CONTINUOUS-TIME DYNAMIC PROGRAMMING FOR ALM
WITH RISK AVERSE LOSS FUNCTIONS

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ABSTRACT

In this paper we extend the continuous-time dynamic programming approach for Asset/Liability Management from Boulier et al. (1995). It is an extension in the sense that we consider objective functions for pension fund management that are different from the standard quadratic loss functions. In particular, we calculate optimal policies for a loss function with Constant Relative Risk Aversion (CRRA) as well as one with Constant Absolute Risk Aversion (CARA). Taking these specific loss functions is based on the work of Merton (1990), as he uses these same functions as utility functions for the consumption/investment framework. For each loss function we solve the associated HJB-equation and obtain closed form solutions.

1. Introduction

In the area of Asset/Liability Management (ALM), much attention is paid to the formulation and optimization of stochastic programming models. An important area of application is ALM for pension funds. See Dert (1995), Cariño et al. (1993) and Boender (1997) for some examples of this approach. See also Ziemba and Mulvey (1998) for a recent overview. Much effort is put in finding efficient algorithms, creating sets of scenarios that are consistent with real world uncertainty, and overcoming computational problems and limitations. It is remarkable that comparatively little attention has been paid to the opposite of this `quantitative' approach, namely the qualitative assessment of the factors that can or should be important for pension fund management in determining the optimal policy. A first step in this direction is taken by Boulier et al. (1995), where a continuous-time dynamic programming model is formulated. It contains all of the basic elements for modeling dynamic pension fund behavior, but can be solved by means of analytical methods. See also the article of Sundaresan and Zapatero (1997), which is specifically aimed at asset allocation and retirement decisions in the case of a pension fund. In the current paper, we extend the approach taken by Boulier et al. (1995,1996), and Cairns (1997) by dropping the assumption of a quadratic loss function.

The paper is set up as follows. In Section 2, we introduce the model of Boulier et al. (1995), without specifying the exact form of the loss function. In section 3 we specify non-quadratic loss functions as in Merton (1990) and in particular study the effect of constant relative and constant absolute risk aversion (CRRA and CARA) on the resulting optimal decision rules. We end the paper in section 4 with some concluding remarks and directions for further research.

2. Model setup

As said in the introduction, we follow the approach as given by Boulier et al. (1995) to model pension fund management. It is the framework of stochastic dynamic optimization. The essence of the approach is that continuous-time relationships between state variables are defined, and then a well-defined objective function. The result is a mathematical formulation of the basic characteristics of a pension fund.

We start with the definition of the state variables. The two state variables in the model are the level of wealth (W) and the amount of pension benefits to be paid (P). To start with the latter, the aggregated amount of pension benefits follows a deterministic process with growth rate α:

$$dp_t = \alpha\, p_t\, dt. \tag{1}$$

As pension benefits are directly subtracted from the wealth of the fund, we will refer to it as pension costs in the rest of the paper.

The evolution of wealth depends on the return on investment, the amount of pension cost and the level of contribution(C) that is paid by the participants to the fund. The investment opportunities are given by one riskless asset, yielding a rate of return of r (continuously compounded) and one risky asset, i.e. stocks. The market value of the risky asset is denoted by S_t and follows a stochastic process given by

$$dS_t / S_t = (\lambda + r)dt + \sigma dB_t, \tag{2}$$

where $\lambda > 0$ is the risk premium for the risky asset and B_t denotes standard Brownian motion. The investment decision is modeled by a decision variable u, representing the fraction of assets that is invested in the risky asset. Given the evolution of the riskless and the risky asset, the evolution of wealth can be described as

$$\begin{aligned} dW_t &= u_t W_t \frac{dS_t}{S_t} + (1-u_t)W_t r dt + (C_t - P_t)dt \\ &= (rW_t + \lambda u W_t + C_t - P_t)dt + u_t \sigma W_t dB_t, \end{aligned} \tag{3}$$

where the second equality follows directly from the definition of S_t in equation (2).

Given the evolution of state variables, and the definition of the decision variables u and C, the problem the pension fund management faces is to find optimal feedback decision rules for the fraction to invest in the risky asset, u(W,P), and the level of contributions, C(W,P), solving the stochastic dynamic optimization problem:

$$J = \min_{u,C} E_0\left[\int_0^\infty e^{-\rho t} L(C)dt\right],\ C > 0, \tag{4}$$

where L(C) is a positive convex function in the amount of contributions and ρ the psychological discount factor. Although many other objective functions could be assumed for a pension fund, it captures the essence of pension fund management, under the restriction that wealth is always strictly positive, which we add to avoid a trivial solution (C≡0).

3. General solution

Given the objective function in (4) and the evolution of the state variables in equations (1) and (3), the appropriate HJB-equation associated with this problem is:

$$
\begin{aligned}
0 &= \inf_{u,C} \varphi(u, C; W, t), \\
&= \inf_{u,C} \left\{ e^{-\rho t} L(C) + J_t + (rW + \lambda u W + C - P) J_W + \alpha P J_P + \tfrac{1}{2} u^2 \sigma^2 W^2 J_{WW} \right\}
\end{aligned}
\tag{5}
$$

where the derivative of a function f with respect to variable x is denoted as f_x. See Merton (1990) for a good introduction to stochastic dynamic programming. He applies it to several instances of a general consumption/investment problem. See also Øksendal (1998) for an exposition on stochastic differential equations and the application to Stochastic Control.

Differentiating φ(u,C;W,t) with respect to C and u gives the first order conditions:

$$
\begin{aligned}
& e^{-\rho t} U_C + J_W = 0, \\
& \lambda W J_W + u \sigma^2 W^2 J_{WW},
\end{aligned}
$$

which yield the following expression for the optimal decision rules for C and u :

$$C^* = U_C^{-1}\left(-e^{\rho t} J_W\right), \tag{6}$$

$$u^* = -\frac{\lambda J_W}{J_{WW} W \sigma^2}. \tag{7}$$

Clearly, these decision rules minimize J if and only if

$$H_{\varphi} = \begin{vmatrix} \varphi_{cc} & 0 \\ 0 & \varphi_{uu} \end{vmatrix} > 0,$$

which is satisfied for $L_{cc} > 0$ and $J_{WW} > 0$. Because the optimal decision rules can be expressed as functions of W and P , when substituting (6) and (7) into the HJB-equation (5), we obtain a differential equation in the two state variables W and P . The theory from stochastic dynamic programming now tells us that a solution to the differential equation (5) is also the optimal solution to the original problem. To this extent, the next two subsections are dedicated to finding a solution to the HJB-equation, given a specific functional form for L(C).

For the solution method to be valid, it follows that we have to assume $r > \alpha$, i.e. the riskfree return exceeds the growth of pension cost. Given this assumption and writing $W_m = P / (r-\alpha)$, it is clear that in the domain $W_m \leq W$ the optimal policy is zero contribution and no risky asset in the portfolio. This means that the optimal policies we derive in the next section hold for $W_m > W$ only.

3.1 Constant Relative Risk Aversion

In this subsection, we solve the HJB-equation associated with the optimization problem formulated in section 1 for a loss function that exhibits Constant Relative Risk Aversion(CRRA). Formally, this means that the Pratt-Arrow *relative risk-aversion function* - L_{CC} / L_C is a constant. See Pratt (1964) for an exposition on the notion of risk aversion. Ingersoll (1987) gives a thorough treatment on the use of utility functions with respect to financial decisions.

We choose the loss function $L(C)=C_{\gamma} / \gamma$ as CRRA loss function, $\gamma > 1$. It is easy to see that it is convex and indeed has constant relative risk aversion. In the appendix, we solve the HJB-equation associated with the optimization problem by guessing the functional form of the optimal value function and solving for the unknown parameters. Given the expression for the optimal value function, the resulting decision rules follow from substitution in equations (6) and (7), which yields:

$$C^*(W,P)=\frac{1}{\gamma-1}\left(r-\frac{p}{\gamma}-\frac{\lambda^2}{2\sigma^2(\gamma-1)}\right)(W_m-W), \tag{8}$$

$$u^*(W,P)=\frac{\lambda}{\sigma^2}\frac{1}{\gamma-1}\frac{W_m-W}{W}. \tag{9}$$

3.1.1 Economic interpretation

First of all, we observe that the amount invested in the risky asset is linear in wealth and goes to zero as W reaches the equilibrium wealth W_m. Following Merton (1990), one could call the fraction $\lambda / (\sigma^2(\gamma-1))$ the optimal-growth fraction invested in the risky asset. It is observed that the optimal fraction in our model is equal to this optimal-growth fraction, multiplied by the *relative* distance to the equilibrium wealth W_m . For $W=W_m/2$ it is exactly equal to the optimal-growth fraction.

Secondly, looking at the condition that holds for C^* , namely that it must be nonnegative, we have that

$$\rho\le\gamma\left(r-\frac{\lambda^2}{2\sigma^2(\gamma-1)}\right). \tag{10}$$

According to Ingersoll (1987), we have here a `transversality condition'. Formally, optimal contribution is negative should ρ violate this condition, but this obviously cannot be true. Even if negative contribution (retribution) is meaningful, its loss (utility) is not defined for power utility functions.

With respect to the influence of uncertainty in the decision rules, we observer the following: If the excess return on stocks as represented by λ is higher, contributions are lower and investment in stocks is higher. On the other hand, if the variability in stock returns increases, this leads to higher contributions and a lower fraction invested in stocks.

Finally, note that one could also obtain optimal decision rules in terms of the time parameter, but we prefer to have the solution in so called *feedback* form, as we are interested in the relations between the state variables and the optimal controls, not in the time path of the controls an sich.

3.2 Constant Absolute Risk Aversion

We can solve the same optimization problem for another type of loss function, one that has constant absolute risk aversion (CARA). Let $L(C) = e_{\eta}{}^{C}$, $\eta>0$, where $-L_{CC}/L_C=\eta$ is Pratt's measure of absolute risk aversion.

Again, we have derived for this specific loss function the optimal decision rules that solve the optimization problem. See the appendix for derivation. The results are:

$$C^* = r(W_m - W) + \frac{1}{\eta r}\left(r - \rho - \frac{\lambda^2}{2\sigma^2}\right), \tag{11}$$

$$u^* = \frac{\lambda}{\sigma^2 \eta r W}. \tag{12}$$

3.2.1 Economic interpretation

The first observation we make is that except for some differences in sign, the optimal policies bear a close resemblance to those derived by Merton. Given the context of pension fund management, it is remarkable that the optimal fraction invested in the risky asset is not dependent on the equilibrium level of wealth, W_m , but only on the current wealth W . Note that the *amount* invested in the risky asset is constant, whereas for the CRRA case the amount invested was linear in wealth.

Uncertainty in state variables is displayed through the term $\lambda^2/(2\sigma^2)$, and the same connections between excess return, volatility and the optimal policies hold as for the CRRA loss function.

4 Conclusion

We have formulated the same model as Boulier et al. (1995) for dynamic pension fund management, but solved it for a much broader class of loss function, namely the CRRA and the CARA class. We observe that the results show a great deal of similarity with the result derived by Merton (1990) for a consumption/investment model.

Besides the similarity, we observe that the expression for the optimal value function in both cases is of the same functional form as the loss function. It was proven by Merton (1990) that this property holds for the consumption/investment framework, but it seems to hold in the asset/liability setting as well.

4.1 Suggestions for further research

Since the seminal paper of Kahneman and Tversky (1979) a lot of research has been done on the subject of how a realistic utility or value function should look like. In the field of prospect theory for example, it is now generally agreed upon that "...the marginal value of both gains and losses generally decreases with their magnitude...". In the context of the present pension fund modeling, this implies that a loss function in the contributions should be concave in the contributions: the loss of asking one extra unit of contribution decreases with the level of contributions. Although this might seem easy to implement, it is obvious that a concave loss function leads to corner solutions with the rest of the model unaltered, i.e. it becomes optimal to ask contributions necessary to reach equilibrium all at once. More research and perhaps a totally different setup is necessary to overcome these difficulties.

Besides the loss function, the evolution of the state variables could need some adjustment. In particular, consider the (deterministic) growth of pension cost: In the present setup, the pension cost is completely hedged. There is no risk associated with the development of the amount of benefits to be paid and the riskfree rate exceeds the pension cost growth rate. Introducing correlation with returns on investment or other exogenous uncertainty would change the outcome of the optimal policies drastically, if the problem can be solved analytically at all.

Finally, for scenario-based optimization models, and in practice it is often observed that the *fraction* of wealth invested in stocks tends to rise when the pension fund becomes wealthier. See for example Sundaresan and Zapatero (1997) where it is mentioned that investing an increasing amount in the riskfree security "...seems to contradict empirical evidence, which suggests that an increasing proportion of the overfunding is actually placed in equity." In our opinion, more research could be done to find simple models that display this behavior.

Appendix

A General power loss function

With U(C)=C_γ/γ , we substitute (6) and (7) in (5) and obtain

$$0=\frac{1-\gamma}{\gamma}\left(-J_W\right)^{\frac{\gamma}{\gamma-1}}\cdot\exp\left(\frac{-\rho t}{1-\gamma}\right)+J_t+(rW-P)J_W+\alpha PJ_P-\frac{\lambda^2}{2\sigma^2}\frac{J_W^2}{J_{WW}}.$$

(13)

To solve (13), we take as trial solution:

$$\frac{1}{\gamma a}e^{-\rho t}(-aW+bP)^{\gamma},$$

(14)

where a and b are constants that have to be determined. Substituting (14) in (13) leads to

$$0=\frac{1-\gamma}{\gamma}(-aW+bP)-\frac{\rho}{\gamma a}(-aW+bP)-(rW-P)+\frac{b}{a}\alpha P-\frac{\lambda^2}{2\sigma^2}\frac{1}{a(\gamma-1)}(-aW+bP)$$

Given that the above equality must hold for *any* W and P , we obtain:

$$a=\frac{\gamma}{\gamma-1}\left(r-\frac{\rho}{\gamma}-\frac{\lambda^2}{2\sigma^2(\gamma-1)}\right)(W_m-W),$$

$$b=\frac{a}{r-\alpha},$$

provided that a>0 and b>0 . If otherwise, then the solution method we have used is not valid. With a>0 it is clear that $J_{WW}>0$. As explained in section 3 this means that we have indeed a minimum for these parameter values. With b>0 we see that we have to demand $r>\alpha$, as we already mentioned at the end of section 3. By substitution in the expressions for the optimal decision rules, we obtain:

$$C^*=C^*(W,P)=\frac{\gamma}{\gamma-1}\left(r-\frac{\rho}{\gamma}-\frac{\lambda^2}{2\sigma^2(\gamma-1)}\right)(W_m-W),$$

$$u^*=u^*(W,P)=\frac{\lambda}{\sigma^2}\frac{1}{\gamma-1}\frac{W_m-W}{W}.$$

B Negative exponential loss function

With loss function $L(C)=e_{\eta}^{C}/\eta$ the expressions for the optimal decision rules become

$$C^* = \frac{1}{\eta}\log\left(-e^{\rho t} J_W\right)$$

$$u^* = -\frac{\lambda J_W}{J_{WW} W \sigma^2},$$

and substitution in the HJB-equation (5) yields

$$0 = -\frac{J_W}{\eta} + J_t + (rW - P)J_W + \frac{1}{\eta}\log\left(-e^{\rho t} J_W\right)J_W + \alpha P J_P - \frac{\lambda^2}{2\sigma^2}\frac{J_W^2}{J_{WW}}.$$

Taking as a trial solution for the value function:

$$J^{trial} = e^{-\rho t}\exp\{-bW + cP + d\}$$

leads to

$$0 = -\frac{b}{\eta} - \rho + (rW - P)b - \frac{b}{\eta}\log(b) - \frac{b}{\eta}(-bW + cP + d) + \alpha c P - \frac{\lambda^2}{2\sigma^2}.$$

Because the above equality must hold for all W and P we can solve for b, c and d, giving:

$$b = \eta r,$$

$$c = \frac{\eta r}{r - \alpha},$$

$$d = 1 - \frac{\rho}{r} - \log b - \frac{\lambda^2}{2r\sigma^2}.$$

Substituting the parameter values in the trial value function leads to the following expression for the optimal value function:

$$J = e^{-\rho t}\exp\left\{-\eta r(W - W_m) + 1 - \frac{\rho}{r} - \log(\eta r) - \frac{\lambda^2}{2r\sigma^2}\right\}.$$

Substitution in equations (6) and (7) gives:

$$C^* = r(W_m - W) + \frac{1}{\eta r}\left(r - \rho - \frac{\lambda^2}{2\sigma^2}\right),$$

$$u^* = \frac{\lambda}{\eta r \sigma^2 W}.$$

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