Dynamic models for investment of pension fund assets

The Sundaresan-model approach

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Summary

1. Asset allocation decisions are a key aspect of pension fund management, given that more than 90% of pension fund returns are a function of asset allocation. A standard quantitative model for determining optimal asset allocation is the efficient-frontier approach. The traditional efficient-frontier approach entails establishing an asset allocation for a predetermined investment period, during which the allocation is not changed. However, in light of the long period of time over which pension fund assets are invested, changes in the structure of pension liabilities should not be ignored. Dynamic models for the investment of pension fund assets, which are currently being debated in Japan, take into consideration the changing liability positions of pension funds. The Sundaresan model, which is based on several assumptions, attempts to apply a dynamic asset allocation approach to ascertain the optimal allocation for pension assets.

2. Under the Sundaresan-model approach, two portfolios are constructed. One portfolio, called the surplus portfolio, is an optimally allocated portfolio for surplus pension fund assets. Because this portfolio is not impacted by liabilities, its allocation to stocks is determined by the risk tolerance of the fund. The second portfolio, called the hedge portfolio, hedges against changes in liabilities. This portfolio holds both stocks and bonds to ensure that pension fund assets do not fall below pension liabilities.

3. The Sundaresan model is based on the assumption that wages and stock prices are perfectly positively correlated. Because wages and stock prices are not perfectly correlated, there is no guarantee that portfolios generated using the Sundaresan model are optimal. In this report, we present a simulation of portfolio performance using the Sundaresan model based on the assumption that wages and stock prices are not perfectly correlated. Our simulation suggests that portfolio performance based on the Sundaresan model is superior to that based on a static model, even assuming that wages and stock prices are not perfectly correlated.

I. Introduction

Asset allocation decisions are a key aspect of pension fund management, given that more than 90% of pension fund returns are a function of asset allocation. Various methodologies are employed to ascertain appropriate asset allocations. A standard quantitative model for determining optimal asset allocation is the efficient-frontier approach. The traditional efficient-frontier approach entails establishing an asset allocation for a predetermined investment period, during which the allocation is not changed. However, in light of the long period of time over which pension fund assets are invested, changes in the structure of pension liabilities should not be ignored.

Dynamic models for the investment of pension fund assets are currently being debated in Japan. These models enable a fund manager to change asset allocation within the predetermined investment period in an attempt to achieve improved performance. The Sundaresan model, which is based on the assumption that stock prices and wages are perfectly positively correlated, attempts to apply a dynamic asset allocation approach to ascertain the optimal allocation for defined-benefit pension plans. The model attempts to limit the risk associated with pension liabilities while dynamically adjusting the optimal asset allocation for pension fund assets.

The Sundaresan model is based on the assumption that wages and stock prices are perfectly positively correlated. Because wages and stock prices are not perfectly correlated, there is no guarantee that portfolios generated using the Sundaresan model are optimal. In this report, we present a Monte Carlo simulation of portfolio performance using the Sundaresan model based on the assumption that wages and stock prices are not perfectly correlated. Our simulation suggests that portfolio performance based on the Sundaresan model is superior to that based on a static model, even assuming that wages and stock prices are not perfectly correlated.

In section II we discuss the differences between static and dynamic optimal asset allocation models. In section III, we examine the Sundaresan model, while in section IV we compare the Sundaresan model with a conventional model using a Monte Carlo simulation.

II. Dynamic models for investment of pension fund assets

Dynamic and static models are used to calculate optimal asset allocations. In this section we will discuss each of these models as well as their merits and demerits.

1. Static models versus dynamic models

A static model is based on mean-variance portfolio theory and is used to calculate an optimal portfolio allocation for a predetermined investment period. As shown in Exhibit II-1 (a), a static model does not vary asset allocation during the investment horizon. Markowitz developed a static model in the 1950s and following the introduction of the Markowitz model, Sharp and Linter introduced CAPM, which is now widely used for portfolio management.

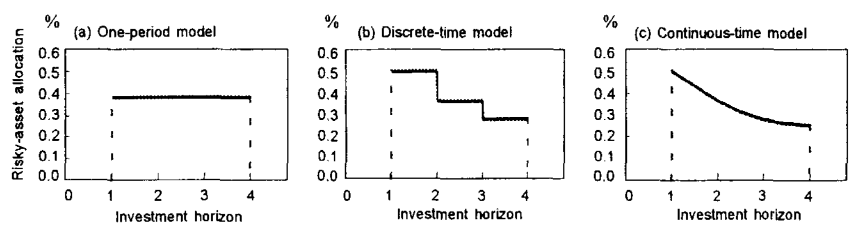
In contrast, dynamic models allow fund managers to change asset allocation over the investment horizon. Exhibit II-1 (b) and (c) are examples of dynamic asset allocation models.

The dynamic models illustrated in the exhibits below are based on the assumption that two asset classes are held, namely risk-free assets and risky assets. The Y axis depicts the allocation to risky assets. As shown, the dynamic models presented have different allocations for risky assets over time. Specifically, the optimal amount allocated to risky assets according to these models declines over time.

In general, there are two types of dynamic models, namely discrete-time models, as shown in Exhibit II-1 (b), and continuous-time models, as shown in Exhibit II-1 (c). Discrete-time models divide the investment time period into several time periods. During each period, the optimal asset allocation remains unchanged. Continuous-time models, in contrast, allow continuous changes to asset allocation.

Unlike static models, dynamic models have for the most part not been employed as a portfolio management tool. The main reasons for the lack of use include difficulties in understanding the models and calculation-related problems. That said, various solutions have been put forth for dealing with the complexities of dynamic models, resulting in increased use. The Sundaresan-model approach we discuss in this report is one example of a dynamic optimal asset allocation model.

II-1 Static model (a) versus dynamic models (b and c)



2. The importance of dynamic models

Dynamic models take into consideration changes in the investment climate, such as changes in expected risk and return. Thus, assuming that the key variables that impact the optimal asset allocation do not change, the allocation generated by static and dynamic models will be the same. However, assuming that key variables, such as the investment climate and the liability structure of a pension fund, change, the optimal asset allocation will also change, making the allocations derived from static and dynamic models different.

As mentioned, static models are the standard models used by most practitioners. This is not a problem as long as the allocations generated by static models are not significantly different from those generated by dynamic models. However, if the difference is large, the allocation generated by a static model may lead to incorrect investment decisions.

Dynamic models are especially important for managing pension fund assets. Many pension funds use a static model to generate an appropriate long-term optimal asset allocation. This allocation is subsequently tweaked every year to take into consideration current changes in the investment environment. Although this approach results in yearly allocation changes, it is essentially a repetitious implementation of a static one-year model. Thus, although this approach incorporates changes in key investment variables, such as expected returns, it does not incorporate longer-term expectations. In contrast, a dynamic model incorporates longer-term changes in expectations.

3. Pension fund management with dynamic models

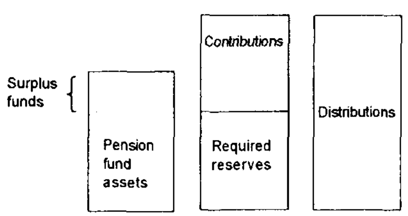
The question is how to apply a dynamic model to pension fund asset/liability management. Pension fund asset/liability management involves the management of pension funds from the perspective of both fund assets and pension liabilities.

One model that incorporates pension liabilities centers on surplus funds, which are pension fund assets minus required reserves, as shown in Exhibit II-2. A healthy pension fund has a positive surplus, but a pension fund can also have a negative surplus if losses are incurred in the management of the fund's assets or if required reserves increase owing to a reduction in the assumed rate of return or a rise in the rate of compensation growth. (Exhibit II-3) Under a pension fund asset/liability management model that focuses on surpluses, the key is to keep an eye on the level of surpluses and maintain an appropriate level of risk in managing pension fund assets so that the pension does not become underfunded.

As mentioned in the previous section, a dynamic model involves adjusting the optimal asset allocation in response to changes in investment opportunities. Such a model is particularly important for managing pensions because of changes in the timing of liabilities. The structure of a pension fund's liabilities varies depending on the maturity of the fund and changes in the level of compensation of pension-plan participants and can be modeled beforehand. Also, it is more important for pension fund managers than for short-term investors to consider such changes because the time horizon for pension funds is typically very long.

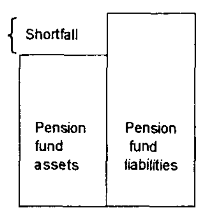
For these reasons, a dynamic model of pension fund asset/liability management is very significant. One such model is the one developed by Prof. Suresh Sundaresan and introduced briefly in the previous section.

II-2 Relationship of pension fund assets/liabilities and surplus funds



Source: Nomura

II-3 Pension liabilities exceed assets



Source: Nomura

III. The Sundaresan model

The Sundaresan model is a dynamic one that seeks the optimal asset allocation for a defined-benefit pension plan, based on changes in the pension fund's liabilities as the fund becomes more mature.

1. The assumptions of the model

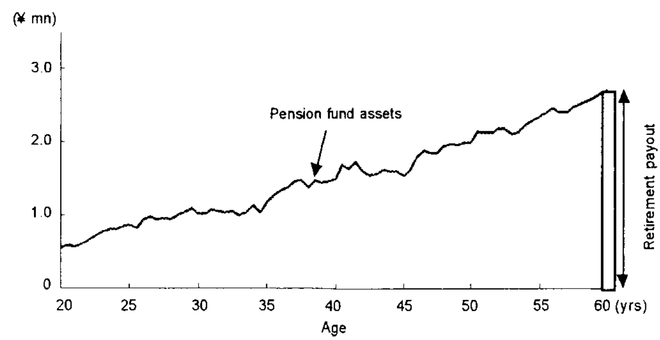
The Sundaresan model is based on several assumptions:

a. *A simplified pension plan.* Employees become eligible to join the pension plan when they are hired and upon retirement they receive their pensions as lump-sum payments, the amounts of which are based on their levels of past compensation. The Sundaresan model seeks the optimal asset allocation based on the ages of the plan participants. Under this simplified model, it is assumed that the plan is funded when an employee becomes a participant under the plan. In other words, the model does not consider income from the contributions put into the plan; a sum equal to the retirement benefits is paid in at once when an employee joins the plan. Exhibit III-1 provides an outline of the simplified model. Employees become participants in the plan when they are hired, at the age of 20, at which time companies pay in the present value of the entire amount of the employees' retirement benefits. Then when employees retire at the age of 60, their retirement benefits are paid out of these funds.

b. *Only two asset classes.* Pension assets are invested in stocks and bonds only. Stocks correspond to risky assets and bonds to risk-free assets. The yield on bonds is set for the duration of the model, so there is no distinction between short- and long-term bonds.

c. *A perfect correlation between wages and stock prices.* The model assumes a perfect correlation between wages and stock prices. Although this assumption is not realistic, changes in stock prices reflect the productivity of companies and changes in wages also reflect the productivity of companies. In the Sundaresan model, wages and stock prices are treated as random variables that exhibit Brownian motion and have a correlation coefficient of 1.

III-1. Change in pension fund assets for one employee under Sundaresan model's simplified plan



2. A model of the liabilities structure

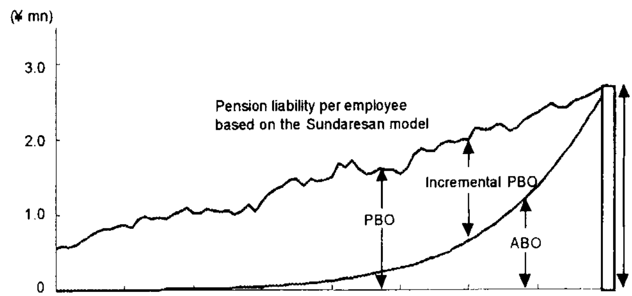
Two methods for calculating pension liabilities are accumulated benefit obligations (ABO) and projected benefit obligations (PBO). Whereas ABO is the present value of future pension benefit obligations based on current wage levels, PBO is based on an estimate of wage increases up until retirement. Because ABO does not factor in wage increases, it is typically a smaller amount than PBO and underestimates the true pension benefit obligations. For risk management purposes, then, PBO is the preferred estimate of pension liabilities.

The Sundaresan model bases the structure of pension liabilities on PBO. Like the percentage change in stock prices, the percentage change in wages is random, and so the present value of pension liabilities can be derived from option theory. Hence, P(t), or the value of the pension liability when an employee is at age t, is as follows:

$$P(t) = PBO(t) = ABO(t) + \text{incremental } PBO(t)$$

The first term on the right side of the equation corresponds to ABO, which is independent of changes in wages. The second term represents the difference between PBO and ABO and is the portion of the future benefit obligations stemming from expected future increases in wages. Thus, the estimate of the incremental PBO changes as wages change. Exhibit III-2 shows a typical sample path based on a calculation of the liability amount as a random variable. The exhibit shows the change in the pension liability for an employee from the ages of 20 to 60. ABO as a percentage of the total pension liability increases as the employee approaches the retirement age of 60, since ABO is the portion of the liability determined at that time, without any consideration of future wage increases. The incremental PBO as a percentage of the total liability, meanwhile, decreases toward retirement age since PBO is the portion that accounts for future wage increases. At retirement, the incremental PBO is zero, since the pension amount is completely fixed.

III-2. Change in pension liability for one employee with liability amount calculated as a random variable

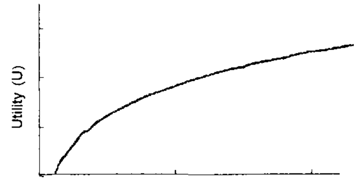


3. Optimal pension fund asset allocation

In the previous subsection, the value of the pension liability for an employee at age t was estimated. The surplus at age t is defined as pension fund assets less pension liabilities at time t. The optimal asset allocation is the one that maximizes the ultimate surplus; at the least, the asset allocation should be maintained such that the surplus does not turn into a deficit during the investment period. With a focus on surpluses, a portfolio that considers the structure of pension liabilities can be constructed.

In the Sundaresan model, the variable for the utility function is the surplus. When the surplus is positive, the utility function is the type of function shown in Exhibit III-3; the higher the surplus, the higher the utility. In addition, as the surplus approaches zero, the utility approaches zero (when $\gamma=1$ it is negative infinity). Thus, the Sundaresan model is similar to portfolio insurance, with the floor for the surplus set at zero.

III-3. Utility function



$$U(S) = \begin{cases} \dfrac{S^{1-\gamma}}{1-\gamma} & (\gamma > 0,\ \gamma \neq 0) \\ \log S & (\gamma = 1) \end{cases}$$

(γ: risk aversion)

Source: Nomura

To analytically arrive at the pension fund asset allocation with the maximum expected utility, Merton's continuous-time optimization method is used. The optimal asset allocation at an employee's age t can thus be described as the sum of a surplus portfolio at time t and a hedge portfolio at time t:

Optimal asset allocation (t) = surplus portfolio (t) + hedge portfolio (t)

These two portfolios each have a component that hedges against changes in the pension liability and a component that corresponds to the surplus amount. The surplus portion is managed using the conventional optimal allocation that considers only assets and not the impact of the pension liability on the optimal allocation. On the other hand, the portion that hedges against changes in the pension liability is managed so that the pension fund assets change in line with the value of the pension liability.

Based on Exhibit III-4, we explain the two types of portfolios as follows:

The surplus portfolio. The amount of this portion allocated to stocks is determined on the basis of risk tolerance, since pension liabilities do not have to be considered.

The hedge portfolio. This portion hedges against changes in pension liabilities. These liabilities consist of an ABO component and an incremental-PBO component, so the hedge portfolio can be regarded as having two parts:

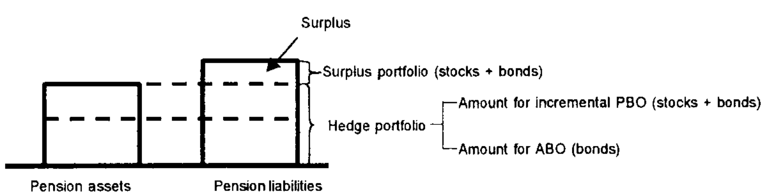
The ABO component: Since ABO is a set amount not affected by changes in wages, the portion of the portfolio for meeting ABO can be invested entirely in bonds.

The incremental-PBO component: The incremental-PBO amount changes depending on the rise in wage levels, so the rise in pension liabilities can be met by investing a certain proportion in stocks, in line with the company's earnings.

With the Sundaresan model, therefore, risk assets that need to be held are divided into a surplus portion and a pension-duplication portion. However, this result requires the bold assumption that stock prices and pension liabilities move together in perfect correlation.

In the following section of this paper, we discuss our simulations using the Sundaresan model and compare the model with a static model. We conclude that the Sundaresan model is still valid even if we try to create more realistic conditions by relaxing some of the original assumptions.

III-4. Breakdown of pension fund assets



IV. Simulations to verify the Sundaresan model

The Sundaresan model is based on the assumption of a perfect positive correlation between wages and stock prices. But because this assumption does not hold in reality, portfolios generated by the model are not guaranteed to be the optimal ones. In this section, we use simulations to compare the investment performance of portfolios based on the Sundaresan model and those based on a conventional static model.

1. Simulation methodology

The key characteristic of the Sundaresan model is that it optimizes asset allocation based on changes in pension liabilities. Considering this point, we conducted Monte Carlo simulations of the model under the following conditions:

* The investment horizon was set at 40 years, from the date of hire at the age of 20 to retirement at 60. The simulation was of a simplified pension plan.

* Income from contributions was not considered. The amount of the future pension benefit was considered paid in when the employee was at the age of 20, so that the initial surplus was zero.

To generate the optimal asset allocation, the Sundaresan model assumes, as noted earlier, that wages and stock prices are perfectly positively correlated. We also did a simulation of more realistic conditions by eliminating this assumption and then investing based on the optimal allocation generated by the model. Specifically, in addition to doing a simulation using a correlation coefficient (rho) for wages and stock prices of 1, we also ran a simulation using a correlation coefficient of 0. When rho was not equal to 1, the hedge portfolio was unable to completely hedge against changes in pension liabilities, resulting in a negative surplus. In these cases, we used an asset allocation that resulted in a surplus of zero.

Other assumptions we used in the simulations were as follows:

* Stocks: annual returns 6.5%, risk 19.8%

* Bonds: annual yield 4%

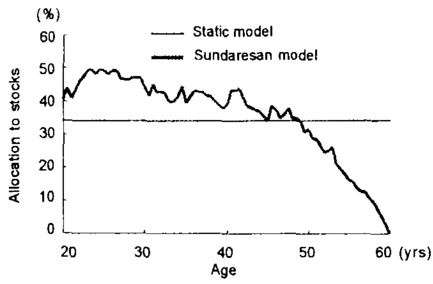
* Monte Carlo simulation frequency: 3,000 times each for the correlation coefficients for wages and stock prices of 0 and 1.

2. A typical sample path

We thus ran the simulation 6,000 times for a simplified pension plan. Here we look at a typical sample path.

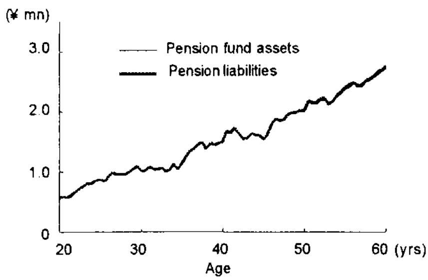
First, Exhibit IV-1 shows the classic change in the stock allocation over time when the correlation coefficient is 0. The stock allocation declines as retirement approaches because the incremental PBO shrinks under the Sundaresan model. If the surplus is negative, the stock allocation at the age of 60 is zero. In contrast, under a static model, the stock allocation is fixed for the entire time horizon.

IV-1. Change in allocation to stocks (rho = 0)

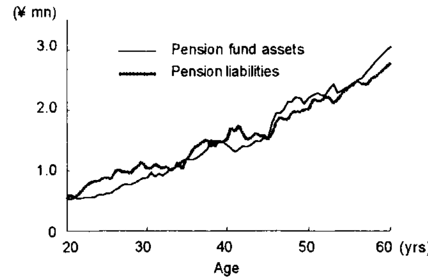


Exhibits IV-2 and IV-3 show the changes in pension fund assets and liabilities over time for the same sample. In Exhibit IV-2, rho is 1 and pension fund assets and liabilities move in lock step. The initial surplus is zero and assets and liabilities are perfectly hedged. In Exhibit IV-3, rho is 0 and thus movements in wages and stock prices are completely unrelated. In this case, pension liabilities exceed pension fund assets at the time of retirement, resulting in an underfunded plan.

IV-2. Change in pension fund assets (rho = 1)



IV-3. Change in pension fund assets (rho = 0)

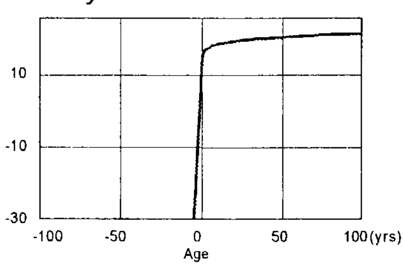


Note: Above data is for one employee

3. *Comparison with a static model*

We compared the Sundaresan model with a static model on the basis of their expected utilities. The utility function must first be defined, but in the case of a negative surplus under the Sundaresan model, the utility function is not defined. Instead, we used a utility function combining a nonlinear function and a straight line extended from the point at which the surplus is close to zero (Exhibit IV-4).

IV-4. Utility function



Based on this utility function, we calculated the expected utility E(U) as the average utility generated by the 3,000 Monte Carlo simulation runs we did for each of the values of rho—0 and 1—for wages and stock prices. For the static model, we calculated the expected utility using stock allocations from 0% to 100%.

We thus define the change in expected utility, delta E(U), as follows:

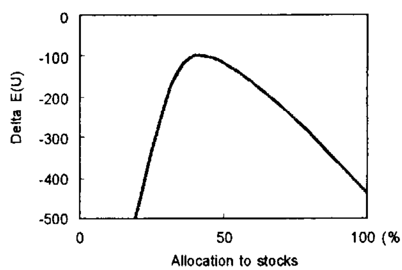
delta E(U) = E(U) static model - E(U) Sundaresan model

When delta E(U) is less than zero, the expected utility of the Sundaresan model is greater, and when delta E(U) is greater than zero, the expected utility of the static model is higher. When the correlation coefficient of wages and stock prices was set at 1, the expected utility of each model and delta E(U) were as shown in Exhibits IV-5 and IV-6. The X axis of the graphs is the stock allocation of the static model; for any stock allocation value, delta E(U) was less than zero. We next calculated delta E(U) when the correlation coefficient was set at 0. As shown in Exhibit IV-7, for any stock allocation value between 0% and 100%, delta E(U) was less than zero. For a correlation coefficient of 1 or 0 and a utility function as shown in Exhibit IV-4, the Sundaresan model had a higher expected utility than a static model.

IV-5. Expected utilities of the two models

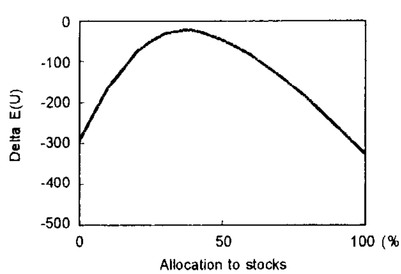
	Allocation to stocks: W %	E(U)
Sundaresan model	-	13
Static model	0 20 40 60	-1007 -465 -87 -151

IV-6. Delta E(U) (rho = 1)



Note: We progressively increased the allocation to stocks by 2 percentage points and calculated the impact on delta E(U).

IV-7. Delta E(U) (rho = 0)



Note: We progressively increased the allocation to stocks by 2 percentage points and calculated the impact on delta E(U).

V. Conclusion

When needs change over time, asset management based on a conventional static model is not necessarily the optimal approach, particularly in the case of pension funds. The optimal pension fund asset allocation model introduced in this paper, developed by Prof. Sundaresan, is a dynamic one for figuring out the optimal stock allocation as PBO changes over time.

However, one of the core assumptions of the model is that wages and stock prices are perfectly positively correlated. In this paper, we used a Monte Carlo simulation to show that the Sundaresan model is valid even without this assumption of a perfect positive correlation between wages and stock prices. We found that the expected utility of the model is always high, no matter what the assumed correlation between wages and stock prices.

Although some aspects of the Sundaresan model still need work—such as the model of asset returns and the simplified liabilities structure—it is significant in that it addresses the biggest weakness of conventional asset management models, namely, a failure to dynamically incorporate changes in pension liabilities. Such a dynamic approach to asset allocation is likely to become important in Japan as well.

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