

Quantile Hedging for Defaultable Securities in an Incomplete Market

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Abstract: In this paper, we aim at

1. giving formulas of prices and replicating-strategies of defaultable securities (e.g., bonds, swaps, derivatives) in incomplete market, and
2. giving “solvable” examples of quantile hedging strategies in incomplete market.

Considering an incomplete market that consists of tradable assets and an unhedgeable defaultable security, whose non-predictable default time has stochastic intensity correlated with the tradable assets-price-processes, we treat the problem of pricing and hedging of the defaultable security on it. We employ the quantile hedging strategy (cf., [F-L]) to replicate “the cumulative dividend process” of the defaultable security by an admissible strategy between the tradable assets. The strategy that maximize the success probability of hedge under the given initial capital and the strategy that minimize the initial capital under the given success probability of hedge are calculated explicitly.

Keywords: quantile-hedging, defaultable security, incomplete market, Neyman-Pearson’s lemma

1 Introduction

One of the major approach to pricing defaultable securities, the so-called “reduced-form approach” (or “intensity-model approach”) regards the default time as “unpredictable” (i.e., totally inaccessible) stopping time (e.g., [D-S,D-S-S,J-T,L1,L2]); so, a new-introduced default event and related securities are unhedgeable by their definitions. This feature is also observed transparently by a model of Duffie and Lando in [D-L]: the reduced-form model appears *naturally* under the setting with restricted information of the firm-value process, even if one starts with another major so-called “structural approach”, originated from Merton’s work [M]. In both situations, equivalent martingale measures are not unique at least in theoretical senses. In the referred papers above, and the all existing studies about reduced-form approach as we know, an equivalent martingale measure is given a priori and fixed, and arbitrage-free pricing and hedging formulas of defaultable securities (e.g., bonds, swaps, and derivatives) are derived under the measure.

In this paper, we will start with an incomplete market setting (not fixing equivalent martingale measure) and price or replicate (new-introduced) defaultable securities. Especially, we will employ the quantile hedging strategy for the replication, which has recently introduced by Föllmer and Leukert in [F-L] in place of perfect or super replication: we will seek the strategy that

1. maximize the probability of success of hedge under a given initial capital, or

2. minimize the initial capital under a given lower bound of success probability of hedge.

These might not be so unnatural interpretations of “hedge” in real-world since they can be regarded as dynamic versions of the VaR (i.e., Value at Risk), the globally standard method for the measurement of market risks, although they also have some drawbacks as suggested in [F-L]. In Corollary 2, as a typical example, we give a simple Jarrow-Turnbull-type defaultable-bond model with deterministic hazard-rate process (cf., [J-T]) and evaluate “default”-yield-spreads with respect to the yield of default-free-bonds. If one gives the lower bound of the probability of successful hedge, say $1 - \epsilon$ ($0 < \epsilon < 1$), for example, the spreads are determined by solving the equations contains ϵ .

To obtain the explicit optimal solutions, the Neyman-Pearson’s fundamental lemma in hypothesis-testing has been effectively utilized (at least in complete market cases) in [F-L], while it might not be so effective in general incomplete market cases. Fortunately, in our defaultable security models, since the equivalent martingale measure that realizes “the worst scenario for hedging” can be characterized explicitly (cf., Lemma 4, and the proof of Lemma 5 in Section 3), we can also obtain the explicit solutions via the Neyman-Pearson’s lemma (by solving a statistical-test-type problem against a *simple* alternative iterately). Financial theoretically, our defaultable security model is an unsatisfactory deformed one as stated in Assumption 2 in the next section. We will restrict the behavior of the security-holder *after the default*, which enables us to concentrate to hedge the “payoff” at the *terminal-date* T :

$$H_T := \begin{cases} d_T, & \text{if default occurs before } T, \\ D, & \text{if default does not occur before } T \end{cases}$$

of the security, and as a result, the problems are simplified and the explicit solutions for this “European-type” defaultable security can be obtained. More proper model(or problem) is may be the one stated in the remark after Problem 1-2, for example, though it remains unsolved.

In the next section, we will state our setup and our main results, in Section 3, we prove them, and in Appendix, we present some numerical examples with a simplest defaultable bond model.

2 Setup and Results

For a fixed constant $T(> 0)$, let us prepare a complete probability space, (Ω, \mathcal{F}, P) , a d -dimensional Brownian motion on it, $w := (w_t)_{t \in [0, T]}$, the Brownian filtration, $(\mathcal{G}_t)_{t \in [0, T]}$, and a random variable e that is independent of \mathcal{G}_T and exponentially distributed (with intensity 1).

Now, consider a financial market on a time interval $[0, T]$ consists of the following elements:

1. the $(d + 1)$ -asset-price-processes:

$$p := (p_t)_{t \in [0, T]}, q^1 := (q_t^1)_{t \in [0, T]}, \dots, q^d := (q_t^d)_{t \in [0, T]}$$

that are \mathcal{G}_t -adapted processes, in particular, p is the price process of a default-free bond maturing at T , i.e., it holds that $p_t > 0$ for all $t \in [0, T]$ and $p_T = 1$ P -a.e.,

2. a defaultable security, expressed as the triplet: (τ, d, D) (cf., [D-S-S]), i.e.,

(a) the default time, τ , defined by the formula:

$$\tau := \inf \left\{ t > 0; e \leq \int_0^t \lambda_u du \right\}$$

with a \mathcal{G}_t -adapted process, $\lambda := (\lambda_t)_{t \in [0, T]}$, which satisfies $\lambda_t \geq 0$ and

$$\Pr \left(\int_0^t \lambda_u du < \infty \right) = 1,$$

P -a.e. for all $t \in [0, T]$,

- (b) the payoff upon default, d_τ , which is determined by the default time τ above and a nonnegative \mathcal{G}_t -predictable process $d := (d_t)_{t \in [0, T]}$,
- (c) the payoff at the terminal date T , say D , which is nonnegative, \mathcal{G}_T -measurable and provided if there has been no default.

Let us denote the default indicator function by

$$N_t := 1_{\{\tau \leq t\}} \quad (t \in [0, T]),$$

set the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ by

$$\mathcal{F}_t := \mathcal{G}_t \vee \sigma\text{-}\{N_s; s \in [0, t]\},$$

and interpret it as the whole information on the market (along the time-evolution). This is a way of introducing *reduced-form* defaultable security model, which follows [L1-2], especially. More generally, [D-S-S] and [K] are referred for example. For simplicity, we assume $\mathcal{F}_0 = \{\emptyset, \Omega\}$. By definition above, it is easy to see that the relation

$$E[1 - N_t \mid \mathcal{G}_t] = \Lambda_t := \exp\left\{-\int_0^t \lambda_u du\right\}$$

holds for $t \in [0, T]$ and that the process: $(M_t)_{t \in [0, T]}$, where

$$M_t := N_t - \int_0^t (1 - N_u)\lambda_u du$$

is an \mathcal{F}_t -martingale obtained from the Doob-Meyer decomposition of the submartingale $(N_t)_{t \in [0, T]}$. Moreover, let us recall the following, which shall be used in the proof of our results:

Lemma 1 (Corollary 3.8 in [K], or Proposition 3.1 in [L2]) For any \mathcal{G}_T -measurable and $L^1(P)$ -random variable F , we have

$$E[F(1 - N_T) \mid \mathcal{F}_t] = (1 - N_t)E[\Lambda_T \Lambda_t^{-1} F \mid \mathcal{G}_t] \quad \text{for any } t \in [0, T].$$

Throughout this paper, we assume the following:

Assumption 1 The normalized assets-prices-process:

$$X = (X^1, \dots, X^d)' := (q^1/p, \dots, q^d/p)',$$

with a numéraire p satisfies the following stochastic differential equation:

$$dX_t^i = X_t^i \left[\sum_{j=1}^d \sigma_t^{ij} (dw_t^j + \gamma_t^j dt) \right] \quad (i = 1, \dots, d, t \in [0, T]).$$

Here, a $d \times d$ -matrix-valued $\sigma = (\sigma_t^{ij})_{1 \leq i, j \leq d}$ and an \mathbb{R}^d -valued $\gamma = (\gamma^1, \dots, \gamma^d)'$ are \mathcal{G}_t -adapted and satisfy the conditions:

1. $c|x|^2 \leq (\sigma_t(w)x, x) \leq C|x|^2$, $P \times dt$ -a.e., for all $x \in \mathbb{R}^d$ and for some $0 < c \leq C$,
2. the space of the probability measures on (Ω, \mathcal{F}) :

$$\mathcal{P} := \{Q : \text{equivalent to } P, \text{ and } X \text{ is a martingale under } Q\}$$

contains \hat{P} , given by the formula:

$$\left. \frac{d\hat{P}}{dP} \right|_{\mathcal{F}_t} = \mathcal{E}_t \left(-\int \gamma dw \right) =: Z_t \quad \text{for all } t \in [0, T].$$

The cumulative dividend process $H := (H_t)_{t \in [0, \tau \wedge T]}$ of the defaultable security (τ, d, D) is defined by

$$\begin{aligned} H_t &:= d_\tau N_t + D(1 - N_T)1_{\{t \geq T\}} \\ &= \int_0^{\tau \wedge t} d_u dN_u + D(1 - N_T)1_{\{t \geq T\}}, \end{aligned}$$

as in [D-S-S]; we will deform the definition:

Assumption 2 *The process d/p is a \hat{P} -martingale, which means that a holder of the security receives some tradable (and priced arbitrage-free) asset in the case of default. After the default $t > \tau$, we will assume that the holder keeps the tradable d , so, we will interpret the value of the cumulative dividend $H_t := d_t$; we will extend the cumulative dividend process H on $[0, \tau \wedge T]$ to $[0, T]$ by redefining*

$$\begin{aligned} H_t &:= d_t N_t + D(1 - N_T)1_{\{t \geq T\}} \\ &= \begin{cases} 0 & \text{on } \{0 \leq t < (\tau \wedge T)\}, \\ d_t & \text{on } \{\tau \leq t \leq T\}, \\ D & \text{on } \{t = T < \tau\}. \end{cases} \end{aligned}$$

Assumption 3 *One of the following is satisfied:*

$$\begin{aligned} (A) \quad & D > d_T \geq 0 \quad \hat{P}\text{-a.e.}, \\ (B) \quad & 0 \leq D \leq d_T \quad \hat{P}\text{-a.e.} \end{aligned}$$

Remark: Assumption 3 can be removed. It is just for the simplicity of the presentation of our results, and it is satisfied in typical examples: e.g.,

- a defaultable (zero-coupon) bond model: $D = 1 > d_T \geq 0$, P -a.e.. This can be interpreted a generalization of defaultable bond model by Jarrow and Turnbull in [J-T]. Upon default, the bond-holder receives $\delta_\tau p_\tau$, where $\delta_\tau := \hat{E}[d_T | \mathcal{G}_\tau]$ is called the recovery-rate upon default.
- a default-swap model: e.g., an insurance on the defaultable-bond above, i.e., the holder is insured the default-loss: $d_t := (1 - \delta_\tau)p_\tau$, and D is set to 0.

Now, consider the situation that a hedger seeks to recover the default-loss of the defaultable security by a self-financing strategy between the assets p, q^1, \dots, q^d , or that a writer of the security who wants to decide the price of this defaultable security. By a standard argument, the value process $(V_t)_{t \in [0, T]}$ of the self-financing hedging portfolio is written as

$$V_t = p_t \left(\frac{V_0}{p_0} + \int_0^t \xi_u dX_u \right),$$

where $V_0 \in \mathbb{R}_+$ is the initial cost and the \mathcal{G}_t -predictable (and X -integrable) process $\xi := (\xi_t)_{t \in [0, T]}$ represents the trading-process of the assets. If $V_t \geq 0$, P -a.e. for all $t \in [0, T]$, then, the strategy is called *admissible* in this paper. Obviously, the hedger cannot replicate perfectly the cumulative dividend process $H := (H_t)_{t \in [0, T]}$ of the defaultable security by the admissible strategy between p, q^1, \dots, q^d , i.e., our market is incomplete. We can observe

$$(1) \quad \bar{H}_t := \operatorname{esssup}_{P^* \in \mathcal{P}} E^* [H_T | \mathcal{F}_t] = \frac{d_t}{p_t} + (1 - N_t) \hat{E} \left[(D - d_T)^+ \mid \mathcal{G}_t \right]$$

(cf., Lemma 6), it provides us the trivial super hedging strategy of H such that:

- starting with the initial cost $\bar{H}_0 = d_0/p_0 + \hat{E} \left[(D - d_T)^+ \right] = \hat{E} \left[\max(d_T, D) \right]$ and choose the trading process $(\xi_t)_{t \in [0, T]}$ of X , such that

$$\frac{V_t}{p_t} = \bar{H}_0 + \int_0^t \xi_u dX_u := \hat{E} \left[\max(d_T, D) \mid \mathcal{G}_t \right]$$

then, the hedger shall be in the safe-side:

$$V_T \geq H_T,$$

at the terminal-date T with probability 1.

Instead of the trivial strategy above, we will employ the quantile hedging strategy that has been proposed by Föllmer and Leukert in [F-L] as more “suitable” strategy and price for the defaultable security; we will seek the following:

Problem 1 (*maximizing the probability of success*) Fix $\tilde{V}_0 \leq \tilde{H}_0$. Among admissible strategies, solve the following optimization-problem:

$$(2) \quad \max \Pr(\{V_T \geq H_T\}) \quad \text{subject to} \quad V_0 \leq \tilde{V}_0,$$

Problem 2 (*minimizing the cost for a given probability of success*) Fix $0 < \epsilon < 1$. Among admissible strategies, solve the following optimization-problem:

$$(3) \quad \min V_0 \quad \text{subject to} \quad \Pr(\{V_T \geq H_T\}) \geq 1 - \epsilon,$$

Remark: It might be more natural to consider the probability at the default time:

$$\Pr(\{V_{\tau \wedge T} \geq H_{\tau \wedge T}\})$$

in place of the probability at the terminal:

$$\Pr(\{V_T \geq H_T\})$$

in the expression (2) and (3) since the defaultable securities are only defined on the time interval $[0, \tau \wedge T]$; for example, in Problem 1, the inequality:

$$\max \Pr(\{V_{\tau \wedge T} \geq H_{\tau \wedge T}\}) \leq \max \Pr(\{V_T \geq H_T\}),$$

is always satisfied, where the maximization is considered over all admissible strategies with the initial cost $V_0 \leq \tilde{V}_0 (\leq \tilde{H}_0)$. Our deformation simplifies our quantile hedging problems, we only have to see the “two states”: N_T and $1 - N_T$, i.e., at the terminal T , if the default occurs or not.

Our results are stated as follows:

Theorem 1 (A) *Let (A) in Assumption 3 hold. For a nonnegative constant k , denote*

$$\begin{aligned} A_1(k) &:= \{1 - \Lambda_T > k d_T Z_T, \Lambda_T \leq k(D - d_T) Z_T\}, \\ A_2(k) &:= \{1 > k D Z_T, \Lambda_T > k(D - d_T) Z_T\}, \end{aligned}$$

and assume that there exists $k^* = k^*(\tilde{V}_0)$ satisfying

$$(4) \quad \hat{E}[1_{A_1^*} d_T + 1_{A_2^*} D] = \tilde{V}_0 / p_0,$$

where we denote by $A_1^* := A_1(k^*)$, $A_2^* := A_2(k^*)$. The super replicating strategy of “the modified claim”:

$$(5) \quad \tilde{H}_T := 1_{A_1^* \cup A_2^*} d_T N_T + 1_{A_2^*} D (1 - N_T)$$

is a solution of Problem 1. We have

$$\begin{aligned} \operatorname{esssup}_{P^* \in \mathcal{P}} E^* \left[\tilde{H}_T \mid \mathcal{F}_t \right] &= \hat{E}[1_{A_1^* \cup A_2^*} d_T \mid \mathcal{G}_t] + (1 - N_t) \hat{E} \left[(1_{A_2^*} D - 1_{A_1^* \cup A_2^*} d_T)^+ \mid \mathcal{G}_t \right] \\ &\leq \hat{E}[1_{A_1^* \cup A_2^*} d_T \mid \mathcal{G}_t] + \hat{E} \left[(1_{A_2^*} D - 1_{A_1^* \cup A_2^*} d_T)^+ \mid \mathcal{G}_t \right] \\ &= \hat{E}[\max(1_{A_1^* \cup A_2^*} d_T, 1_{A_2^*} D) \mid \mathcal{G}_t] \\ &= \hat{E}[1_{A_1^*} d_T + 1_{A_2^*} D \mid \mathcal{G}_t], \end{aligned}$$

and we can construct an optimal strategy (\tilde{V}_0, ξ^*) by defining

$$\hat{E} [1_{A_1^*} d_T + 1_{A_2^*} D \mid \mathcal{G}_t] = \frac{\tilde{V}_0}{p_0} + \int_0^t \xi_u^* dX_u \quad \text{for } t \in [0, T].$$

(B) Let (B) in Assumption 3 hold. For a nonnegative constant k , denote

$$\begin{aligned} B_1(k) &:= \{1 > kd_T Z_T, 1 - \Lambda_T > k(d_T - D)Z_T\}, \\ B_2(k) &:= \{\Lambda_T > kDZ_T, 1 - \Lambda_T \leq k(d_T - D)Z_T\}, \end{aligned}$$

and assume that there exists $k^* = k^*(\tilde{V}_0)$ satisfying

$$(6) \quad \hat{E} [1_{B_1^*} d_T + 1_{B_2^*} D] = \tilde{V}_0/p_0,$$

where we denote by $B_1^* := B_1(k^*)$, $B_2^* := B_2(k^*)$. The super replicating strategy of “the modified claim”:

$$(7) \quad \tilde{H}_T := 1_{B_1^*} d_T N_T + 1_{B_1^* \cup B_2^*} D(1 - N_T)$$

is a solution of Problem 1. We have

$$\begin{aligned} \operatorname{esssup}_{P^* \in \mathcal{P}} E^* [\tilde{H}_T \mid \mathcal{F}_t] &= \hat{E} [1_{B_1^*} d_T \mid \mathcal{G}_t] + (1 - N_t) \hat{E} [(1_{B_1^* \cup B_2^*} D - 1_{B_1^*} d_T)^+ \mid \mathcal{G}_t] \\ &\leq \hat{E} [1_{B_1^*} d_T \mid \mathcal{G}_t] + \hat{E} [(1_{B_1^* \cup B_2^*} D - 1_{B_1^*} d_T)^+ \mid \mathcal{G}_t] \\ &= \hat{E} [\max(1_{B_1^* \cup B_2^*} D, 1_{B_1^*} d_T) \mid \mathcal{G}_t] \\ &= \hat{E} [1_{B_1^*} d_T + 1_{B_2^*} D \mid \mathcal{G}_t], \end{aligned}$$

and we can construct an optimal strategy (\tilde{V}_0, ξ^*) by defining

$$\hat{E} [1_{B_1^*} d_T + 1_{B_2^*} D \mid \mathcal{G}_t] = \frac{\tilde{V}_0}{p_0} + \int_0^t \xi_u^* dX_u \quad \text{for } t \in [0, T].$$

Theorem 2 (A) Let (A) in Assumption 3 hold and assume that the equation:

$$(8) \quad E [1_{A_1(k)} (1 - \Lambda_T) + 1_{A_2(k)}] = 1 - \epsilon$$

with respect to k is solved for some $k^* = k^*(\epsilon)$. Then, the super replicating strategy of “the modified claim” defined by (5) is a solution of Problem 2.

(B) Let (B) in Assumption 3 hold and assume that the equation:

$$(9) \quad E [1_{B_1(k)} + 1_{B_2(k)} \Lambda_T] = 1 - \epsilon$$

with respect to k is solved for some $k^* = k^*(\epsilon)$. Then, the super replicating strategy of “the modified claim” defined by (7) is a solution of Problem 2.

Remark: The existence of the sets $A_1^*, A_2^*, B_1^*, B_2^* \in \mathcal{G}_T$ satisfying (4), (6), (8), or (9) is assured if, for example,

$$\begin{aligned} &E [1_{\partial A_1(k)} (1 - \Lambda_T) + 1_{\partial A_2(k)}] = 0 \\ \text{and} &E [1_{\partial B_1(k)} + 1_{\partial B_2(k)} \Lambda_T] = 0, \\ \text{where} &\partial A_1(k) := \{1 - \Lambda_T = kd_T Z_T, \Lambda_T = k(D - d_T)Z_T\}, \\ &\partial A_2(k) := \{1 = kDZ_T, \Lambda_T = k(D - d_T)Z_T\}, \\ &\partial B_1(k) := \{1 = kDZ_T, 1 - \Lambda_T = k(d_T - D)Z_T\}, \\ \text{and} &\partial B_2(k) := \{\Lambda_T = kDZ_T, 1 - \Lambda_T = k(d_T - D)Z_T\} \end{aligned}$$

are satisfied for arbitrary $k \geq 0$ (cf., e.g., [Sc] Chapter III.3). If the sets do not exist, we can reformulate our quantile-hedging procedure as stated in [F-L]: for instance, in Problem 1, we will modify “the success-set-maximization” to “the success-ratio-maximization”.

Further, we add two corollaries of the theorems above without proofs. First, we observe the following “trivialized” situations:

Corollary 1 (A) 1. if $0 < (1 - \Lambda_T)D \leq d_T < D$ holds \hat{P} -a.e., we have

$$A_1^* = \emptyset, \quad A_2^* = \left\{ Z_T < \frac{1}{k^* D} \right\} \quad \text{satisfying} \quad \hat{E}[1_{A_2^*} D] = \bar{V}_0/p_0,$$

2. if $0 < (1 - \Lambda_T)D \leq d_T < D$ holds P -a.e., we have

$$A_1^* = \emptyset, \quad A_2^* = \left\{ Z_T < \frac{1}{k^* D} \right\} \quad \text{satisfying} \quad P(A_2^*) = 1 - \epsilon,$$

(B) 1. if $0 < \Lambda_T d_T \leq D \leq d_T$ holds \hat{P} -a.e., we have

$$B_1^* = \left\{ Z_T < \frac{1}{k^* d_T} \right\} \quad \text{satisfying} \quad \hat{E}[1_{B_1^*} d_T] = \bar{V}_0/p_0, \quad B_2^* = \emptyset,$$

2. if $0 < \Lambda_T d_T \leq D \leq d_T$ holds P -a.e., we have

$$B_1^* = \left\{ Z_T < \frac{1}{k^* d_T} \right\} \quad \text{satisfying} \quad P(B_1^*) = 1 - \epsilon, \quad B_2^* = \emptyset,$$

In each cases, (conditional) default probability Λ_T has no effect on the optimal solutions of quantile hedging.

Secondly, we give an explicit calculation in the case of Jarrow-Turnbull-type defaultable bond model.

Corollary 2 Let $0 < d_T = \delta < D = 1$ and Λ (or λ) be deterministic. We have

$$A_1^* = \left\{ \frac{\Lambda_T}{k^*(1-\delta)} \leq Z_T \leq \frac{1-\Lambda_T}{k^*\delta} \right\}, \quad A_2^* = \left\{ Z_T < \frac{\Lambda_T}{k^*(1-\delta)} \right\}$$

in the case of $\Lambda_T + \delta < 1$, and

$$A_1^* = \emptyset, \quad A_2^* = \left\{ Z_T < \frac{1}{k^*} \right\}$$

in the case of $\Lambda_T + \delta \geq 1$, as given in Corollary 1 (A). Setting $d = 1$ and the risk-premium process γ constant, we observe

1. in Problem 1, the equations (4) is reexpressed as

$$\begin{aligned} (1-\delta)\hat{F}_T^\gamma\left(\frac{\Lambda_T}{k^*(1-\delta)}\right) + \delta\hat{F}_T^\gamma\left(\frac{1-\Lambda_T}{k^*\delta}\right) &= \bar{V}_0/p_0 \quad \text{if } \Lambda_T + \delta < 1, \\ \hat{F}_T^\gamma(1/k^*) &= \bar{V}_0/p_0 \quad \text{if } \Lambda_T + \delta \geq 1, \end{aligned}$$

where \hat{F}_T^γ denote the distribution functions of Z_T under \hat{P} , i.e.,

$$\begin{aligned} \hat{F}_T^\gamma(z) &:= \hat{P}(Z_T < z) := \int_0^z g_T(h_\gamma^-(x)) |h_\gamma^-(x)| dx, \\ g_T(x) &:= \frac{1}{\sqrt{2\pi T}} e^{-x^2/2T}, \quad h_\gamma^-(x) := \frac{1}{\gamma} \left(\log x - \frac{\gamma^2 T}{2} \right), \end{aligned}$$

2. in Problem 2, the equations (6) is reexpressed as

$$\begin{aligned}\Lambda_T F_T^\gamma \left(\frac{\Lambda_T}{k^*(1-\delta)} \right) + (1-\Lambda_T) F_T^\gamma \left(\frac{1-\Lambda_T}{k^*\delta} \right) &= 1-\epsilon \quad \text{if } \Lambda_T + \delta < 1, \\ F_T^\gamma (1/k^*) &= 1-\epsilon \quad \text{if } \Lambda_T + \delta \geq 1,\end{aligned}$$

where F_T^γ denote the distribution functions of Z_T under P , i.e.,

$$\begin{aligned}F_T^\gamma(z) &:= \mathbb{P}(Z_T < z) := \int_0^z g_T(h_\gamma^+(x)) |h_\gamma^+(x)'| dx, \\ h_\gamma^+(x) &:= \frac{1}{\gamma} \left(\log x + \frac{\gamma^2 T}{2} \right).\end{aligned}$$

The initial cost:

$$\bar{V}_0(\epsilon) := p_0 \left[\delta \hat{P}(A_1(k^*(\epsilon))) + \hat{P}(A_2(k^*(\epsilon))) \right]$$

of the quantile hedging strategy under the success probability constraint, $\geq 1-\epsilon$, is equal to

$$\bar{V}_0(\epsilon) = p_0 \left[\delta \hat{F}_T^\gamma \left(\frac{1-\Lambda_T}{k^*(\epsilon)\delta} \right) + (1-\delta) \hat{F}_T^\gamma \left(\frac{\Lambda_T}{k^*(\epsilon)(1-\delta)} \right) \right]$$

in the case of $\Lambda_T + \delta < 1$, or equal to

$$\bar{V}_0(\epsilon) = p_0 \hat{F}_T^\gamma \left(\frac{1}{k^*(\epsilon)} \right)$$

in the case of $\Lambda_T + \delta \geq 1$, respectively.

In Appendix, numerical examples of Corollary 2.2 above are given.

Remark: Computable examples with stochastic Λ_T , (or λ) seem to be necessary to lead more (financial) considerations and implementations since the corollaries above treat only trivialized situations. For example, we are now considering to adopt modeling the functionals:

$$G_n(\omega) := \Lambda_{(n+1)l} \Lambda_{nl}^{-1} = \exp \left\{ - \int_{nl}^{(n+1)l} \lambda_u(\omega) du \right\} \quad (n = 0, 1, \dots)$$

for some fixed $l(> 0, \text{e.g., } = 0.25, 0.5, 1)$ in place of modeling process λ nor Λ , since we may not have explicit expression of the joint distribution of Z_T and Λ_T generally; nor even in a simplest example:

$$\begin{aligned}Z_T &= \mathcal{E}(\gamma w_T) \quad \text{with } \gamma \in \mathbb{R}^d, \\ \text{and } \lambda_t &= a(Y_t + b)^2 + c \quad \text{with } dY_t = dw_t - KY_t dt,\end{aligned}$$

$b \in \mathbb{R}^d$, positive constants a, c and positive-definite $d \times d$ -matrix K , which is called the squared Gaussian hazard-rate-model, and since we may not observe sample-path, $\lambda(\omega)$, nor $\Lambda(\omega)$.

3 Proofs

First, we will prove Theorem 1. Let us consider

Problem 1'

$$\begin{aligned}\max_{A, B \in \hat{\mathcal{G}}_T} E[1_A(1-\Lambda_T) + 1_B \Lambda_T], \\ \text{subject to } E^*[1_A d_T N_T + 1_B D(1-N_T)] \leq \bar{V}_0/p_0, \quad \text{for all } P^* \in \mathcal{P},\end{aligned}$$

and, via similar discussions to Proposition 2.8 in [F-L], see the following:

Lemma 2 *Let us denote a solution of Problem 1' by A^* and $B^* \in \mathcal{G}_T$. The super replicating strategy of "the modified claim" $H_T^* := 1_{A^*} d_T N_T + 1_{B^*} D(1 - N_T)$ is a solution of Problem 1.*

Proof: For any admissible strategy (V_0, ξ) with $V_0 \leq \tilde{V}_0$, the associated "success set":

$$1_{\{V_T \geq H_T\}} = 1_{\{V_T \geq d_T\}} N_T + 1_{\{V_T \geq D\}} (1 - N_T)$$

satisfies

$$\begin{aligned} V_T &\geq H_T 1_{\{V_T \geq H_T\}} \\ &= 1_{\{V_T \geq d_T\}} d_T N_T + 1_{\{V_T \geq D\}} D(1 - N_T) \end{aligned}$$

P -a.s., so,

$$\tilde{V}_0/p_0 \geq V_0/p_0 \geq E^*[V_T] \geq E^*[1_{\{V_T \geq d_T\}} d_T N_T + 1_{\{V_T \geq D\}} D(1 - N_T)]$$

for any $P^* \in \mathcal{P}$ since nonnegative local P^* -local martingale V/p is a super martingale, this implies

$$E[1_{\{V_T \geq H_T\}}] \leq E[1_{A^*} N_T + 1_{B^*} (1 - N_T)] = E[1_{A^*} (1 - \Lambda_T) + 1_{B^*} \Lambda_T]$$

On the other hand, the super replicating strategy of "the modified claim" H_T^* is obviously admissible,

$$\operatorname{esssup}_{P^* \in \mathcal{P}} E^*[H_T^* | \mathcal{F}_t] = \operatorname{esssup}_{P^* \in \mathcal{P}} E^*[1_{A^*} d_T N_T + 1_{B^*} D(1 - N_T) | \mathcal{F}_t] \geq 0,$$

and has a maximal "success set". i.e., in the expression

$$1_{\{H_T^* \geq H_T\}} = 1_{\{1_{A^*} d_T \geq d_T\}} N_T + 1_{\{1_{B^*} D \geq D\}} (1 - N_T),$$

we observe

$$\tilde{A} := \{1_{A^*} d_T \geq d_T\} \supset A^*, \quad \text{and} \quad \tilde{B} := \{1_{B^*} D \geq D\} \supset B^*,$$

hence, \tilde{A} and \tilde{B} are solutions of Problem 1', and the optimality in Problem 1 follows. □

Moreover, we set

Problem 1''

(A) *In the case of $D > d_T \geq 0$, \tilde{P} -a.e.,*

$$\max_{A, B \in \mathcal{G}_T, A \supset B} E[1_A(1 - \Lambda_T) + 1_B \Lambda_T], \quad \text{subject to} \quad \tilde{E}[\max(1_A d_T, 1_B D)] \leq \tilde{V}_0/p_0,$$

(B) *in the case of $0 \leq d_T \leq D$, \tilde{P} -a.e.,*

$$\max_{A, B \in \mathcal{G}_T, A \subset B} E[1_A(1 - \Lambda_T) + 1_B \Lambda_T], \quad \text{subject to} \quad \tilde{E}[\max(1_A d_T, 1_B D)] \leq \tilde{V}_0/p_0,$$

and observe

Lemma 3 *Problem 1' is equivalent to Problem 1''.*

Proof: We have

$$\begin{aligned} E^* [1_A d_T N_T + 1_B D(1 - N_T)] &= E^* [1_A d_T] + E^* [(1_B D - 1_A d_T)(1 - N_T)] \\ &\leq E^* [1_A d_T] + E^* [(1_B D - 1_A d_T)^+] \\ &= \hat{E} [\max(1_A d_T, 1_B D)] \end{aligned}$$

for any $P^* \in \mathcal{P}$, since $P^* = \hat{P}$ on \mathcal{G}_T . If we use $\left(Q_\epsilon^{(1_B D - 1_A d_T)}\right)_{\epsilon > 0} (\subset \mathcal{P})$, which shall be defined in Lemma 4 below, we can approximate the trivial upper bound as

$$\lim_{\epsilon \rightarrow 0} E_\epsilon^{(1_B D - 1_A d_T)} [(1_B D - 1_A d_T)(1 - N_T)] = \hat{E} [(1_B D - 1_A d_T)^+],$$

so

$$\sup_{P^* \in \mathcal{P}} E^* [1_A d_T N_T + 1_B D(1 - N_T)] = \hat{E} [(1_B D - 1_A d_T)^+].$$

Further, if $D > d_T \geq 0$ \hat{P} -a.e., for example, for any $A, B \in \mathcal{G}_T$ satisfying the condition $\hat{E} [\max(1_A d_T, 1_B D)] \leq \hat{V}_0/p_0$, set $\bar{A} := A \cup B$, and recall that the relation

$$\begin{aligned} E [1_A(1 - \Lambda_T) + 1_B \Lambda_T] &\leq E [1_{\bar{A}}(1 - \Lambda_T) + 1_B \Lambda_T], \\ \hat{E} [\max(1_A d_T, 1_B D)] &= \hat{E} [\max(1_{\bar{A}} d_T, 1_B D)] \end{aligned}$$

hold, hence follows the lemma. □

Lemma 4 For arbitrary $F \in L^1(\hat{P})$, $\alpha, \beta > 0$ and $\epsilon \in (0, T)$, let us define an equivalent martingale measure Q_ϵ^F by the formula

$$\begin{aligned} \frac{dQ_\epsilon^F}{dP} \Big|_{\mathcal{F}_t} &= \rho_t, \\ \text{where } \rho_t &= 1 + \int_0^t \rho_u (-\gamma_u dw_u + \kappa_u dM_u), \\ \kappa_t &:= \begin{cases} -1 + \epsilon/\lambda_t & \text{if } t < T - \epsilon \\ -1 + (\epsilon^{-\alpha-1} - (\epsilon^{-\alpha-1} + e^{\beta-1}) \hat{E} [1_{\{F \geq 0\}} | \mathcal{G}_{T-\epsilon}]) / \lambda_t & \text{if } T - \epsilon \leq t \leq T, \end{cases} \end{aligned}$$

Then,

$$(10) \quad \lim_{\epsilon \rightarrow 0} E_\epsilon^F [F(1 - N_T)] = \hat{E} [F^+]$$

holds, where we have denoted the expectation with respect to Q_ϵ^F by $E_\epsilon^F[\cdot]$.

Proof: We will only show “ \geq -side” inequality in (10), since the relation

$$\lim_{\epsilon \rightarrow 0} E_\epsilon^F [F(1 - N_T)] \leq E_\epsilon^F [F^+] = \hat{E} [F^+]$$

is obvious. Under Q_ϵ^F ,

$$\left(N_t - \int_0^t (1 - N_u)(1 + \kappa_u) \lambda_u du \right)_{t \in [0, T]}$$

is a martingale, and the relation

$$E_\epsilon^F [F(1 - N_T)] = \hat{E} [F \hat{\Lambda}_\epsilon^F]$$

holds, where we have denoted

$$\begin{aligned}
\log \bar{\Lambda}_\epsilon^F &:= -\int_0^T (1 + \kappa_t) \lambda_t dt \\
&= -\epsilon(T - \epsilon) - \left(\epsilon^{-\alpha} - (\epsilon^{-\alpha} + \epsilon^\beta) \hat{E} [1_{\{F \geq 0\}} | \mathcal{G}_{T-\epsilon}] \right) \\
&= -\epsilon(T - \epsilon) - \epsilon^{-\alpha} 1_{G_\epsilon} + \epsilon^\beta 1_{G_\epsilon} - (\epsilon^{-\alpha} + \epsilon^\beta) L_\epsilon^F \\
\text{with } G_\epsilon = G(F, \alpha; \epsilon) &:= \left\{ \hat{E} [1_{\{F \geq 0\}} | \mathcal{G}_{T-\epsilon}] \geq 1 - \epsilon^{\alpha+1} \right\} \\
\text{and } L_\epsilon^F &:= 1_{G_\epsilon} - \hat{E} [1_{\{F \geq 0\}} | \mathcal{G}_{T-\epsilon}].
\end{aligned}$$

(cf., [K], for example.) So, by using the inequality $e^{-x} \geq 1 - x$ ($x \in \mathbb{R}$), we have

$$\begin{aligned}
\bar{\Lambda}_\epsilon^F &\geq \exp \left\{ -\epsilon(T - \epsilon) - \epsilon^{-\alpha} 1_{G_\epsilon} + \epsilon^\beta 1_{G_\epsilon} \right\} \left(1 - (\epsilon^{-\alpha} + \epsilon^\beta) L_\epsilon^F \right) \\
&= \left[\exp(-\epsilon^{-\alpha}) 1_{G_\epsilon} + \exp(\epsilon^\beta) 1_{G_\epsilon} \right] e^{-\epsilon(T-\epsilon)} \left(1 - (\epsilon^{-\alpha} + \epsilon^\beta) L_\epsilon^F \right)
\end{aligned}$$

therefore,

$$\begin{aligned}
E_\epsilon^F [F(1 - N_T)] &\geq \exp \left\{ -\epsilon(T - \epsilon) - \epsilon^{-\alpha} \right\} \hat{E} [F 1_{G_\epsilon}] \\
&\quad + \exp \left\{ -\epsilon(T - \epsilon) + \epsilon^\beta \right\} \hat{E} [F 1_{G_\epsilon}] \\
&\quad - (\epsilon^{-\alpha} + \epsilon^\beta) \exp \left\{ -\epsilon(T - \epsilon) - \epsilon^{-\alpha} \right\} \hat{E} [F L_\epsilon^F 1_{G_\epsilon}] \\
&\quad - (\epsilon^{-\alpha} + \epsilon^\beta) \exp \left\{ -\epsilon(T - \epsilon) + \epsilon^\beta \right\} \hat{E} [F L_\epsilon^F 1_{G_\epsilon}]
\end{aligned}$$

follows. The second term of the right-hand-side above converges to $\hat{E} [F^+]$ as $\epsilon \rightarrow 0$, and the rest of all terms go to 0 as $\epsilon \rightarrow 0$, since $x e^{-x} \rightarrow 0$ as $x \rightarrow \infty$ and since the relation:

$$\hat{E} [F L_\epsilon^F 1_{G_\epsilon}] = \hat{E} \left[F \left(1 - \hat{E} [1_{\{F \geq 0\}} | \mathcal{G}_{T-\epsilon}] \right) 1_{G_\epsilon} \right] \leq \hat{E} [F 1_{G_\epsilon}] \epsilon^{\alpha+1}$$

is observed. Hence follows the lemma. □

To obtain Theorem 1, we show the following

Lemma 5 (A) *The sets $A_1^* \cup A_2^*$ and A_2^* defined in Theorem 1 (A) is a solution of Problem 1" (A).*

(B) *The sets B_1^* and $B_1^* \cup B_2^*$ defined in Theorem 1 (B) is a solution of Problem 1" (B).*

Proof: We only show (A), since (B) can be seen similarly. For the constant $k^* = k^*(\bar{V}_0)$ given in Theorem 1 (A), define

$$\begin{aligned}
B^* &:= \{ \Lambda_T > k^*(D - d_T) Z_T \}, \\
A^* &:= \{ (1 - \Lambda_T + 1_B \Lambda_T) > k^*(d_T + 1_B (D - d_T)) Z_T \},
\end{aligned}$$

and note that the relations

$$A^* \setminus B^* = A_1^* \quad \text{and} \quad A^* \cap B^* = A_2^*$$

hold. For any \mathcal{G}_T -measurable $A \supset B$ satisfying $\hat{E} [\max(1_A d_T, 1_B D)] \leq \bar{V}_0/p_0$, we have

$$\begin{aligned}
&E [1_A (1 - \Lambda_T) + 1_B \Lambda_T] - k^* \bar{V}_0/p_0 \\
&\leq E [1_A (1 - \Lambda_T) + 1_B \Lambda_T] - k^* \hat{E} [\max(1_A d_T, 1_B D)] \\
&= E [1_A ((1 - \Lambda_T) + 1_B \Lambda_T)] - k^* \hat{E} [1_A (d_T + 1_B (D - d_T))] \\
&= E [1_A \{ (1 - \Lambda_T) - k^* d_T Z_T + 1_B (\Lambda_T - k^* (D - d_T) Z_T) \}]
\end{aligned}$$

$$\begin{aligned}
&\leq E[1_A \{(1 - \Lambda_T) - k^* d_T Z_T + 1_{B^*} (\Lambda_T - k^* (D - d_T) Z_T)\}] \\
&= E[1_A \{(1 - \Lambda_T + 1_{B^*} \Lambda_T) - k^* (d_T + 1_{B^*} (D - d_T)) Z_T\}] \\
&\leq E\{1_{A^*} \{(1 - \Lambda_T + 1_{B^*} \Lambda_T) - k^* (d_T + 1_{B^*} (D - d_T)) Z_T\}\} \\
&= E[1_{A^*} (1 - \Lambda_T + 1_{B^*} \Lambda_T)] - k^* \hat{E}[1_{A^*} (d_T + 1_{B^*} (D - d_T))] \\
&= E[1_{A_1^* \cup A_2^*} (1 - \Lambda_T) + 1_{A_2^*} \Lambda_T] - k^* \hat{E}[1_{A_1^* \cup A_2^*} d_T + 1_{A_2^*} (D - d_T)] \\
&= E[1_{A_1^* \cup A_2^*} (1 - \Lambda_T) + 1_{A_2^*} \Lambda_T] - k^* \tilde{V}_0 / p_0,
\end{aligned}$$

therefore the optimality is derived, hence follows the lemma. \square

\square

Now, Theorem 2 can be obtained straightforwardly. Following to the discussion in [F-L], we can reduce solving Problem 2 to solving

Problem 2'

(A) If $D > d_T \geq 0$, \hat{P} -a.e.,

$$\min_{A \supset B \in \mathcal{G}_T} \hat{E}[\max(1_A d_T, 1_B D)] \quad \text{subject to} \quad E[1_A (1 - \Lambda_T) + 1_B \Lambda_T] \geq 1 - \epsilon,$$

(B) If $0 \leq d_T \leq D$, \hat{P} -a.e.,

$$\min_{A \subset B \in \mathcal{G}_T} \hat{E}[\max(1_A d_T, 1_B D)] \quad \text{subject to} \quad E[1_A (1 - \Lambda_T) + 1_B \Lambda_T] \geq 1 - \epsilon.$$

and we can give solutions of these problems via similar "Neyman-Pearson-like" discussion as Lemma 5, although we omit the detail. At the last of this section, we give the relation, which is used to describe the super hedging strategy of H_T .

Lemma 6 *The relation (1) holds.*

Proof: For any $P^* \in \mathcal{P}$, we have

$$E^* [d_T N_T + D(1 - N_T) \mid \mathcal{F}_t] = \hat{E} [d_T \mid \mathcal{G}_t] + E^* [(D - d_T)(1 - N_T) \mid \mathcal{G}_t],$$

so obviously,

$$\bar{H}_t \leq \frac{d_t}{p_t} + (1 - N_t) \hat{E} [(D - d_T)^+ \mid \mathcal{G}_t].$$

If we prepare $(Q_\epsilon^{D-d_T})_{\epsilon>0}$ defined in Lemma 4 above, we observe

$$\text{esssup}_{\epsilon>0} E_\epsilon^{D-d_T} [H_T \mid \mathcal{G}_t] = \frac{d_t}{p_t} + (1 - N_t) \hat{E} [(D - d_T)^+ \mid \mathcal{G}_t],$$

hence actually, the trivial upper bound above is estimated arbitrary. \square

\square

Appendix: Numerical Examples of Corollary 2.2

Set $\Lambda_T := e^{-\lambda T}$, fix $\gamma = 0.03$, and compute the so-called default-yield-spread-curve: $(Sp(T))_{T \geq 0}$, where

$$Sp(T) := -\frac{1}{T} \log \left(\frac{\tilde{V}_0(\epsilon)}{p_0} \right) = -\frac{1}{T} \log \left(\delta \hat{P}(A_1(k^*(\epsilon))) + \hat{P}(A_2(k^*(\epsilon))) \right).$$

Fig.1: $\lambda = 0.03, \delta = 0.2$, and $\epsilon = 0.01, 0.05, 0.1$.

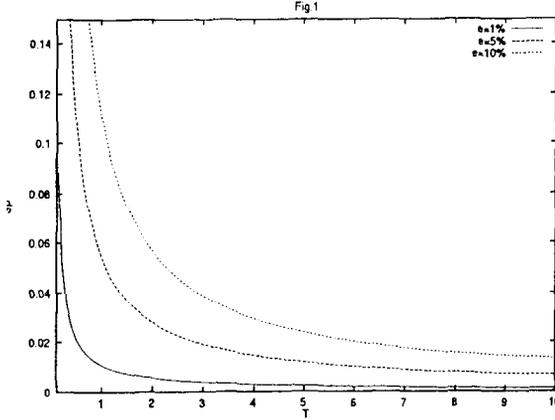


Fig.2: $\lambda = 0.05, \epsilon = 0.01$, and $\delta \rightarrow 0, \delta = 0.2, \delta \geq 1 - \exp(-0.05 \times 10)$.

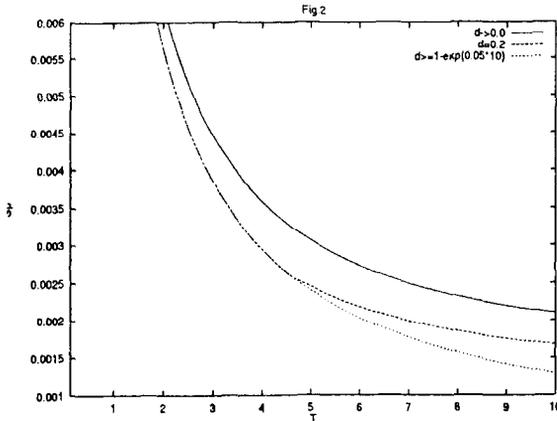
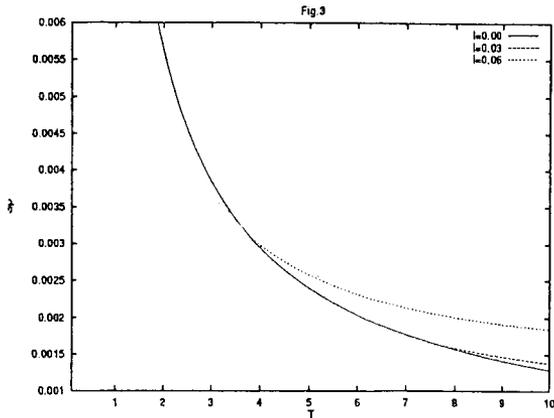


Fig.3: $\delta = 0.2$, $\epsilon = 0.01$, and $\lambda = 0, 0.03, 0.06$.



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ASSET-LIABILITY MODELLING FOR PENSION SCHEMES

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This paper is closely based on a talk presented, on behalf of the European Federation for Retirement Provision, to Insurance and Pensions Supervisors from the Member States of the European Union, at the Offices of the Commission of the European Union, on 23 September 1998. It is reproduced by permission of the European Federation for Retirement Provision.

ABSTRACT

The background to this paper is that what are called "pension funds" in some Member States of the European Union (EU) are in effect specialised life insurance companies who write individual pension policies anyone who wishes to subscribe to such a contract. They are sometimes referred to as "open pension funds" and they are quite different from the "closed pension funds", common in the Anglo-Saxon countries and in the Netherlands (and elsewhere), which are sponsored by one employer for that employer's employees only. Often the employer stands behind the closed pension fund, and guarantees (up to a point) the benefits that have been promised.

The regulations for supervising pension funds within the EU have not been harmonised, although the supervision of insurance companies has been through the Third Life and Non-life Directives. There is some pressure on the one hand for it to be possible to arrange pension funds on a cross-border basis (which would help multi-national employers) and on the other hand for pension funds to be supervised in a similar way in all Member States.

The European Federation for Retirement Provision (EFRP) represents the associations of pension funds in the separate Member States. Its officers were anxious that the form of regulation that is found for some of the open type of pension funds, some of which is quite appropriate for life insurance companies, should not be applied to closed pension funds, which are seen as being quite different.

In particular, in some Member States there are restrictions on the maximum proportion of the assets that an open pension fund may invest in "risky" securities, such as ordinary shares, and in some there are requirements that a certain minimum proportion of the assets should be invested in government bonds, which are seen as safer investments.

The EFRP was invited to make a presentation at a meeting of the Insurance and Pensions Supervisors of the Member States of the EU in September 1998, and I was asked by the EFRP to make such a presentation.

In the talk, after some preliminary background, I explained what AFIR readers will recognise as the standard Markowitz mean-variance model, but with the full boundary calculated, including the outer edge, which is commonly just sketched in. I show that restrictions such as a maximum fraction in risky securities, or a minimum fraction in what may be thought of as relatively safe securities, worsen the position for the prudent investor, but do reduce the possibility of imprudent investors choosing very inefficient portfolios.

In the final paragraphs, which I omitted at the talk, I show that index-linked securities, where these are available, are a better match to final salary liabilities than conventional bonds, and I demonstrate the numerical effect of this, on the given assumptions.

The calculation of the complete boundary of the feasible region is not a trivial task. I adapted the programme given by Harry Markowitz in *Mean-variance analysis in portfolio choice and capital*

amrkets (Blackwell, 1987), and programmed also the calculations for the outer boundary in accordance with Markowitz's explanations therein.