

Closed-Form Formulas for Generalized Cox, Ingersoll and Ross Models

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Abstract

This paper gives a closed-form formula for pricing default-free zero-coupon bonds in a generalization of the Cox, Ingersoll and Ross (1985) model, in which the “normal level” of the short rate is an exponentially weighted average of past short rates. The model is further extended to the case in which the initial yield curve can be prescribed exogenously, and a closed-form formula for bond prices is also found.

Keywords: Term structure of interest rates; Yield curve; Zero-coupon bonds; Malkiel models; Cox, Ingersoll and Ross model

1. Introduction

Fixed income securities comprise a major portion of assets owned by the insurance industry. Many products it sells are highly sensitive to interest rate fluctuations. For example, more than two thirds of the assets held by U.S. life insurance companies are bonds and mortgages. Certain deferred annuity products, such as fixed annuities, sold by U.S. life insurance companies are basically bonds, wrapped around with some insurance features. Hence models of the term structure of interest rates are important tools for actuaries. Actuaries need such models for pricing assets and liabilities, for designing and executing hedging strategies, and for generating interest rate scenarios used in solvency-testing exercises (Panjer, et al., 1998). Some recent surveys on term structure of interest rates models are Ang and Sherris (1997), Babbel and Merrill (1996), Back (1996), Björk (1997), Chen (1996), Cheyette (1997), Ho (1995), Marsh (1995), Rebonato (1998), Rogers (1995), and Vetzal (1994).

One of the most important term structure models in the literature is the Cox, Ingersoll and Ross (1985) model. In this model, the dynamics of the short rate, $r(t)$, are governed by the stochastic differential equation

$$dr(t) = k[x - r(t)]dt + \sigma\sqrt{r(t)} dW(t), \quad (1.1)$$

where $\{W(t)\}$ is a standard Wiener process, and k , x , and σ are positive constants. Because of the drift term $k[x - r(t)]$, the short rate process, $\{r(t)\}$, is mean reverting; the current value of the short rate process is pulled towards the long-run mean x with a speed proportional to the difference from the mean. The volatility term, $\sigma\sqrt{r}$, approaches zero as r approaches zero, ensuring that the short rate stays positive. Also, the volatility increases as the short rate increases. These are reasonable properties for a short rate process.

Cox, Ingersoll and Ross (1981, Section VII) suggested that the quantity x in (1.1) can be generalized as a state variable $x(t)$,

$$dr(t) = k[x(t) - r(t)]dt + \sigma\sqrt{r(t)} dW(t), \quad (1.2)$$

with $x(t)$ being an exponentially weighted average (with parameter $\beta > 0$) of the past short rates,

$$x(t) = \beta \int_{-\infty}^t r(s) e^{-\beta(t-s)} ds. \quad (1.3)$$

If for some t_0 , the value of $x(t_0)$ is known, then, for $t > t_0$, we have

$$x(t) = e^{-\beta(t-t_0)}x(t_0) + \beta \int_{t_0}^t r(s) e^{-\beta(t-s)} ds. \quad (1.4)$$

The differential form of (1.3) and (1.4) is

$$dx(t) = \beta[r(t) - x(t)]dt. \quad (1.5)$$

Cox, Ingersoll and Ross (1981, p. 791) pointed out that Malkiel (1966) had proposed the discrete time equivalent of (1.3) by using a moving average as “normal range” for the interest rate, as substitute for the Keynesian concept of a normal level. We call a term structure of interest rates model involving (1.5) a *Malkiel model*. We call the model prescribed by (1.2) and (1.5) the *CIR-Malkiel model*. Although the short rate itself is not a Markov process, equations (1.2) and (1.5) mean that the short rate and the exponentially weighted average are jointly modeled as a Markovian system. Note that (1.2) and (1.5) are a special case of

$$dr = \mu_1(r, x, t) dt + \sigma_1(r, x, t) dW_1, \quad (1.6)$$

$$dx = \mu_2(r, x, t) dt + \sigma_2(r, x, t) dW_2, \quad (1.7)$$

with the volatility term in (1.7), σ_2 , being identically zero.

An important feature of the Cox, Ingersoll and Ross (1985) model is that there is a closed-form formula for pricing default-free zero-coupon bonds. As Babbel and Merrill (1998, p. 14) pointed out in an article for actuaries, an important criterion for selecting an interest rate model is whether it has closed-form formulas for bond prices. “For models with such capabilities, the economies in modeling and computer intensity are simply enormous, especially for scenario analyses. ... Without this feature, a model would need to incur many tens of millions of additional simulation runs to achieve a similar level of richness and to provide for a rigorous and consistent depiction of asset/liability management.” However, Ingersoll (1987a, p. 408; 1987b, p. 881) wrote that there was no known closed-form formula for default-free zero-coupon bond prices for the CIR-Malkiel model. A main purpose of this paper is to give such a closed-form formula.

For valuation purposes, a term structure model should produce bond prices consistent with those actually observed in the marketplace. In Section 4, we show how the CIR-Malkiel model can be extended so that it is consistent with an exogenously given initial term structure of interest rates.

2. Notation

A default-free zero-coupon bond maturing at time T is a security that will pay one unit of currency at time T and nothing at any other time. For $t < T$, we denote the time- t *price* of the

bond as $P(t, T)$. Hence, $P(T, T) = 1$. The time- t *yield* (or yield to maturity) of the bond, $y(t, T)$, is determined by

$$P(t, T) = \exp[-\tau y(t, T)], \quad (2.1)$$

where $\tau = T - t$ is the remaining time to maturity. That is,

$$y(t, T) = -\frac{\log P(t, T)}{\tau}. \quad (2.2)$$

With t fixed, the graph of $y(t, T)$, with τ as the horizontal axis, is usually called the *yield curve* at time t .

The *short rate* or *instantaneous spot rate* prevailing at time t , $r(t)$, is defined by

$$r(t) = y(t, t) \equiv \lim_{T \rightarrow t^+} y(t, T). \quad (2.3)$$

For $t < T$, the time- T (*instantaneous*) *forward rate* prevailing at time t , $f(t, T)$, is defined as

$$f(t, T) = -\frac{\partial}{\partial T} \log P(t, T). \quad (2.4)$$

It follows from the last equation that

$$P(t, T) = \exp\left[-\int_t^T f(t, s) ds\right]. \quad (2.5)$$

3. A closed-form formula for bond prices in the CIR-Malkiel model

In the CIR-Malkiel model, the state variables are the short rate, $r(t)$, and the exponentially weighted average, $x(t)$. The dynamics of the two state variables are governed by (1.2) and (1.5). With the assumption that the default-free zero-coupon bond price, $P(t, T)$, is a function also of $r(t)$ and $x(t)$, we have the partial differential equation:

$$\frac{1}{2}\sigma^2 r \frac{\partial^2 P}{\partial r^2} + [k(x - r) - \lambda(r, x)] \frac{\partial P}{\partial r} + \beta(r - x) \frac{\partial P}{\partial x} + \frac{\partial P}{\partial t} - rP = 0 \quad (3.1)$$

(Cox, Ingersoll and Ross, 1981, p. 792; Ingersoll, 1987a, p. 408, (68)). Cox, Ingersoll and Ross (1981, p. 790) called $\lambda(r, x)$ a *risk compensation factor*. We now guess that a solution for the bond price equation (3.1) is of the *exponential-affine* form,

$$P(t, T) = \exp[A(\tau) + B(\tau)r + C(\tau)x], \quad t < T, \quad (3.2)$$

where $\tau = T - t$, $x = x(t)$ and $r = r(t)$; see Duffie and Kan (1996, p. 383) and Björk (1997, Section 3.4). Substituting (3.2) into (3.1) and canceling out the $P(t, T)$ terms gives

$$\frac{1}{2}\sigma^2 r B^2 + [k(x - r) - \lambda(r, x)]B + \beta(r - x)C - \frac{dA}{d\tau} - \frac{dB}{d\tau}r - \frac{dC}{d\tau}x - r = 0. \quad (3.3)$$

Motivated by equation (69) in Cox, Ingersoll and Ross (1981), we assume that $\lambda(r, x)$ is a linear function of r and x ,

$$\lambda(r, x) = \lambda_0 + \lambda_r r + \lambda_x x, \quad (3.4)$$

where λ_0 , λ_r and λ_x are constants to be specified. Then (3.3) is a linear equation of r and x . Because (3.3) is an equation not dependent on specific values of r and x , the coefficients of r and x (and hence the remaining term) must each be zero:

$$\frac{1}{2}\sigma^2 B^2 - (k + \lambda_r)B + \beta C - \frac{dB}{d\tau} - 1 = 0, \quad (3.5a)$$

$$(k - \lambda_x)B - \beta C - \frac{dC}{d\tau} = 0, \quad (3.5b)$$

$$-\lambda_0 B - \frac{dA}{d\tau} = 0.$$

$$(3.5c)$$

Duffie and Kan (1996, p. 383) called this the *matching principle*.

Because $P(T, T) = 1$, we have the boundary conditions

$$A(0) = B(0) = C(0) = 0. \quad (3.6)$$

To find a solution we impose the condition

$$\frac{1}{2}\sigma^2 B^2 + \beta C = 0, \quad (3.7)$$

which simplifies (3.5a), and also choose

$$\lambda_r = \frac{\beta}{2} - k \quad (3.8a)$$

and

$$\lambda_x = k - \frac{\sigma^2}{\beta}. \quad (3.8b)$$

Then we find that

$$B(\tau) = -\frac{2}{\beta}(1 - e^{-\beta\tau/2}) \quad (3.9)$$

satisfies (3.6) and

$$-\frac{\beta}{2}B - \frac{dB}{d\tau} - 1 = 0, \quad (3.10)$$

and that

$$C(\tau) = -\frac{\sigma^2}{2\beta} [B(\tau)]^2 \quad (3.11)$$

satisfies (3.6), (3.7) and

$$\frac{\sigma^2}{\beta} B - \beta C - \frac{dC}{d\tau} = 0. \quad (3.12)$$

It follows from (3.5c), (3.6) and (3.9) that

$$A(\tau) = -\lambda_0 \int_0^\tau B(s) ds = \frac{2\lambda_0}{\beta} [\tau + B(\tau)]. \quad (3.13)$$

Thus (3.2) becomes

$$P(t, T) = \exp\left\{\frac{2\lambda_0}{\beta} [\tau + B(\tau)] + B(\tau)r(t) - \frac{\sigma^2}{2\beta} [B(\tau)]^2 x(t)\right\}, \quad t < T, \quad (3.14)$$

where $\tau = T - t$, $B(\tau)$ is given by (3.9), and $x(t)$ is given by (1.3) or (1.4), and λ_0 is a free parameter. To check that (3.14) is a solution, we substitute it into (3.1) and apply (3.4), (3.8a) and (3.8b).

Remark Cox, Ingersoll and Ross (1981) also suggested the following model:

$$\begin{aligned} dr(t) &= \{k_1[\theta - r(t)] + k_2[x(t) - r(t)]\}dt + \sigma\sqrt{r(t)}dW(t), \\ dx(t) &= \beta[x(t) - r(t)]dt. \end{aligned} \quad (3.15)$$

To find a closed-form formula for bond prices, we would choose

$$\lambda_r = \frac{\beta}{2} - (k_1 + k_2)$$

and

$$\lambda_x = k_2 - \frac{\sigma^2}{\beta}$$

in place of (3.8a) and (3.8b). Then a formula pricing default-free zero-coupon bonds is

$$P(t, T) = \exp\left\{\frac{2(\lambda_0 - k_1\theta)}{\beta} [\tau + B(\tau)] + B(\tau)r(t) - \frac{\sigma^2}{2\beta} [B(\tau)]^2 x(t)\right\}, \quad t < T, \quad (3.16)$$

where $\tau = T - t$, $B(\tau)$ is given by (3.9), and $x(t)$ is given by (1.3) or (1.4).

4. Matching the initial yield curve

Hull and White (1990) extended the Cox, Ingersoll and Ross (1985) model so that it is consistent with the initial ($t = 0$) term structure of interest rates. That the initial yield curve or equivalently, the time-0 bond prices $\{P(0, T), T > 0\}$ can be prescribed exogenously is of much practical importance. Cheyette (1997, p. 5) wrote: "To be used for valuation, a model must be 'calibrated' to the initial spot rate curve. That is, the model structure must accommodate an exogenously determined spot rate curve, typically given by fitting to bond prices, or sometimes to futures prices and swap rates. All models in common use are of this type."

There is no closed-form formula for bond prices in Hull and White's extension of the Cox, Ingersoll and Ross model. In this section we give a closed-form formula for bond prices in an extended CIR-Malkiel model, which is defined by (1.5) together with

$$dr(t) = \{a(t) + k[x(t) - r(t)]\}dt + \sigma\sqrt{r(t)}dW(t). \quad (4.1)$$

The deterministic function $a(t)$ provides the flexibility for matching the exogenously given initial term structure of interest rates.

Generalizing (3.2), we assume that the default-free zero-coupon bond prices are of the form

$$P(t, T) = \exp[A(t, T) + B(t, T)r(t) + C(t, T)x(t)], \quad t < T, \quad (4.2)$$

with $A(T, T) = B(T, T) = C(T, T) = 0$. (There is a slight abuse of notation here. The functions A , B and C introduced in the last section are functions of a single variable $\tau = T - t$, the remaining time to maturity, while those in (4.2) are functions of two variables, time t and maturity date T .) Corresponding to (3.3), we have

$$\frac{1}{2}\sigma^2 r B^2 + [a + k(x - r) - \lambda(r, x)]B + \beta(r - x)C + \frac{\partial A}{\partial t} + \frac{\partial A}{\partial t} r + \frac{\partial C}{\partial t} x - r = 0. \quad (4.3)$$

Again, we assume

$$\lambda(r, x) = \lambda_0 + \left(\frac{\beta}{2} - k\right)r + \left(k - \frac{\sigma^2}{\beta}\right)x$$

and impose the condition (3.7). Then, with $\tau = T - t$, we have

$$B(t, T) = B(\tau) = -\frac{2}{\beta}(1 - e^{-\beta\tau/2}), \quad (4.4)$$

$$C(t, T) = C(\tau) = -\frac{\sigma^2}{2\beta}[B(\tau)]^2, \quad (4.5)$$

and

$$\begin{aligned}
A(t, T) &= \int_t^T [a(s) - \lambda_0]B(T - s) ds \\
&= \int_t^T a(s)B(T - s) ds + \frac{2\lambda_0}{\beta} [T + B(\tau)]. \tag{4.6}
\end{aligned}$$

Now we show how to choose the function $a(t)$ to replicate the initial ($t = 0$) term structure of interest rates. It follows from (4.2) that

$$A(0, T) = \log P(0, T) - B(0, T)r(0) - C(0, T)x(0).$$

Define

$$g(T) = \log P(0, T) - \frac{2\lambda_0}{\beta} [T + B(T)] - B(T)r(0) - C(T)x(0). \tag{4.7}$$

Then it follows from (4.6) that

$$\int_0^T a(s)B(T - s) ds = g(T). \tag{4.8}$$

Because $B(0) = 0$, differentiating (4.8) with respect to T yields

$$\int_0^T a(s)B'(T - s) ds = g'(T). \tag{4.9}$$

Noting that the function B satisfies (3.10), we multiply (4.8) with $\beta/2$ and add it to (4.9) to obtain

$$\int_0^T a(s) ds = -\left[\frac{\beta}{2}g(T) + g'(T)\right]. \tag{4.10}$$

Differentiating (4.10) we have

$$a(T) = -\left[\frac{\beta}{2}g'(T) + g''(T)\right]. \tag{4.11}$$

Differentiating (3.10) yields

$$\frac{\beta}{2}B'(T) + B''(T) = 0. \tag{4.12}$$

It follows from (2.4) and (4.12) that (4.11) becomes

$$\begin{aligned}
a(T) &= \frac{\beta}{2}f(0, T) + \frac{\partial}{\partial T}f(0, T) + \frac{2\lambda_0}{\beta} + \left[\frac{\beta}{2}C'(T) + C''(T)\right]x(0) \\
&= \frac{\beta}{2}f(0, T) + \frac{\partial}{\partial T}f(0, T) + \frac{2\lambda_0}{\beta} - \frac{\sigma^2}{\beta}e^{-\beta T}x(0). \tag{4.13}
\end{aligned}$$

To evaluate $A(t, T)$ according to (4.6), we evaluate the integral on the right-hand side of (4.6) using (4.13). Integrating by parts and applying (3.6) yields

$$\int_t^T \left[\frac{\partial}{\partial s}f(0, s)\right]B(T - s)ds = -f(0, t)B(T - t) - \int_t^T f(0, s) \frac{\partial}{\partial s}B(T - s) ds. \tag{4.14}$$

Hence it follows from (4.14) and (3.10) that

$$\begin{aligned}
 & \frac{\beta}{2} \int_t^T f(0, s)B(T-s) ds + \int_t^T \left[\frac{\partial}{\partial s} f(0, s) \right] B(T-s) ds \\
 &= -f(0, t)B(T-t) - \int_t^T f(0, s) ds \\
 &= -f(0, t)B(T-t) + \log[P(0, T)/P(0, t)].
 \end{aligned} \tag{4.15}$$

Similarly, we get

$$\begin{aligned}
 \frac{\beta}{2} \int_t^T C'(s)B(T-s) ds + \int_t^T C''(s)B(T-s) ds &= -C'(t)B(T-t) - C(T) + C(t) \\
 &= -\left[\frac{\sigma^2}{2} B(t)e^{-\beta v^2} \right] B(T-t) - C(T) + C(t)
 \end{aligned}$$

by (3.11) and (3.9). Hence (4.2) becomes

$$\begin{aligned}
 A(t, T) &= \log[P(0, T)/P(0, t)] - f(0, t)B(T-t) + \frac{2\lambda_0}{\beta} \left(1 - \frac{2}{\beta}\right) [T-t + B(T-t)] \\
 &\quad - \left\{ C(T) - C(t) + \frac{\sigma^2}{2} B(t)B(T-t)e^{-\beta v^2} \right\} x(0).
 \end{aligned} \tag{4.16}$$

We now have an extended CIR-Malkiel model in which the dynamic movements of the two state variables are determined by (1.5) and

$$dr(t) = \left\{ \frac{\beta}{2} f(0, t) + \frac{\partial}{\partial t} f(0, t) + \frac{2\lambda_0}{\beta} - \frac{\sigma^2}{\beta} e^{-\beta T} x(0) + k[x(t) - r(t)] \right\} dt + \sigma \sqrt{r(t)} dW(t),$$

where $f(0, t)$, $t \geq 0$, are the initial forward rates based on observed bond prices. Thus we have the bond price formula

$$\begin{aligned}
 P(t, T) &= \frac{P(0, T)}{P(0, t)} \exp(B(T-t)[r(t) - f(0, t)] + \frac{2\lambda_0}{\beta} \left(1 - \frac{2}{\beta}\right) [T-t + B(T-t)] + C(T-t)x(t) \\
 &\quad - \left\{ C(T) - C(t) + \frac{\sigma^2}{2} B(t)B(T-t)e^{-\beta v^2} \right\} x(0)), \quad 0 \leq t < T.
 \end{aligned} \tag{4.17}$$

Remarks (i) Cox, Ingersoll and Ross (1985, p. 395) also discussed how to adjust their model so that it can match an exogenously given initial yield curve. They would generalize (1.1) as (1.2), but without (1.3). The function $x(t)$ would be such that the initial yield curve is matched.

(ii) Dybvig (1997) had proposed a method to adjust a term structure model to match an exogenously given initial yield curve or equivalently, time-0 bond prices $\{P(0, T), T > 0\}$. For $0 \leq t \leq T$, let $P^*(t, T)$ denote the bond price given by a model before adjustment. Then the

adjusted bond prices are

$$P(t, T) = \frac{P(0, T) P^*(0, t)}{P(0, t) P^*(0, T)} P^*(t, T), \quad 0 \leq t \leq T.$$

(iii) Hull and White (1990) also discussed extending the Cox, Ingersoll and Ross (1985) model to match the future volatilities of the short rate. Thus we would change the positive constant σ in (4.1) to a positive deterministic function $\sigma(t)$. To get a closed-form formula for bond prices, we would assume the risk compensation factor to be of the form

$$\lambda(r, x, t) = \lambda_0(t) + \lambda_r(t)r + \lambda_x(t)x,$$

with

$$\lambda_r(t) = \frac{\sigma'(t)}{\sigma(t)} + \frac{\beta}{2} - k$$

and

$$\lambda_x(t) = k - \frac{[\sigma(t)]^2}{\beta}.$$

(iv) Many practitioners would call an interest rate evolution model “arbitrage-free” if it can reproduce an exogenously given initial yield curve, irrespective of its other properties; this usage of the term “arbitrage-free” can lead to confusion.

5. Conclusion

We have presented a closed-form formula for pricing default-free zero-coupon bonds in the CIR-Malkiel model for a particular family of risk compensation factors, with λ_0 being the parameter. The model was further extended so that it can match an exogenously given initial yield curve, and a closed-form formula for bond prices was also found. Such closed-form formulas provide an expeditious method for generating interest rate scenarios or yield curve paths for cash flow testing and dynamic solvency analysis.

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