0, r'}\left(\frac{r}{r'}X\right).$$

Proof. Let us remark that for each $H \in \mathcal{M}_0, \mathcal{A} + \mathcal{M}_0 + H = \mathcal{A} + \mathcal{M}_0$ which ensures that for each $H \in \mathcal{M}_0$ each X and each $m \in \mathbb{R}, X + mH + m \in \mathcal{A} + \mathcal{M}_0$ if and only if $X + m \in \mathcal{A} + \mathcal{M}_0$. We then write

$$\begin{aligned} \rho_{\mathcal{A}' + \mathcal{M}'_0, r'}\left(\frac{r}{r'}X\right) &= \inf\left\{m \mid \frac{r}{r'}X + m \in \mathcal{A}' + \mathcal{M}'_0\right\} \\ &= \inf\left\{m \mid \frac{r}{r'}X + m \in \frac{r}{r'}\mathcal{A} + \frac{r}{r'}\mathcal{M}_0\right\} \\ &= \inf\left\{m \mid X + m + \frac{r' - r}{r}m \in \mathcal{A} + \mathcal{M}_0\right\} \\ &= \inf\{m \mid X + m \in \mathcal{A} + \mathcal{M}_0\}, \end{aligned}$$

by the remark above and the fact that $H = \frac{r' - r}{r} \in \mathcal{M}_0$ since $rH = r' - r$ has zero initial cost.

3. THE EXAMPLE OF CURRENCY RISK

Suppose we are dealing explicitly with exchange rate risk, with e_0 and e_1 the number and the \mathcal{F} -adapted random variable which describe how many units of foreign currency one unit of domestic currency buys at date 0 and date 1.

Let us specialize and adapt the definitions and results from Section 2 to handle the case of two admissible assets r and r^f in domestic *and* foreign currencies respectively. We take therefore $S_1^0 = 1, S_1^1 = r, S_2^0 = 1, S_2^1 = \frac{r^f}{e_1}$

A future net worth reported in the “domestic” admissible asset as X shall be reported by $X^f = e_1 \frac{r}{r^f} X$ in the admissible asset linked to the foreign currency.

It is easy to derive from the acceptance set \mathcal{A} devoted to future “values” expressed in the first admissible asset r and the domestic currency, the acceptance set \mathcal{A}' denoted now by \mathcal{A}^f , relative to the second admissible asset $\frac{r^f}{e_1}$ and the foreign currency. We have $\mathcal{A}^f = e_1 \frac{r}{r^f} \mathcal{A}$ (coordinate-wise multiplication).

The intuition about appropriate risk *management* in this special case, is to add, at date 0, to an unacceptable future value X , δ and ϕ units of the admissible assets

respectively, providing $r\delta$ and $r^f\phi$ units of date 1 domestic and foreign currencies respectively. This provides for the future net worth the random variable $X + \delta + \frac{1}{e_1} \frac{r^f}{r} \phi$ which is in \mathcal{A} if and only if the random variable $X^f + e_1 \frac{r}{r^f} \delta + \phi$ is in \mathcal{A}^f .

The augmented acceptance sets in either currency are $\mathcal{A} + \mathcal{M}_0$ the vector sum of \mathcal{A} and of the subspace \mathcal{M}_0 of all $\delta + \frac{1}{e_1} \frac{r^f}{r} \phi$ and $\mathcal{A}^f + \mathcal{M}_0^f$ the vector sum of \mathcal{A}^f and of the subspace $\mathcal{M}_0^f = e_1 \frac{r}{r^f} \mathcal{M}_0$ of all $e_1 \frac{r}{r^f} \delta + \phi$, where δ and ϕ satisfy $e_0\delta + \phi = 0$.

Definition 1’. For currency risk the risk measures $\rho^{\mathcal{A}}$ and $\rho^{\mathcal{A}^f}$ are defined by the relations

$$\begin{aligned} \rho^{\mathcal{A}}(X) &= \inf\{m \mid X + \delta + \frac{1}{e_1} \frac{r^f}{r} \phi \in \mathcal{A}, \text{ for some } (\delta, \phi) \text{ with } \delta + \frac{1}{e_0} \phi = m\}, \\ &= \inf\{m \mid X + m \in \mathcal{A} + \mathcal{M}_0\}, \end{aligned}$$

and

$$\begin{aligned} \rho^{\mathcal{A}^f}(Y) &= \inf\{n \mid Y + e_1 \frac{r}{r^f} \delta + \phi \in \mathcal{A}^f, \text{ for some } (\delta, \phi) \text{ with } e_0\delta + \phi = n\}, \\ &= \inf\{n \mid Y + n \in \mathcal{A}^f + \mathcal{M}_0^f\}. \end{aligned}$$

Precluding some market arbitrage would have led to assume that $\inf(e_0r^f - e_1r) < 0 < \sup(e_0r^f - e_1r)$, that is the line generated by $\mathbf{1} - \frac{e_0}{e_1} \frac{r^f}{r}$ does not intersect \mathbb{R}_{++}^Ω . but the absence of “acceptability arbitrage”, to avoid that *any* unacceptable position becomes acceptable by a mere cost free addition of a long and a short position in the two tradable admissible assets, takes the stronger form:

Assumption NAA for currency risk. No acceptability arbitrage: the space \mathcal{M}_0 , which happens to be generated by $\mathbf{1} - \frac{e_0}{e_1} \frac{r^f}{r}$, does not intersect the interior of \mathcal{A} .

The risk measures $\rho^{\mathcal{A}}$ and $\rho^{\mathcal{A}^f}$ defined above are linked to the coherent acceptance cones $\mathcal{A} + \mathcal{M}_0$ and $\mathcal{A}^f + \mathcal{M}_0^f$ in a way which makes them “admissible asset invariant”.

Definition 2. Currency invariance: given two currencies with exchange rates e_0 and e_1 at dates 0 and 1 respectively, two admissible assets r and r^f in these respective currencies, as well as coherent acceptance sets \mathcal{A} , $\mathcal{A}^f = e_1 \frac{r}{r^f} \mathcal{A}$, the coherent risk measure $\rho_{\mathcal{A},r}$ is called currency and admissible assets invariant if and only if for each $X \in \mathbb{R}^\Omega$, $e_0\rho_{\mathcal{A},r}(X) = \rho_{\mathcal{A}^f,r^f}(e_1 \frac{r}{r^f} X)$.

Proposition 3’. Under assumption NAA the coherent risk measure $\rho^{\mathcal{A}} = \rho_{\mathcal{A} + \mathcal{M}_0, r}$ is currency and admissible assets invariant.

APPENDICES

An inefficient use of capital.

The following example illustrates an inefficient use of capital in presence of a currency risk.

For one period of uncertainty with three equally weighted states of nature suppose the current exchange rate e_0 being 1 and the future rate e_1 being $(\frac{1}{2}, 1, 2)$ in the respective three states. Assume zero interest rate in either currency, i.e.

$r = r^f = 1$. For simplicity use the, admittedly extreme, risk measurement defined as “worst case scenario”, $\rho(X) = -\min(X(\omega_1), X(\omega_2), X(\omega_3))$, i.e. the acceptance cone \mathcal{A} is just the positive orthant \mathbb{R}_+^3 .

Consider a “domestic” random future net worth $X = (-2, 6, -1)$. Making it acceptable using domestic currency requires two units of currency, while a units of current foreign currency will provide acceptance as long as $X + (2a, a, \frac{a}{2})$ is in the positive orthant \mathcal{A} , that is $\frac{a}{2} - 1 \geq 0$.

It is more efficient to use a bit of foreign currency to offset the loss should the first state obtain and domestic currency to offset the loss in the third state: actually $\frac{2}{3}$ units of domestic currency together with $\frac{2}{3}$ units of foreign currency, at a total cost of $\frac{4}{3}$, suffice for acceptability. One can say that the planned business X being “global” it is not surprising that a “buffer” made of a mix of currencies is cheaper than a single currency buffer.

Distribution invariance and absence of currency invariance of tail-value-at-risk.

Given the same simple probability space as above the coherent risk measure $\rho = TailVaR_{0.5}$ (see [D]) is given by:

$$\rho(X) = -\frac{2}{3} \inf_i X(\omega_i) - \frac{1}{3} X(\omega_{\sigma(2)})$$

where σ is a permutation of $1, 2, 3$ such that $X(\omega_{\sigma(1)}) \leq X(\omega_{\sigma(2)}) \leq X(\omega_{\sigma(3)})$. It is “distribution invariant” as its acceptance set is permutation invariant in \mathbb{R}^3 . This very fact prevents $TailVaR$ being currency invariant under a non constant exchange rate e_1 .

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