

# Threshold Models of the Term Structure of Interest Rates

Jeffrey Pai

Warren Centre for Actuarial Studies and Research

Faculty of Management

University of Manitoba

180 Freedman Crescent

Winnipeg, MB R3T 5V4

Canada

Telephone: (204) 474-6412

FAX: (204) 474-7545

E-Mail: [jpai@ms.umanitoba.ca](mailto:jpai@ms.umanitoba.ca)

Hal W. Pedersen

Actuarial Science Program

Department of Risk Management and Insurance

J. Mack Robinson College of Business

Georgia State University

P.O. Box 4036

Atlanta, Georgia 30302-4036

U.S.A.

Telephone: (404) 651-0962

FAX: (404) 651-4219

E-Mail: [inshwp@panther.gsu.edu](mailto:inshwp@panther.gsu.edu)

**Summary** In this paper we examine and estimate threshold models for the short-rate of interest. Threshold models are capable of picking up important stylised facts in interest-rate time series without a significant loss in tractability nor the introduction of complicated unobservable variables. We fit the model to the Federal Reserve Bank's H15 data set and compare the results of our estimation of one, two, and three regime models. Our statistical techniques are based on the Bayesian approach and computed using Markov chain Monte Carlo (MCMC) techniques.

## 1. INTRODUCTION

One of the goals of the classical approach to interest-rate modelling is to fit an empirically representative and tractable process to the short-rate of interest. This is fundamental since the short-rate is one of the key observables that drives overall term structure dynamics in classical interest-rate models. One-factor models are not able to capture empirical aspects of the short-rate data series. Indeed, one-factor models tend to do particularly poor jobs of capturing the volatility properties and conditional dependence structures of the short-rate. Traditionally, financial economists have used multi-factor models to better capture the empirical properties of the short-rate. Multi-factor model can improve the fit of the model to the data but they come at the cost of increased numerical complexity and the introduction of latent variables. Furthermore, multi-factor models often based on the introduction of more of the same processes that are known not to fit the short-rate well in the one-dimensional case.

Another way to improve the empirical fit of the traditional one-factor model without introducing a second factor into the model is to use a threshold model. By keeping the model to one factor, it is possible to avoid some of the most significant numerical difficulties that are encountered in generating numerical results for multi-factor models. However, a single-factor threshold model is locally subject to many of the key shortcomings of traditional one-factor models. Chief among these is the perfect correlation of yield rate movements for changes in the short-rate which remain within the same striations. When the short-rate process is near the threshold boundaries this perfect correlation will be broken however. One-factor threshold models are able to pick up some of the changes in volatility of the short-rate that appear to depend on the level (or size) of the short-rate and this is one of the most compelling reasons to study these models.

The two earliest diffusion models of the short-rate appeared in [16], known as the Vasicek model and in [4] (the original draft of the CIR working paper appeared in 1977), known as the CIR model. The Vasicek model uses the following specification for the short rate:

$$\text{(VAS)} \quad dr_t = (\alpha + \beta r_t)dt + \sigma dW_t.$$

The Cox-Ingersoll-Ross model uses a different specification of the short-rate:

$$\text{(CIR)} \quad dr_t = (\alpha + \beta r_t)dt + \sigma \sqrt{r_t}dW_t.$$

An empirical study of the CIR model and some suggestions on how the model should be extended to better capture the properties of the

empirical data may be found in [2]. A generalisation of both of these models is to allow an additional parameter (which we will call  $\gamma$ ) and specify that the short-rate follow the equation:

$$(CKLS) \quad dr_t = (\alpha + \beta r_t)dt + \sigma r_t^\gamma dW_t.$$

This equation was studied in [3]. This one-factor model is the most general form of the one-dimensional diffusion model for the short-rate that has been studied. This equation can also be written as

$$dr_t = -\beta \left( \frac{\alpha}{-\beta} - r_t \right) dt + \sigma r_t^\gamma dW_t,$$

from which  $-\beta$  is seen to be the speed of mean reversion and  $-\alpha/\beta$  is the mean reversion level.

In Figure 1 is presented a graph of the three month US treasury rate based on the Federal Reserve Bank's H15 data set.

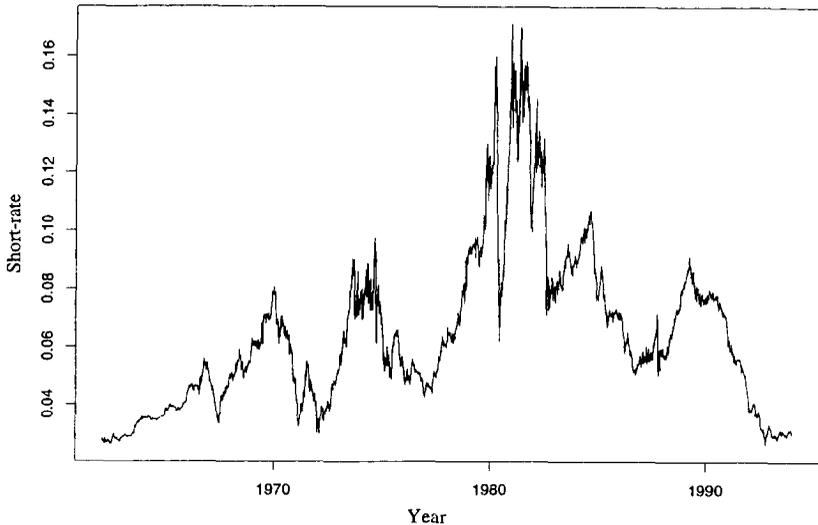


FIGURE 1. The US Short-rate.

Without getting involved in a detailed statistical analysis of each of these models, let us offer some intuition on the shortcomings and motivations for each model.

All models have mean reversion. More precisely, each of these models can accommodate mean reversion and the estimated values of  $\alpha$  and  $\beta$  will determine whether or not there is mean reversion in the

model. There is a general belief that under typical economic conditions there should be a degree of mean reversion in the short-rate and thus a reasonable model of the short-rate should allow for this feature. The Vasicek model has the drawback that the short-rate can assume negative values. In practice, the probability of the short-rate becoming negative will be relatively small for short time horizons if the process is calibrated to typical US data. The CIR model has the advantage that the short-rate process is always positive if the parameters are within certain ranges.

Perhaps the most significant drawback of the Vasicek model is that the volatility of the short-rate is constant at the value  $\sigma$ . Even a casual look at Figure 1 suggests that the volatility of the short-rate is far from constant. As a matter of fact, it appears that the volatility of the short-rate depends on the level the short-rate is at. In the roughest of terms, the short-rate appears to have lower volatility when the short-rate is low and higher volatility when the short-rate is high. Unlike the Vasicek model, the volatility of the CIR model is not constant, in fact, it is equal to  $\sigma\sqrt{r_t}$  and thus is proportional to the square root of the level of the short-rate. Unfortunately, even though this volatility specification does a better job of picking up the volatility in the short-rate than does the Vasicek model, statistical analysis can be used to show that the data does not give strong support for volatility of the form specified in the CIR model. The CKLS model offers an additional degree of freedom in that the volatility is permitted to be of the form  $\sigma r_t^\gamma$ . Evidently, the CIR model is the special case of this model with  $\gamma = 1/2$  and the Vasicek model is a special case of this model with  $\gamma = 0$ . If one estimates the CKLS model for a fairly wide set of data, one generally finds that  $\gamma$  is in the interval  $[0.9, 1.5]$ , the value which you obtain depending on which years in the data you based your estimation on. Apparently, a much higher value of  $\gamma$  is required to pick up the volatility in the short-rate data than is allowed in the CIR model. Even when the CKLS model is fit to the data, one finds that there is no single value of  $\gamma$  that can adequately pick up the volatility of the short-rate data. Short of introducing more parameters, we are pretty much at the limits of what can be done with a simple single-factor model. As we have noted, one approach is to now introduce other factors. Perhaps a more sophisticated approach is to allow for “regimes”. One can allow for regimes using a Markov switching process as was done in [7], [6], [9], [10], and others. This approach is capable of capturing very rich interest-rate dynamics. The two chief disadvantages of this approach are that a key latent (or unobservable) variable is introduced into the model and the term structure dynamics are difficult to analyse because

of this latent variable. Another approach to allowing for regimes is to permit the level of the short-rate to determine the current regime. This leads us to threshold models as pioneered by Tong (and presented in [14] and [15]) and first applied to interest-rate modelling by [11].

Threshold models are an economical way to accommodate regimes without admitting the additional complexity of a latent regime switching process. The basic idea is that the level of the short-rate will determine the regime itself. This makes some degree of economic sense. For instance, one would expect the behaviour of the short-rate to exhibit very different properties when it is historically high compared to when it is in its historical range. Indeed, when rates are high the economy is likely to perform differently, the Fed is likely to be very actively pursuing economic policies, and there is probably a great deal of general economic uncertainty. In a threshold model of the short-rate, the instantaneous dynamics of the short-rate depends on which region of the state space the short-rate process currently lies in. A two regime threshold model is illustrated Figure 2. Suppose that our single threshold level (resulting in two regimes) is denoted by  $u$ .

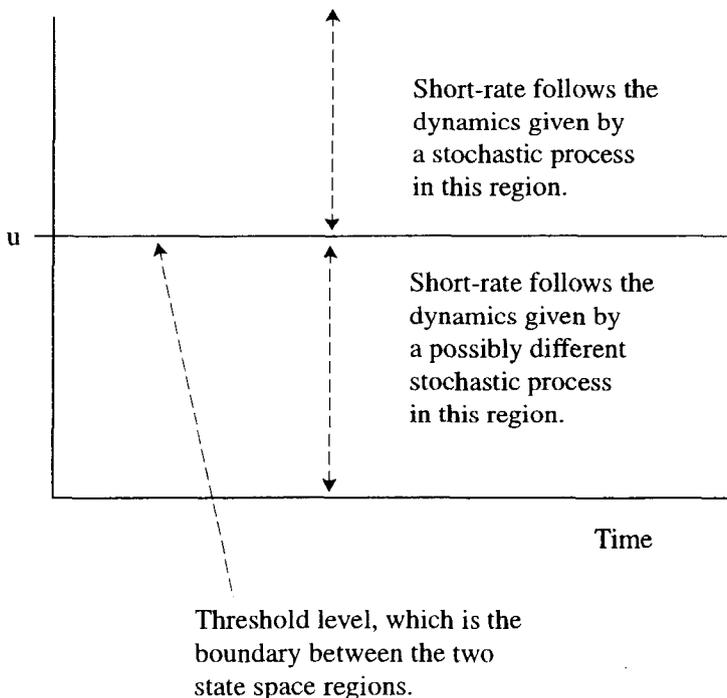


FIGURE 2. Dynamical behaviour of a simple threshold model.

The basic theoretical construct of a threshold model is to split the dynamics of the process up into striations which depend on the region of the state space in which the interest-rate process is currently located.

In this paper, we explore a one-factor self exciting threshold autoregressive (SETAR) model of the short-rate of interest. This model can have as many threshold levels as one desires. The number of threshold levels is limited by our ability to estimate the models and a recognition of the principle of parsimony. The inclusion of too many regimes will chop up the data into many different strata and the resulting model will tell us little about how the short-rate behaves. On the other hand, if too few strata are used then the model will not be sufficiently refined to pick up the various degrees of variation in the short-rate data. We will estimate models with one, two, and three regimes and this will provide a sufficient degree of differentiation to tell if the use of thresholds levels can help us in studying the volatility structure of the short-rate.

These models can also be used to readily generate the full term structure via Monte Carlo simulation. A knowledge of the market price of risk process is required, however, if this simulation is to be fully calibrated to the data. Parametric techniques for estimating the market price of risk in sophisticated models without closed formulas for bond prices have not been discovered at this time. In practice, one will have to obtain values for the market price of risk process by trial and error or by nonparametric methods. This is an important area for future research. It is possible to approach the modelling of the short-rate and the entire term structure from a nonparametric perspective. This technique can yield an estimate of the market price of risk parameter, which is necessary to obtain the term structure from the short-rate process. This approach has been studied in [1], [13] and [5].

We now turn our attention to the investigation of threshold models for the short-rate of interest.

## 2. THRESHOLD MODELS FOR THE SHORT-RATE PROCESS

A one-factor threshold model for the short-rate with single threshold level  $u$  is defined by the process:

$$dr_t = \begin{cases} (\alpha_1 + \beta_1 r_t) dt + \sigma_1 r_t^{\gamma_1} dW_t & r_t \leq u \\ (\alpha_2 + \beta_2 r_t) dt + \sigma_2 r_t^{\gamma_2} dW_t. & r_t > u \end{cases}$$

This particular specification has only one threshold level but one can accommodate multiple threshold levels. Additionally, it is not necessary to have the threshold level governed by the current value of the short-rate process. Instead, the threshold changes can be triggered by lagged variables or even an adapted functional of the short-rate history. We will not investigate these more general specifications in this paper. These short-rate dynamics can be written out in stochastic differential form using indicator functions:

$$dr_t = \mathbf{1}_{\{r_t \leq u\}} [(\alpha_1 + \beta_1 r_t) dt + \sigma_1 r_t^{\gamma_1} dW_t] \\ + \mathbf{1}_{\{r_t > u\}} [(\alpha_2 + \beta_2 r_t) dt + \sigma_2 r_t^{\gamma_2} dW_t].$$

It is preferable to write this out in the following form:

$$dr_t = [ \mathbf{1}_{\{r_t \leq u\}} (\alpha_1 + \beta_1 r_t) + \mathbf{1}_{\{r_t > u\}} (\alpha_2 + \beta_2 r_t) ] dt \\ \text{(SIM)} \quad + [ \mathbf{1}_{\{r_t \leq u\}} \sigma_1 r_t^{\gamma_1} + \mathbf{1}_{\{r_t > u\}} \sigma_2 r_t^{\gamma_2} ] dW_t,$$

as this shows the dynamics in terms of traditional drift and diffusion coefficients.

This model is sometimes referred to as a two regime model because there are two regimes implicit in the model. Which regime (or interest rate dynamics) are being followed depends on which side of the threshold the short-rate is currently on. This is not the same as the Markov switching models that are customarily referred to as “regime switching models.” However, there is regime switching that takes place in a threshold model. The difference is that in a threshold model the regime is determined by the level of the observable variable  $r$  whereas in Markov switching models the regime is determined by an unobservable switching variable.

A three regime model can be defined as in the following equation. In this case we assume there are two threshold levels,  $u_1$  and  $u_2$ , with

$u_1 < u_2$ .

$$dr_t = \begin{cases} (\alpha_1 + \beta_1 r_t) dt + \sigma_1 r_t^{\gamma_1} dW_t & r_t \leq u_1 \\ (\alpha_2 + \beta_2 r_t) dt + \sigma_2 r_t^{\gamma_2} dW_t. & u_1 < r_t \leq u_2 \\ (\alpha_3 + \beta_3 r_t) dt + \sigma_3 r_t^{\gamma_3} dW_t. & r_t > u_2 \end{cases}$$

As you will note, both the two regime and three regime threshold models are nonlinear models but we have retained a standard linear model of the CKLS form in each strata. The nonlinearity arises because of the changes in regimes.

### 3. ESTIMATION OF THE SHORT-RATE PROCESS

We used Bayesian estimation techniques and the Markov chain Monte Carlo (MCMC) technique to compute the posterior distributions for the various model parameters (including the threshold parameters). Our short-rate data is the three month rate from the Federal Reserve Bank’s H15 data set.

The single regime (or CKLS model) has no threshold parameter and the unknown parameters  $\alpha$ ,  $\beta$ ,  $\sigma^2$ , and  $\gamma$ . These estimated values are reported in Table 1. This model serves as a frame of reference to compare the estimation results of the two and three regime models with.

$\alpha$	$\beta$	$\gamma$	$\sigma^2$
0.000346	-0.00817	0.975	0.00168
(0.000277)	(0.00781)	(0.0529)	(0.000641)

TABLE 1. Posterior Mean Estimates of the Single Regime (or CKLS) Model for the Short-Term Interest Rate. Posterior standard deviations are in parentheses.

The results for the two regime model are presented in Table 2 and Table 3. A graph of the posterior density for  $u$  is shown in Figure 3.’

<b>Regime I</b>	$\alpha$	$\beta$	$\sigma^2$	$\gamma$
	0.0000745 (0.0000924)	-0.00127 (0.00280)	0.000732 (0.000449)	0.936 (0.139)
<b>Regime II</b>	$\alpha$	$\beta$	$\sigma^2$	$\gamma$
	0.0151 (0.0201)	-0.106 (0.134)	0.00445 (0.0159)	0.625 (0.507)

TABLE 2. Posterior Mean Estimates of the Two Regime Model for the Short-Term Interest Rate. Posterior standard deviations are in parentheses.

	Posterior Mean	Standard Deviation
$u$	0.110	0.0278

TABLE 3. Posterior Mean and Standard Deviation Estimates of the Threshold Parameter for the Two Regime Model.

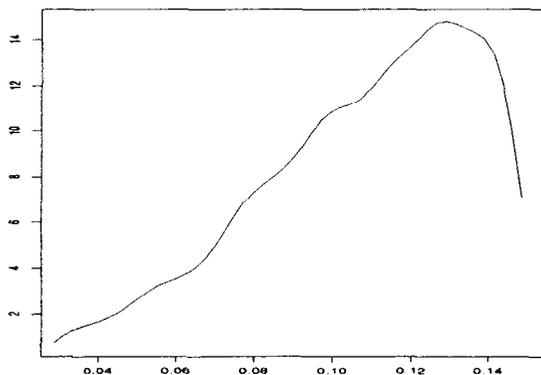


FIGURE 3. Posterior Density of  $u$  for the Two Regime Threshold Model.

The results for the three regime model are presented in Table 4 and Table 5 where Table 5 gives the estimated value of the threshold parameters  $u_1$  and  $u_2$ . A graph of the posterior density for  $u_1$  is shown in Figure 4 and a graph of the posterior density for  $u_2$  is shown in Figure 5.

<b>Regime I</b>	$\alpha$	$\beta$	$\sigma^2$	$\gamma$
	0.000143 (0.000190)	-0.00378 (0.00649)	0.000233 (0.000250)	0.733 (0.146)
<b>Regime II</b>	$\alpha$	$\beta$	$\sigma^2$	$\gamma$
	-0.000377 (0.00415)	0.00370 (0.0356)	0.00730 (0.0286)	0.981 (0.419)
<b>Regime III</b>	$\alpha$	$\beta$	$\sigma^2$	$\gamma$
	0.0223 (0.0218)	-0.155 (0.144)	0.000272 (0.00115)	0.287 (0.198)

TABLE 4. Posterior Mean Estimates of the Three Regime Model for the Short-Term Interest Rate. Posterior standard deviations are in parentheses.

	Posterior Mean	Standard Deviation
$u_1$	0.0713	0.0247
$u_2$	0.122	0.0210

TABLE 5. Posterior Mean and Standard Deviation Estimates of the Threshold Parameters for the Three Regime Model.

For estimation of the three regime model we imposed the prior restrictions:

**Restriction (a)**  $P(u_1 < u_2) = 1$ ,

**Restriction (b)**  $P(0.03 < u_1, u_2 < 0.15) = 1$ , and

**Restriction (c)**  $P(u_2 - u_1 > 0.02) = 1$ .

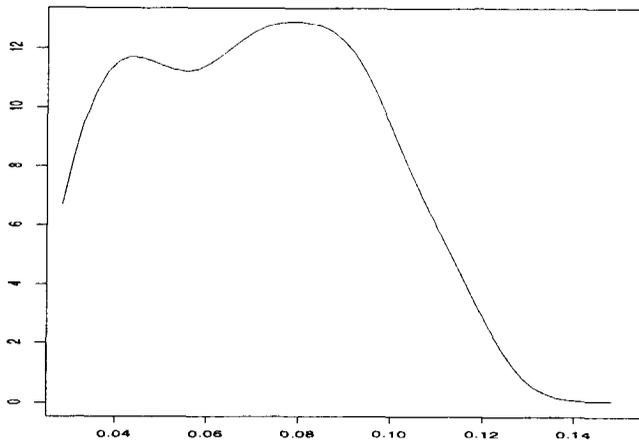


FIGURE 4. Posterior Density of  $u_1$  in the Three Regime Threshold Model.

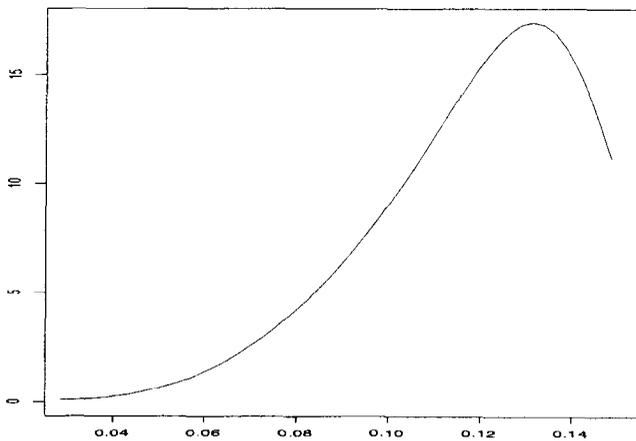


FIGURE 5. Posterior Density of  $u_w$  in the Three Regime Threshold Model.

The exact details of the estimation of the threshold model are based on dividing the data into the strata defined by the threshold levels as shown in Figure 6, which would apply to a data sample path for estimation in a three regime model.

If the indicated sample path were the data that was being used in the estimation then we would in effect be estimating three distinct processes for three separate blocks of data which are separated into the strata

**lower strata: B**

**middle strata: A and C**

**upper strata: D.**

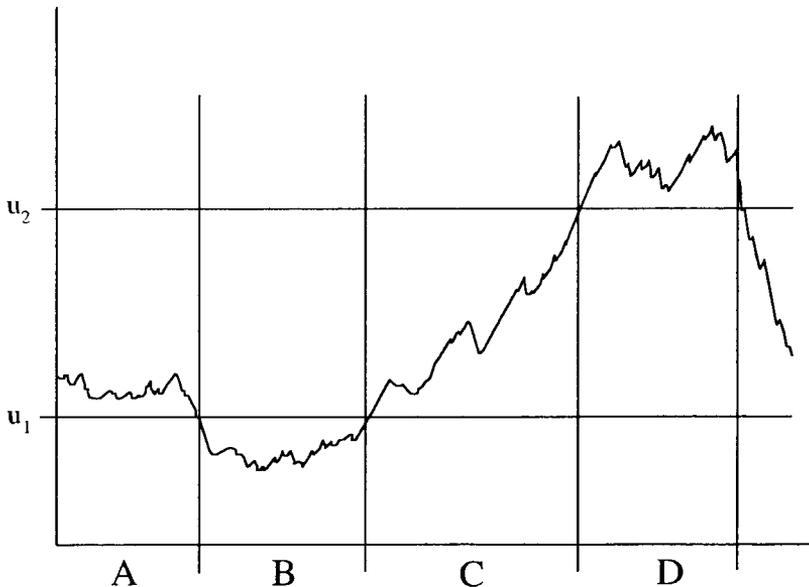


FIGURE 6. Data is partitioned into strata which are then estimated.

#### 4. CONCLUSIONS

In this paper we did not provide illustrations of the full term structure model based on a threshold model for the short-rate. As in all arbitrage-free models of the term structure, the default-free zero coupon bond prices are given by the equation:

$$P(t, T) = \mathbb{E}_t^{\mathbb{Q}} \left[ \exp \left( - \int_t^T r_u du \right) \right].$$

Monte Carlo simulation can be performed to evaluate this equation once the short-rate process is calibrated and an assumption is made about the market price of risk. The dynamical form in (SIM) is easiest to implement for the Monte Carlo simulation.

Some financial economists would argue that the mean reversion behaviour that the short-rate seems to exhibit when away from its historical range is an artifact of regime switches and that if one looks at shorter periods of data, as done in [8], then this strong mean reversion will disappear. This is an interesting and important econometric issue and at its heart is the question of whether more sophisticated modelling is an appropriate way to obtain models that can generate volatility changes like those that have been observed in the historical time series for the short-rate.

The work of [12] employs macroeconomic aspects of the term structure to investigate the behaviour of long-term and short-term interest rates. Real interest rates are linked to a model of the business cycle in this model. In order to obtain better models of the term structure, these structural relationships need to be investigated, understood, and modelled. Generally, it seems useful and important to extend the threshold behaviour of the type of model we have discussed to allow for thresholds that depend on the level of the inflation rate and other key macroeconomic variables. Related suggestions are made in [11]. Investigation of multi-factor threshold models involving the inflation rate will be the subject of future research.

## REFERENCES

- [1] Y. Ait-Sahalia (1992). Nonparametric pricing of interest rate derivative securities. *Econometrica* 64, 527-560.
- [2] Brown, S. and P. Dybvig (1986). The Empirical Implications of the Cox, Ingersoll, Ross Theory of the Term Structure of Interest Rates. *Journal of Finance* 41, 617-632.
- [3] Chan, K., Karolyi, G., Longstaff, F., and A. Saunders (1992). An Empirical Comparison of Alternative Models of the Short-Term Interest Rate. *Journal of Finance* 47, 1209-1227.
- [4] Cox, J., Ingersoll, J., and S. Ross (1985). A Theory of the Term Structure of Interest Rates. *Econometrica* 53, 385-407.
- [5] Cox, S. and H. Pedersen (1999). Nonparameteric Estimation of Interest Rate Term Structure and Insurance Applications. Proceedings of the 1999 ASTIN Colloquium, Tokyo, Japan (to appear).
- [6] Gray, S. (1996). Modeling the Conditional Distribution of Interest Rates as a Regime-Switching Process. *Journal of Financial Economics* 42, 27-62.
- [7] Hamilton, J. (1988). Rational Expectations Econometric Analysis of Changes in Regime: An Investigation of the Term Structure of Interest Rates. *Journal of Economic Dynamics and Control* 12, 385-423.
- [8] Lamoureux, C. and D. Witte (1999). CIR. Working paper: College of Business, University of Arizona
- [9] Naik, V. and M. Lee (1997). Yield Curve Dynamics with Discrete Shifts in Economic Regimes: Theory and Estimation. Working paper, University of British Columbia.
- [10] Pai, J. and H. Pedersen (1998). The Bayesian Approach to Regime Switching Models of the Short Interest Rate: Theory, Calibration, and Implementation. Working paper, Warren Centre for Actuarial Studies and Research, University of Manitoba.
- [11] Pfann, G., Schotman, P., and R. Tschernig (1996). Nonlinear Interest Rate Dynamics and Implications for the Term Structure. *Journal of Econometrics* 74, 149-176.
- [12] Roma, A. and W. Torous (1997). The Cyclical Behavior of Interest Rates. *Journal of Finance* 52, 1519-1542.
- [13] Stanton, R. (1997) A Nonparametric Model of Term Structure Dynamics and the Market Price of Interest Rate Risk. *Journal of Finance* 52, 1973-2002.
- [14] Tong, H. (1983). *Threshold Models in Non-linear Time Series Analysis*. Lecture Notes in Statistics volume 21, Springer-Verlag, New York.
- [15] Tong, H. (1990). *Non-Linear Time Series*. Oxford University Press, New York.
- [16] Vasicek, O.A. (1977). An Equilibrium characterization of the Term Structure. *Journal of Financial Economics* 5, 177-188.