Increasing Markovian Families in Multi-Factor Heath-Jarrow-Morton Models And Pricing Credit Derivatives

Nobuhiro Nakamura *
MTB Investment Technology Institute, Co., Ltd.
5-6, SHIBA 2-CHOME,MINATO-KU, TOKYO 105, JAPAN
Tel: +81-03-3457-1977
Fax: +81-03-3457-0910
e-mail: nob_nkmr@mxc.mesh.ne.jp

Abstract

This paper studies the Markovian framework of the multi-factor Heath-Jarrow-Morton (HJM) model of the interest rate processes to incorporate increasing Markovian families associated with the finite (say, p-th) order ordinary differential equations ($p = 1, 2, \cdots$), which is a more general extension of earlier works such as Bhar-Chiarella(1997), Inui-Kijima(1998) and so on. We reveal the allowable volatility functions, the number of state variables and the resulting Markovian expression of the discount bond price. As a viable application we demonstrate the pricing of a total-rate-of-return swap among a variety of popular credit derivatives, modelling the hazard rates for relevant defaultable entities by those multi-factor Markovian type stochastic processes of HJM model.

Keywords: Heath-Jarrow-Morton models, term structure dynamics, Markovian models, credit derivatives

JEL Classification: G12, G13

1 Introduction

In this paper we study the sufficient conditions such that the multi-factor Heath-Jarrow-Morton (HJM) model (1992)[11] for the forward interest rate process with more general stochastic volatility functions belongs to the Markovian framework. It is well known that, in general, the evolution of the spot interest rate process in the HJM framework is not Markovian, as discussed in, for example, Carverhill(1994) [4] within one-factor HJM model.

*The analysis and conclusions set forth are those of the author and do not indicate concurrence by MTEC. Of course, the author is soley responsible for all remaining errors.

Among earlier works of the Markovian framework of HJM model Inui-Kijima(1998)[12] and Bhar-Chiarella(1997)[2] considered the multi-factor formulations of HJM model, in which they assumed that the volatility function has an ad hoc structure of some bounded measurable function of the spot interest rate times an exponential damping factor with respect to the time-to-maturity, or that further multiplied by some deterministic polynomial of the time-to-maturity, respectively. Our primary concern is to shed light on a sequential structure of the volatility functions for which we introduce ordinary differential equations with the appropriate (say p-th) order and to give insight into an increasing number of families of the HJM framework leading to the finite dimensional Markovian systems, varying the order $p = 1, 2, \cdots$. In this respect our framework might be thought of as a naive extension of Inui-Kijima(1998)[12] and Bhar-Chiarella(1997)[2] in a way to synthesize the extant other literature on Markovian HJM modelling.

For the purpose of empirical research we provide useful Markovian expressions of the discount bond price in terms of appropriate numbers of state variables, which allow us to price most of European type contingent claims over the term structure. That is the case even if the valuation problem is of rather high-dimensionality, when we employ the recently developed quasi-Monte-Carlo simulation technique, using some generalized low-discrepancy sequences (LDS), as demonstrated in Ninomiya-Tezuka(1996)[17]. Similarly to Bhar-Chiarella(1997)[2] and de Jong-Santa-Clara(1999)[6] our formulation can be transformed to the state space representation which enables us to perform the estimation of relevant stochastic processes by means of the Kalman filtering technique and so forth.

The second aim of this paper is to apply this Markovian framework of HJM model to the hazard rate processes according to the reduced-form-approach of Duffie-Singleton(1996)[10], and to confirm its validity of pricing credit derivatives such as default swaps, total-rate-of-return(TROR) swaps and so on. We shall provide useful lemmas about the conditional expectations of the state variables which can be repeatedly used for the analytical calculation of almost credit derivatives, and derive a (nearly) closed-form expression of the TROR swap, as an illustrative example, with a specific but fairly general covenant, using a minimal Markovian type HJM modelling of the hazard rate processes of relevant defaultable entities and default-free interest rate process.

The paper is organized as follows. Section 2 presents the Markovian framework of HJM model, section 3 deals with interest rate and hazard rate processes for pricing credit derivatives, and subsequently gives closed-form expressions of popular credit derivatives in real financial markets. Final section is devoted to summary and concluding remarks.

2 Markovian Framework of HJM Model

2.1 General description

First of all we provide a short description of the multi-factor HJM model for the notational convenience. The starting point of the HJM term structure model of interest rates is the stochastic differential equation(SDE) of the instantaneous forward interest rate:

$$df(t,T) = \mu(t,T)dt + \sum_n \gamma_n(t,T)dW_n^0(t), \quad 0 \le t \le T, \tag{1}$$

where the instantaneous drift $\mu(t,T)$ and volatility $\gamma_n(t,T)$ at time t of the forward interest rate with maturity date T ($\mu, \gamma_n : [0,\infty) \times \mathbf{R}^+ \to \mathbf{R}$) are jointly measurable and satisfy some regularity conditions as shown in HJM(1992)[11], and $W^0 = (W_1^0, \cdots, W_N^0)$ is an N-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P}^0)$, with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which is the $\mathbf{P}^0$-augmentation of the natural filtration $\mathcal{F}^{W_t^0} := \sigma\{W_s^0 : 0 \leq s \leq t\}$ generated by W^0.

We assume that there is no arbitrage opportunity in the complete bond market. Let the money market account be denoted by

$$B(t) := e^{\int_0^t r(s)ds}. \tag{2}$$

Under some regularity conditions HJM prove that there exists a unique equivalent martingale measure P under which a new process $W(t)$ defined by $W^0(t)$ through the market price of risk $\phi(t)$ of the forward interest rate becomes P-Brownian motion and the default-free discount bond price process,

$$P(t,T) := \exp\Big(-\int_t^T f(t,s)ds\Big), \quad 0 \leq t \leq T \tag{3}$$

divided by $B(t)$ is a P-martingale. Under P-measure the SDE of the forward interest rate becomes

$$df(t,T) = \sum_n [b_n(t,T) + \gamma_n(t,T)dW_n(t)], \tag{4}$$

or, in the integrated form,

$$f(t,T) = f(0,T) + \sum_n [\int_0^t b_n(s,T)ds + \int_0^t \gamma_n(s,T)dW_n(s)], \tag{5}$$

where

$$b_n(t,T) = \gamma_n(t,T)\int_t^T \gamma_n(t,u)du. \tag{6}$$

The spot rate is written as

$$\begin{aligned} r(t) &= f(0,t) + Y_0, \\ dr(t) &= [f^{(1)}(0,t) + Y_1 + \Gamma_{00}]dt + dX_0(t), \end{aligned} \tag{7}$$

where we let the superscript (a) denote the a-th order partial derivative of the function with respect to the second argument such as $\gamma^{(a)}(t,T) = \partial^a\gamma(t,T)/\partial T^a$, and introduced several notations:

$$\begin{aligned} \Gamma_{ab}^{mn}(t) &= \int_0^t \gamma_m^{(a)}(s,t)\gamma_n^{(b)}(s,t)ds, \\ \hat{\Gamma}_a^{mn}(t) &= \int_0^t \{\gamma_m^{(a)}(s,t)\int_s^t \gamma_n(s,u)du\}ds, \\ X_a^n(t) &= \int_0^t \gamma_n^{(a)}(s,t)dW_n(s), \\ Y_a^n(t) &= \hat{\Gamma}_a^{nn}(t) + X_a^n(t), \\ X_a &= \sum_n X_a^n, \\ Y_a &= \sum_n Y_a^n. \end{aligned} \tag{8}$$

The discount bond prices are given by

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\exp\Big\{-\sum_n\Big[\int_t^T\int_0^t b_n(s,u)dsdu + \int_t^T\int_0^t \gamma_n(s,u)dW_n(s)du\Big]\Big\}. \tag{9}$$

Applying Ito's formula, we have

$$\begin{aligned}
d\Gamma_{ab}^{mn}(t) &= \gamma_m^{(a)}(t,t)\gamma_n^{(b)}(t,t)dt + \Gamma_{a+1b}^{mn}(t)dt + \Gamma_{ab+1}^{mn}(t)dt, \\
d\hat{\Gamma}_a^{mn}(t) &= \hat{\Gamma}_{a+1}^{mn}(t)dt + \Gamma_{a0}^{mn}(t)dt, \\
dX_a(t) &= \sum_n \gamma_n^{(a)}(t,t)dW_n(t), \\
dY_a^n(t) &= d\hat{\Gamma}_a^{nn}(t) + dX_a^n(t) \\
&= \hat{\Gamma}_{a+1}^{nn}(t)dt + \Gamma_{a0}^{nn}(t)dt + \gamma_n^{(a)}(t,t)dW_n(t).
\end{aligned} \tag{10}$$

As it is often useful to change the equivalent martingale measure from P to the so-called forward-measure $Q(T)$ for which the numéraire is changed from a money market account $B(t)$ to a discount bond price $P(t,T)$ with an appropriate future maturity date T:

$$\begin{aligned}
\frac{dQ(T)}{dP} &:= Z^T(t) = \mathcal{E}_t\Big(\int_0^{\cdot} \sigma_P(u,T)dW(u)\Big), \quad 0 \le t \le T \\
Q(T)[A] &:= \mathbf{E}[Z^T(t)\mathbf{1}_{\{A\}}], \quad A \in \mathcal{F}(t), \\
W^{Q(T)}(t) &= W(t) + \int_0^t \sigma_P(u,T)du,
\end{aligned} \tag{11}$$

where $\mathcal{E}_t(\cdot)$ stands for the Doléans-Dade exponential and

$$\sigma_{Pn}(t,T) := -\int_t^T \gamma_n(t,u)du. \tag{12}$$

Under $Q(T)$-measure the above equalities associated with $X_a^n(t)$ are modified slightly as

$$\begin{aligned}
X_a^n(t) &= \int_0^t \gamma_n^{(a)}(s,t)dW_n^{Q(T)}(s) - \int_0^t \{\gamma_m^{(a)}(s,t)\int_s^T \gamma_n(s,u)du\}ds, \\
dX_a(t) &= \sum_n \gamma_n^{(a)}(t,t)dW_n^{Q(T)}(t) - \sum_n \{\gamma_n^{(a)}(t,t)\int_t^T \gamma_n(t,u)du\}dt.
\end{aligned} \tag{13}$$

Note that the coefficients of the r.h.s. of $dX_a(t)$ are still Markovian processes under $Q(T)$-measure, so that the subsequent argument shown later holds even for a change of measure of this sort.

2.2 Increasing Markovian families

In this subsection we introduce a concept of the *p-th minimal* Markovian framework. We notice at first glance that the SDE's of Eqs.(7),(10) are not closed system. Therefore, in order to make those Markovian, it is necessary to introduce an infinite number of state variables, that is evidently intractable and computationally inefficient. To remove the disadvantage, we suppose that up to p-th order partial derivatives, $\gamma_n^{(0)}, \cdots, \gamma_n^{(p)}$ satisfy an ordinary differential equation

$$\gamma^{(p)}(t,T) = \sum_{a=0}^{p-1} c_a(T)\gamma^{(a)}(t,T) \tag{14}$$

with, for example, appropriate boundary conditions

$$\begin{aligned} \gamma(t,t) &= \sigma_0(r(t),t), \\ \gamma^{(1)}(t,t) &= \sigma_1(r(t),t), \\ &= \cdots \\ \gamma^{(p-1)}(t,t) &= \sigma_p(r(t),t). \end{aligned} \tag{15}$$

This additional condition results in a closed SDE system of the interest rates, i.e.,Markovian framework by expanding the state space with a certain truncated finite number of state variables that have a role of accumulating the information of the path of the whole term structure. We know that this differential equation (14) admits some solutions of certain general form according to the given specifications. Consider, e.g., the constant coefficient case, i.e., $c_a(T) = c_a$. Suppose that the characteristic equation associated with this ODE has K real roots, $\lambda_1, \cdots, \lambda_j, \cdots, \lambda_K$ (their multiplicities are denoted by m_j, $j = 1, \cdots, K$) and $(M - K)$ imaginary roots $\lambda_{K+1}, \cdots, \lambda_j, \cdots, \lambda_M$ (their multiplicities are denoted by m_j, $j = K+1, \cdots, M$, and $\lambda_j = \alpha_j + \sqrt{-1}\beta_j$) and $\sum_{j=1}^{M} m_j = p$. Let δ_j stand for any phase determined by the boundary condition. Then the solution $\gamma(\cdot, T)$ is proportional to a linear combination of the following functions depending on T:

$$\begin{aligned} &T^{m_j-1}e^{\lambda_j T} \quad \text{for } j = 1, \cdots, K, \\ &T^{m_j-1}e^{\alpha_j T}\cos(\beta_j T + \delta_j) \quad \text{for } j = K+1, \cdots, M. \end{aligned} \tag{16}$$

Note that the formulation of Bhar-Chiarella(1997)[2] does not incorporate a possibility of solutions with imaginary roots. In other words, our formulation extends Bhar-Chiarella(1997)[2] so as to allow for somewhat general volatility structure including the above as a particular specification. In that sense our treatment of Markovian framework based upon the p-th order ODE would be more comprehensive than those.

In our formulation we find that the solution of the p-th order more general ODE is of a generic form:

$$\gamma_n(t,T) = \sigma_n(r(t),t) f_n(T) \tag{17}$$

with $f_n(T)$ being any function with respect to T like Eq.(16). It is also understood within the volatility structures following Eq.(14) that when $p > 1$ there are $(2 + p + p(p+1)/2)N$ states of (X_0, X_1; $\hat{\Gamma}_a$ $(a = 0, 1, \cdots, p-1)$; Γ_{ab} $(a, b = 0, 1, \cdots, p-1$, symmetric with respect to a,b)). Let this class based upon a p-th order ODE be denoted by $\mathcal{D}_p$. Then it turns out from the above consideration that there arises an increasing number of families of Markovian framework, $\mathcal{D}_1 \subset \mathcal{D}_2 \subset \cdots$.

In order to motivate the usefulness of our formulation of Markovian framework let us consider some heuristic examples of the lower order. First we demonstrate the minimal case, $\mathcal{D}_1$, which is just identical to the volatility specification, as studied in Inui-Kijima(1998)[12]. The case of $p = 1$ in Eq.(14) is that

$$\gamma_n^{(1)}(t,T) = -\kappa_n(t)\gamma_n(t,T), \tag{18}$$

for which $Y_1 = \hat{\Gamma}_1 + X_1 = -\sum_n \kappa_n(t)(\hat{\Gamma}_0^{nn} + X_0^n) = -\sum_n \kappa_n(t) Y_0^n$, $\Gamma_{10} = -\sum_n \kappa_n(t)\Gamma_{00}^{nn}$. Hence one needs $2N$ state variables, $(Y_0^n, \Gamma_{00}^{nn})$ instead of $3N$ states $(\hat{\Gamma}_1^{nn}, X_1^n, \Gamma_{00}^{nn})$, which is due to a remarkable reduction of state variables by pairing $\hat{\Gamma}_0^{nn}$ and X_0^n together. That may be interpreted as a peculiarity of order one case. The simple form of ODE(18) yields a solution

$$\gamma_n(t,T) = \sigma_n(t, r(t)) e^{-\int_t^T \kappa(u)du}. \tag{19}$$

Next let us proceed to a second-minimal case ($p = 2$). It is important to stress that the significance of our formulation manifests itself rather considerably in the case of order two, the ODE of which is given by [1]

$$\gamma^{(2)}(t,T) = \sum_{a=0}^{1} c_a(T)\gamma^{(a)}(t,T). \tag{20}$$

This admits a solution of the general form

$$\gamma_n(t,T) = \sigma_n(t,r(t))e^{-\int_t^T \kappa(u)du}\cos(\omega(T-t)+\delta), \tag{21}$$

which allows us to easily incorporate a humped term structure curve of the volatility function, although there is no specific reduction of state variables in this case, differently from the foregoing order one case. Indeed, $dY_1 = \hat{\Gamma}_2 dt + \Gamma_{10}dt + dX_1 = \sum_{a=0}^{1} c_a\hat{\Gamma}_a + \Gamma_{10}dt + dX_1$. Hence we need $7N$ state variables of $(X_0^n, X_1^n; \hat{\Gamma}_0^{nn}, \hat{\Gamma}_1^{nn}; \Gamma_{00}^{nn}, \Gamma_{10}^{nn}, \Gamma_{11}^{nn})$. In this way we can continue this procedure up to the preferable order so as to fit more complex volatility term structures.

Now let us move on to the Markovian expressions of the discount bond price. Since the p-th order ODE of Eq.(14) has a solution of the generic form (17), one has for $t \le s \le T$

$$\gamma_n(t,T) = \gamma_n(t,s)\frac{f_n(T)}{f_n(s)} \equiv \gamma_n(t,s)F_n(s,T). \tag{22}$$

Let

$$G_n(t,T) \equiv \int_t^T F_n(t,u)du. \tag{23}$$

In Eq.(9) we have

$$\begin{aligned}
&\int_t^T b_n(s,u)du \qquad (24)\\
&= \int_t^T \gamma_n(s,t)F_n(t,u)\left[\int_s^t \gamma_n(s,v) + \int_t^u \gamma_n(s,t)F_n(t,v)dv\right]\Big\}du\\
&= G_n(t,T)b_n(s,t) + \frac{1}{2}G_n^2(t,T)\gamma_n^2(s,t),
\end{aligned}$$

so that

$$\begin{aligned}
\int_0^t \left\{\int_t^T b_n(s,u)du\right\}ds &= G_n(t,T)\int_0^t b_n(s,t)ds + \frac{1}{2}G_n^2(t,T)\int_0^t \gamma_n^2(s,t)ds \qquad (25)\\
&= G_n(t,T)\hat{\Gamma}_0^n(t) + \frac{1}{2}G_n^2(t,T)\Gamma_{00}^n(t).
\end{aligned}$$

While, from Eq.(22)

$$\begin{aligned}
\int_t^T \int_0^t \gamma_n(s,u)dW_n(s)du &= \int_0^t \left\{\int_t^T \gamma_n(s,t)F_n(t,u)du\right\}dW_n(s) \qquad (26)\\
&= G_n(t,T)X_0^n(t).
\end{aligned}$$

[1] If one takes a certain series of orthogonal polynomials, for example, confluent hypergeometric function, $L_n^{(\alpha)}$ (See Abramowitz-Stegun(1970) [1]) such as $\gamma_n(t,T;a_m) = \sigma_n(t)\sum_m a_m L_m^{(\alpha)}(T)$, then it satisfies

$$\gamma_n^{(2)}(t,T;a_m) = -\gamma_n^{(1)}(t,T;\frac{\alpha+1-T}{T}a_m) - \gamma_n^{(0)}(t,T;\frac{m}{T}a_m).$$

Therefore those above lead to the proposition.

Proposition 1
The default-free discount bond price, Eq.(9), is expressed in terms of $G(\cdot,\cdot)$ *as*

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\Big[-\frac{1}{2}\sum_n G_n^2(t,T)\Gamma_{00}^n(t) - \sum_n G_n(t,T) Y_0^n(t) \Big]. \tag{27}$$

If we delete one of X_0^n*'s, e.g.,* X_0^N*, using the spot rate equation (7),* [2] *then the above equality is rewritten as*

$$\begin{aligned} P(t,T) &= \frac{P(0,T)}{P(0,t)} \exp\Big[-\frac{1}{2}\sum_n G_n^2(t,T)\Gamma_{00}^n(t) - \sum_n G_n(t,T)(\hat{\Gamma}_0^{nn} + X_0^n) \Big] \quad (28) \\ &= \frac{P(0,T)}{P(0,t)} \exp\Big[-\frac{1}{2}\sum_n G_n^2(t,T)\Gamma_{00}^n(t) - \sum_{n=1}^{N-1} G_n(t,T) Y_0^n(t) \\ & \quad - G_N(t,T)\Big(r(t) - f(0,t) - \hat{\Gamma}_0(t) - \sum_{n=1}^{N-1} X_0^n(t) + \hat{\Gamma}_0^{NN}(t) \Big) \Big]. \end{aligned}$$

Based upon the SDE's of relevant state variables, one can easily simulate their future values along the forward direction with the aid of the Markovian nature, and thereby obtain the future discount bond prices appearing in the valuation of contingent claims. To complete this section we remark that there is a volatility condition under which the forward interest rate starting in a certain domain continues to stay within such a domain as the time passes. [3]

3 Interest Rate and Hazard Rate Processes

In this section we take the Duffie-Singleton's(1996)[10] reduced-form-approach, establish the modelling of hazard rate processes in terms of the above-mentioned Markovian HJM model, and attempt to value a typical credit derivative among a variety of popular ones. [4] To this end it is usually necessary to specify the relevant stochastic processes. The first subsection presents a

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[2] That was conducted in Inui-Kijima(1998)[12].

[3] It can be stated *a la* Kusuoka(1997) [14]. *Let* V_1^p *be a finite dimensional vector space of* $C^\infty([0,\infty);\mathbf{R})$ *spanned by basis* γ *such that when* $\gamma(T) \in V_1^p, \cdots, \partial^{p-1}\gamma/\partial T^{p-1} \in V_1^p$, $\partial^p\gamma/\partial T^p \in V_1^p$*. Let* V_2^p *be a finite dimensional vector space of* $C^\infty([0,\infty);\mathbf{R})$ *spanned by basis* $\gamma_1(T)\gamma_2(T), \cdots, \gamma_1^{(a)}(T)\gamma_2^{(b)}(T), \cdots, \gamma_1^{(p)}(T)\gamma_2^{(p)}(T)$ *(symmetric with respect to a, b), and* $\gamma_1(T)\int_0^T \gamma_2(s)ds, \cdots, \gamma_1^{(p)}(T)\int_0^T \gamma_2(s)ds$*. Let* $V^p := V_1^p + V_2^p$*. If* $\sigma_n(t,T) = \sum_{i=1}^{M_n} \sigma_{n,i}(t,r(t))\gamma_{ni}(T)$, $\gamma_{ni}(T) \in V_1^p$ *for some integer* M_n*, then* $f(0,T) \in V^p$ *results in* $f(t,T) \in V^p$ *for* $0 \le t$. That might be a slightly extended statement of Kusuoka(1997).

[4] According to the Duffie-Singleton approach Duffie-Liu(1997)[9] computes the credit spread of the defaultable swap contract. Schönbucher(1996)[18] describes a spread term structure dynamics of defaultable bonds within a framework of defaultable HJM model, and Schönbucher(1997) [19] attempts to value a variety of credit derivatives. Recently Duffie(1998)[7] treats the valuation of the default basket swap contract in a setting of the exponential affine model (Duffie-Kan(1996)[8]). The seminar document, Nakamura(1998)[16], also discusses the hazard rate modelling by exponential affine, quadratic Gaussian processes for the valuation of credit derivatives, and highlights the valuation issue of high-dimensional American type credit derivatives by means of the stochastic mesh method of Broadie-Glasserman(1997)[3], enhanced by the generalized low-discrepancy sequence.

general setting of pricing credit derivatives in sufficiently more general form which is valid even for a provision of either the basket type reference assets or defaultable credit risk taker, and next subsection addresses itself to the valuation issue, taking the TROR swap as an illustrative example.

3.1 Markovian framework of multi-dimensional defaultable HJM model

It is assumed in our case that there are multiple defaultable entities which may be either reference firms or a credit risk taker. From the tractability and ease of computation we further assume that the default-free interest rate and the hazard rates of defaultable entities are described in the Markovian framework of the multi-dimensional HJM model mentioned earlier. To begin with, let us introduce the dynamics of different forward interest rates, reflecting the time evolution of the credit qualities of different defaultable entities:

$$df^l(t,T) = \mu^l(t,T)dt + \sum_n \gamma_n^l(t,T)dW_n^0(t), \quad 0 \le t \le T, \tag{29}$$

for $l = 0, 1, \cdots, L$ and $n = 0, \cdots, N$, where we let $l = 0$ correspond to the default-free interest rate, $l = 1, \cdots, L-1$ to the reference defaultable firms, $l = L$ to the credit risk taker, and assume the complete market, so that $N = L$.

The benefit of the Markovian setting of the multi-dimensional HJM model is that it yields analytically tractable expressions of contingent claims and, more importantly, it can facilitate to work out a valuation scheme of contingent claims such as American option which usually requires backward calculation. We assume for simplicity a minimal Markovian volatility function as described in detail in Inui-Kijima(1997)[12]:

$$\gamma_n^l(t,T) = \sigma_n(r^l(t),t)e^{-\kappa_n^l(T-t)} \tag{30}$$

for $l, n = 0, \cdots, L$. Define

$$\begin{aligned} \beta_n^l(t,T) &:= \int_t^T e^{-\kappa_n^l(u-t)}du \\ &= \frac{1-e^{-\kappa_n^l(T-t)}}{\kappa_n^l}, \\ b_n^l(s,t) &:= \gamma_n^l(s,t)\int_s^t \gamma_n^l(s,u)du \\ &= \frac{\sigma_n^{l\,2}}{\kappa_n^l}(e^{-\kappa_n^l(t-s)} - e^{-2\kappa_n^l(t-s)}), \\ \int_0^t b_n^l(s,t)ds &= \frac{\sigma_n^{l\,2}}{\kappa_n^l}\left\{\frac{1-e^{-\kappa_n^l t}}{\kappa_n^l} - \frac{1-e^{-2\kappa_n^l t}}{2\kappa_n^l}\right\} \\ &= \frac{\sigma_n^{l\,2}}{2\kappa_n^l}(1-e^{-\kappa_n^l t})^2. \end{aligned} \tag{31}$$

In terms of the Brownian motion $W(t)$ under a risk-neutral probability measure P, which is assumed to exist here, we denote the stochastic integrals by

$$\begin{aligned} Y_n^l(t) &:= \int_0^t \gamma_n^l(s,t)dW_n(s), \\ X_n^l(t) &:= \int_0^t e^{\kappa_n^l s}dW_n(s). \end{aligned} \tag{32}$$

Note that when $\sigma(\cdot)$ in Eq.(30) is deterministic, $Y_n^l(t)$ is reduced to $\sigma_n^l e^{-\kappa_n^l t} X_n^l(t)$, and with the help of such a setting we can obtain the closed-form expressions of the popular credit derivatives in actual market. Evidently these state variables satisfy SDE's:

$$\begin{aligned} dX_n^l(t) &:= e^{\kappa_n^l t} dW_n(t), \\ dY_n^l(t) &:= -\kappa_n^l Y_n^l(t)dt + \gamma_n^l(t,t)dW_n(t), \end{aligned} \tag{33}$$

having for $t' > t$ integral forms

$$\begin{aligned} X_n^l(t') &:= X_n^l(t) + \int_t^{t'} e^{\kappa_n^l s} dW_n(s), \\ Y_n^l(t') &:= e^{-\kappa_n^l(t'-t)} Y_n^l(t) + \int_t^{t'} \sigma_n^l(r(s),s) e^{-\kappa_n^l(t'-s)} dW_n(s), \end{aligned} \tag{34}$$

and obeying the conditional distribution

$$X_n^l(T)|_{\mathcal{F}(t)} \sim N_N(X_n^l(t), \Sigma_{nm}(t,T)) \tag{35}$$

with $\Sigma_{nm}(t,T) := (\exp(2\kappa_n^l T) - \exp(2\kappa_n^l t))/(2\kappa_n^l)$, and $N_N(\cdot)$ being an N-dimensional normal distribution function.

Let L_l be a loss rate equal to one minus a recovery rate δ_l of a corporate bond issued by the l-th defaultable firm. The defaultable spot rate and discount bond are expressed as

$$\begin{aligned} r^l(t) &= r^0(t) + L_l h^l(t) \\ &= f^l(0,t) + \sum_n \int_0^t b_n^l(s,t)ds + \sum_n \int_0^t \gamma_n^l(s,t)dW_n(s) \\ &= f^l(0,t) + \sum_n \phi_{N+n}^l(t), \\ P^l(t,T) &= \frac{P^l(0,T)}{P^l(0,t)} \exp\left[-\frac{1}{2}\sum_n \beta_n^{l\,2}(t,T)\phi_n^l(t) - \sum_n \beta_n^l(t,T)\phi_{N+n}^l(t)\right], \end{aligned} \tag{36}$$

where letting $< \cdot, \cdot >$ denote the quadratic variation, we defined

$$\begin{aligned} \phi_n^l(t) &:= < Y_n^l(t), Y_n^l(t) > \\ &= \int_0^t \gamma_n^{l\,2}(s,t)ds \\ &= \frac{\sigma_n^{l\,2}}{2\kappa_n^l}\left(1 - e^{-2\kappa_n^l t}\right), \\ \phi_{N+n}^l(t) &:= \int_0^t [b_n^l(s,t)ds + \gamma_n^l(s,t)dW_n(s)] \\ &= \int_0^t b_n^l(s,t)ds + Y_n^l(t). \end{aligned} \tag{37}$$

Then the l-th firm's hazard rate is given by

$$\begin{aligned} h^l(t) &= \frac{1}{L_l}(r^l(t) - r^0(t)) \\ &= \frac{1}{L_l}(f^l(0,t) - f^0(0,t)) + \frac{1}{L_l}\sum_n \int_0^t (b_n^l(s,t) - b_n^0(s,t))ds \\ &\quad + \frac{1}{L_l}(Y_n^l(t) - Y_n^0(t)). \end{aligned} \tag{38}$$

It should be noticed that the hazard rates are not guaranteed to be positive with probability one, which seems to be apparently a serious drawback. Due to the inheritance of the desirable nature of HJM model, this formulation is tractable and easy to calibrate to the market, while, for instance, exponential affine or quadratic Gaussian formulation, not allowing for a negative hazard rate, seems to be difficult to do so, in particular, when it is modelled for the mutually correlated multi-defaultable entities. Thus, at present there would not exist any model to reconcile with these features in valuing credit derivatives underlying multiple assets exposed to the credit risks of those defaultable entities. This paper attaches more importance to the ease of calibration from the practical viewpoint. That is why we explore the HJM modelling of the hazard rate processes in what follows.

Then the SDE's of the spot rate and the l-th defaultable discount bond price $P^l(t,T)$ are of the forms:

$$\begin{aligned} dr^l(t) &= \Big[f^{l(1)}(0,t) - \sum_n \kappa_n^l \phi_{N+n}(t) + \sum_n \phi_n(t)\Big]dt + \sum_n \gamma_n^l(t,t)dW_n(t), \qquad (39) \\ \frac{dP^l(t,T)}{P^l(t,T)} &= r^l(t)dt + \sum_n \sigma_{P_l n}(t,T)dW_n(t), \end{aligned}$$

with

$$\sigma_{P_l n}(t,T) := -\int_t^T \gamma_n^l(t,s)ds. \qquad (40)$$

As a key lemma for pricing both the fixed rate and the recovery side present values, we now provide two conditional expectations:

Lemma 1

For $\Phi : \mathbf{R}^{L\times N} \times \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$, *and* $\Psi : \mathbf{R}^{L\times N} \times \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}$, *conditional on* $\mathcal{F}_t$,

$$\begin{aligned} \Phi(X(t),t,T;\alpha,\beta) &:= \mathbf{E}[e^{-\int_t^T \alpha^\top(u)X(u)du - \beta^\top X(T)}|\mathcal{F}_t] \qquad (41) \\ &= e^{-a(\tau)-b^\top(\tau)X(t)}, \end{aligned}$$

$$\begin{aligned} \Psi(X(t),t,T;\alpha,\beta,\gamma,\delta) &:= \mathbf{E}[\Big(\delta + \gamma^\top(T)X(T)\Big)e^{-\int_t^T \alpha^\top(u)X(u)du - \beta^\top X^\top(T)}|\mathcal{F}_t] \qquad (42) \\ &= \Phi(X(t),t,T;\alpha,\beta)\Big(d(\tau) + c^\top(\tau)X(t)\Big) \end{aligned}$$

hold for $t \in [0,T]$, $(\tau := T-t)$, *where the coefficients are given by*

$$\begin{aligned} a(\tau) &= -\frac{1}{2}\sum_{n,m}\int_t^T du\sigma_n(u)\sigma_m(u)\Big\{\int_u^T \alpha_n(v)dv + \beta_n\Big\}\Big\{\int_u^T \alpha_m(v)dv + \beta_m\Big\}, \qquad (43) \\ b_n(\tau) &= \int_t^T \alpha_n(u)du + \beta_n, \\ c_n(\tau) &= \gamma_n, \\ d(\tau) &= -\sum_{n,m}\int_t^T du\sigma_n(u)\sigma_m(u)\Big\{\int_u^T \alpha_n(v)dv + \beta_n\Big\}\gamma_m + \delta \end{aligned}$$

with initial condition, $a(0)=0$, $b_n(0)=1$, $c_n(0)=\gamma_n$, $d(0)=\delta$.

In particular, if α and γ are constant, and $\sigma_n(t) = e^{\kappa_n t}$, then these coefficients become in slightly simple form:

$$\begin{aligned} a(\tau) &= -\frac{1}{2}\sum_{n,m} e^{(\kappa_n+\kappa_m)T}\Big[\alpha_n\alpha_m I_2(\tau) + 2\alpha_n\beta_m I_1(\tau) + \beta_n\beta_m I_0(\tau)\Big], \\ b(\tau) &= \alpha_n\tau + \beta_n, \\ c(\tau) &= \gamma_n, \\ d(\tau) &= -\sum_{n,m}\gamma_m e^{(\kappa_n+\kappa_m)T}\Big\{\alpha_n I_1(\tau) + \beta_n I_0(\tau)\Big\} + \delta, \end{aligned} \tag{44}$$

where the calculation of the integral $I_n(\cdot)$ $(n = 0, \cdots)$ defined by

$$I_n(\tau) := \int_0^\tau e^{-(\kappa_n+\kappa_m)u} u^n du \tag{45}$$

is straightforward, the first three of which are given by

$$\begin{aligned} I_0(\tau) &= \frac{1}{K}(1 - e^{-K\tau}), \\ I_1(\tau) &= -\frac{1}{K}e^{-K\tau}(\tau + \frac{1}{K}) + \frac{1}{K^2}, \\ I_2(\tau) &= -\frac{1}{K}e^{-K\tau}(\tau^2 + \frac{2}{K}\tau + \frac{2}{K^2}) + \frac{2}{K^3} \end{aligned} \tag{46}$$

with $K := \kappa_n + \kappa_m$.

This lemma is proved easily by the Feynman-Kac formula. See Karatzas-Shreve(1991)[13].

3.2 Total rate of return swap

In this subsection we shall study the valuation of a total rate of return (TROR) swap contract, in which a counterparty pays the so-called "*total return*" (=income plus capital gains) of one or more reference assets over another counterparty. Let us call the former a total-return payer, and the latter a total-return receiver who plays a role of taking a credit risk of defaultable reference assets. In valuing this sort of contract, we typically assume a following covenant:

- There are two parties entering into a contract, which terminates at the time of a credit event stipulated in a covenant, or at a prespecified maturity, whichever is first.
- A credit event is primarily default by one or more reference firms. [5] Furthermore, we take into account the event of termination triggered by the default of a total-return receiver (credit risk taker).
- If credit event does not occur until time t_i(i-th cash flow time;$i = 1, \cdots, N_l$ for the l-th defaultable bond), then total-return payer pays the total return, $c + \Pi(t_i) - \Pi(t_{i-1})$ at time t_i, where we let c be a total income gain(i.e., sum of coupons), and let $\Pi(t)$ stand for a market value of the reference coupon bond portfolio. Note that if the total return is negative, then the total-return payer receives such absolute amount.

[5] As to the definition of a credit event it is also possible to alter the default of a single reference firm to the first-to-default of any one among the basket of reference firms.

- If credit event occurs at time τ, then the total-return receiver (credit-risk-taker) has to pay a certain amount of coverage against a capital loss incurred by the default of reference assets, $\Pi(t_{i^*(\tau)}) - \delta\Pi(\tau)$, where δ stands for an aggregate recovery rate, and $t_{i^*(\tau)}$ means the cash flow time just before the time τ of credit event. Note also that if such amount of capital loss at the default time is negative, then the total-return receiver pays its absolute amount.

Here the market value of bond portfolio at time t denoted by $\Pi(t)$ is represented as

$$\Pi(t) := \sum_{l=1}^{L-1} x_l \sum_{u_j>t}^{N_l} c_l P_l(t, u_j). \tag{47}$$

Before entering into details of the calculation, we prepare somewhat general notations associated with L defaultable entities,

$$\hat{\gamma}_n^l(t,v) := \int_t^v \gamma_n^l(t,u)du, \tag{48}$$

$$b_n^{ll'}(s,t) := \gamma_n^l(s,t)\int_s^t \gamma_n^{l'}(s,u)du = \gamma_n^l(s,t)\hat{\gamma}_n^l(s,t), \tag{49}$$

$$b_n^{ll'}(s,t,v) := \gamma_n^l(s,t)\int_s^v \gamma_n^{l'}(s,u)du = \gamma_n^l(s,t)\hat{\gamma}_n^l(s,v). \tag{50}$$

Let an $(L+1)$-vector be $\alpha_l := (1-\sum_{l=1}^{L}\frac{1}{L_l}, \frac{1}{L_1}, \cdots, \frac{1}{L_L})$. With these notations we shall provide a useful lemma and proposition as shown later. [6] The quantity related to a cumulative non-default probability is represented by

$$\Lambda(t,v) = e^{-\int_t^v (r^0(u)+\sum_{l=1}^{L} h^l(u))du} = e^{-\sum_{l=0}^{L}\int_t^v \alpha_l r^l(u)du},$$

and its conditional expectation is

$$\begin{aligned}\Phi(t,v) &:= \mathbf{E}[\Lambda(t,v)|\mathcal{F}_t] \\ &= e^{-\sum_{l=0}^{L}\alpha_l\int_t^v \mu_{r_l}(u)du - \sum_{l=0}^{L}\alpha_l\int_0^t\{\hat{\gamma}_n^l(s,v)-\hat{\gamma}_n^l(s,t)\}dW_n^P(s)+\frac{1}{2}q^2(t,v)},\end{aligned} \tag{51}$$

where $\mu_{r_l}(u)$ is non-stochastic term of $r_l(u)$, and

$$q^2(t,v) := \sum_{l,l'=0}^{L}\alpha_l\alpha_{l'}\sum_n\int_t^v \hat{\gamma}_n^l(u,v)\hat{\gamma}_n^{l'}(u,v)du.$$

The change of numéraire technique by Davis(1998) [5] will be useful in this calculation. For

[6] Our notation is as follows; as stated earlier, $l = 0$ corresponds to the default-free interest rate, $l = 1, \cdots, L-1$ to the reference bonds, and $l = L$ to the total return receiver (credit risk taker). For the simplicity of exposition we ignore the default risk of the total return payer (insured counterparty). This setting is also comprehensive, which is applicable for the default basket contract with $L-1$ defaultable reference bonds and a defaultable guarantor, although a typical TROR swap contracted in the actual credit derivative market is probably $L = 2$, i.e., the simple case of a single defaultable asset reference and a defaultable credit risk taker.

$v > t$ we take $\Lambda(t,v)$ as a numéraire and denote by $Q(v)$ the associated probability measure, under which

$$W^{Q(v)}(t) = W^P(t) + \sum_{l=0}^{L} \int_0^t \alpha_l \hat{\gamma}_n^l(u,v)du \tag{52}$$

becomes an $(L+1)$-dimensional Brownian motion by Girsanov's theorem.

Under the $Q(v)$-measure relevant stochastic processes are rewritten as follows:

$$\begin{aligned}
r_l(t) &= f^l(0,t) + \int_0^t [b_n^{ll}(s,t) - \sum_{l'=0}^{L} \alpha_{l'} b_n^{ll'}(s,t,v)]ds + Y^l(t), && (53)\\
h_l(t) &= \frac{1}{L_l}(f^l(0,t) - f^0(0,t)) && (54)\\
&\quad + \frac{1}{L_l}\sum_n \int_0^t \{(b_n^{ll}(s,t) - b_n^{00}(s,t)) - \sum_{l'=0}^{L} \alpha_{l'}(b_n^{ll'}(s,t,v) - b_n^{0l'}(s,t,v))\}ds \\
&\quad + \frac{1}{L_l}(Y^l(t) - Y^0(t)) \\
&:= \mu_{h_l}^{Q(v)}(t) + \frac{1}{L_l}(Y^l(t) - Y^0(t)), \\
\phi_{N+n}^l(t) &= \int_0^t b_n^l(s,t)ds - \sum_{l=0}^{L} \alpha_l \int_0^t \gamma_n^l(s,t)\hat{\gamma}_n^l(s,v)ds + Y_n^l(t) && (55)\\
&:= \mu_{\phi_{N+n}^l}^{Q(v)}(t) + Y_n^l(t), \\
P^l(t,T) &= \frac{P^l(0,T)}{P^l(0,t)} \exp\Big(\sum_n \Big[-\frac{1}{2}\beta_n^{l\,2}(t,T)\phi_n^l(t) && (56)\\
&\quad - \beta_n^l(t,T)\int_0^t \{b_n^{ll}(s,t) - \sum_{l'=0}^{L} \alpha_{l'} b_n^{ll'}(s,t,v)\}ds - \beta_n^l(t,T)Y^l(t)\Big]\Big) \\
&= \frac{P^l(0,T)}{P^l(0,t)} \exp\Big(\sum_n \Big[-\frac{1}{2}\beta_n^{l\,2}(t,T)\phi_n^l(t) - \beta_n^l(t,T)\mu_{\phi_{N+n}^l}^{Q(v)}(t) - \beta_n^l(t,T)Y^l(t)\Big]\Big) \\
&:= F_l(0,t,T)\exp\Big(M_l^{Q(v)}(t,T,v) - \sum_n \beta_n^l(t,T)Y_n^l(t)\Big).
\end{aligned}$$

Now we give another lemma used for calculating conditional expectations involving the present value of the recovery side.

Lemma 2
Under the $Q(v)$-measure

$$\begin{aligned}
J_{lm}(t,v,T) &:= \mathbf{E}^{Q(v)}[h_l(v)P^m(v,T)|\mathcal{F}_t] && (57)\\
&= \mu_{h_l}^{Q(v)}(v)F_m(0,v,T)e^{M_m^{Q(v)}(v,T,v)}\Phi_m(t,v,T) \\
&\quad + \frac{1}{L_l}F_m(0,v,T)e^{M_m^{Q(v)}(v,T,v)}(\Psi_{lm}(t,v,T) - \Psi_{0m}(t,v,T))
\end{aligned}$$

for $l,m = 0,\cdots,L$ and $t \le v \le T$, where

$$\begin{aligned}
\Phi_m(t,v,T) &:= \mathbf{E}^{Q(v)}[e^{-\sum_n \beta_n^m(v,T)Y_n^m(v)}|\mathcal{F}_t], && (58)\\
\Psi_{lm}(t,v,T) &:= \mathbf{E}^{Q(v)}[\Big(\sum_n Y_n^l(v)\Big)e^{-\sum_n \beta_n^m(v,T)Y_n^m(v)}|\mathcal{F}_t],
\end{aligned}$$

and $M_m^{Q(v)}(\cdot,\cdot,\cdot)$ *is the same one as defined above, i.e.,*

$$M_m^{Q(v)}(t,T,v) := -\frac{1}{2}\sum_n \beta_n^{m\,2}(t,T)\phi_n^m(t) - \sum_n \beta_n^m(t,T)\mu_{\phi_{N+n}^m}^{Q(v)}(t). \tag{59}$$

Note that the first and second expectations, $\Phi_m(\cdot)$ and $\Psi_{lm}(\cdot)$, are both explicitly computed by lemma 1.

Since the hazard rate process is given by

$$h_l(t) = \mu_h^l(t) + \frac{1}{L_l}(Y_l(t) - Y_0(t)), \tag{60}$$

it follows that

$$\mathbf{E}[h_l(u)|\mathcal{F}_t] = h_l(t) + \mu_h^l(u) - \mu_h^l(t) \tag{61}$$

for $u > t$. Let $i^*(u) := \sup\{i|t_i < u, i = 1, \cdots, N\}$. For the recovery side we have

$$\begin{aligned} V_{rec}^b(t) &= \mathbf{E}^P[\Lambda_0(t,\tau)\Big(\Pi(t_{i^*(\tau)} - \delta\Pi(\tau))\Big)\mathbf{1}_{\{\tau\leq T\}}] \\ &= \int_t^T du\mathbf{E}^P\Big[\Lambda(t,u)\Big(\sum_{l=1}^L h_l(u)\Big)\Big(\Pi(t_{i^*(u)}) - \delta\Pi(u)\Big)\Big]. \end{aligned} \tag{62}$$

Note that for $t < s < u$

$$\begin{aligned} \mathbf{E}^P\Big[\Lambda(t,u)\Big(\sum_{l=1}^L h_l(u)\Big)\Pi(u)\Big] &= \mathbf{E}^P[\Lambda(t,u)]\mathbf{E}^{Q(u)}[\Big(\sum_{l=1}^L h_l(u)\Big)\Pi(u)], \\ \mathbf{E}^P\Big[\Lambda(t,u)\Big(\sum_{l=1}^L h_l(u)\Big)\Pi(s)\Big] &= \mathbf{E}^P[\Lambda(t,u)]\mathbf{E}^{Q(u)}[\Big(\sum_{l=1}^L h_l(u)\Big)\Pi(s)] \\ &= \mathbf{E}^P[\Lambda(t,u)]\mathbf{E}^{Q(u)}[\mathbf{E}^{Q(u)}[\Big(\sum_{l=1}^L h_l(u)\Big)|\mathcal{F}_s]\Pi(s)] \\ &= \mathbf{E}^P[\Lambda(t,u)]\mathbf{E}^{Q(u)}[\sum_{l=1}^L\Big(h_l(s) + \mu_h^l(u) - \mu_h^l(s)\Big)\Pi(s)]. \end{aligned} \tag{63}$$

In calculating the present value of the recovery side, we substitute the above expectations into the integrand of Eq.(62), employ lemma 2 and finally conduct the numerical evaluation of that integral.

Next we proceed to the calculation of the present value of the fixed rate side of the TROR swap contract. Let n_c denote the number of premiums paid periodically until the maturity of contract. The present value is given by

$$\begin{aligned} V_{fix}(t) &= \sum_i^{n_c}\mathbf{E}[\Lambda(t_i)\Big\{-\Big(L(t_i)+\alpha\Big) + \Big(c + \Pi(t_i) - \Pi(t_{i-1})\Big)\Big\}] \\ &= \sum_i^{n_c}\Big\{-\mathbf{E}\Big[\Lambda(t_i)\frac{1}{P_0(t_{i-1},t_i)}\Big] + \mathbf{E}[\Lambda(t_i)\Pi(t_i)] - \mathbf{E}[\Lambda(t_i)\Pi(t_{i-1})] + (1+c-\alpha)\mathbf{E}[\Lambda(t_i)]\Big\}. \end{aligned} \tag{64}$$

Note that for $t < v < u$

$$
\begin{aligned}
\mathbf{E}^{Q(v)}[P^l(t,u)] &= F_l(0,t,u)\mathbf{E}^{Q(v)}[e^{M_l^{Q(v)}(t,u,v)-\sum_n \beta_n^l(t,u)Y_n^l(t)}] && (65)\\
&= F_l(0,t,u)\exp\Big(M_l^{Q(v)}(t,u,v)+\frac{1}{2}\sum_n \beta_n^{l\,2}(t,u)\sigma_n^{Y_l 2}(t)\Big)\\
&:= \Xi^l(t,u;v),\\
\mathbf{E}^{Q(v)}[\frac{1}{P^l(t,u)}] &= \frac{\exp\Big(\sum_n \beta_n^{l\,2}(t,u)\sigma_n^{Y_l 2}(t)\Big)}{\Xi^l(t,u;v)}, && (66)\\
\mathbf{E}[\Lambda(t,v)] &= \Phi(t,v), && (67)
\end{aligned}
$$

with

$$
\Phi(t,v) := e^{-\sum_{l=0}^{L}\alpha_l\int_t^v \mu_{r_l}(u)du-\sum_{l=0}^{L}\alpha_l\int_0^t\{\hat{\gamma}_n^l(s,v)-\hat{\gamma}_n^l(s,t)\}dW_n^P(s)+\frac{1}{2}q^2(t,v)}, \quad (68)
$$

$$
\begin{aligned}
\sigma_n^{Y_l 2}(t) &:= \delta_{nn'} < Y_n^l(t), Y_{n'}^l(t) >, && (69)\\
M_l^{Q(v)}(t,u,v) &:= \sum_n\Big[-\frac{1}{2}\beta_n^l(t,T)\phi_n^l(t)-\beta_n^l(t,T)\int_0^t\{b_n^{ll}(s,t)-\sum_{l'=0}^{L}\alpha_{l'}b_n^{ll'}(s,t,v)\}ds\Big],
\end{aligned}
$$

$$
q^2(t,v) := \sum_{l,l'=0}^{L}\alpha_l\alpha_{l'}\sum_n\int_t^v \hat{\gamma}_n^l(u,v)\hat{\gamma}_n^{l'}(u,v)du, \quad (70)
$$

and $\mu_{r_l}(u)$ being a non-stochastic term of $r_l(u)$. Let $J_{lm}(t,u) := \mathbf{E}^{Q(s)}[h_l(t)P_m(t,u)]$. Note that

$$
\begin{aligned}
\mathbf{E}[\Lambda(t_i)\frac{1}{P(t_{i-1},t_i)}] &= \mathbf{E}^{Q(t_i)}[\frac{1}{P(t_{i-1},t_i)}]\Phi(0,t_i), && (71)\\
\mathbf{E}[\Lambda(t_i)\Pi(t_i)] &= \mathbf{E}^{Q(t_i)}[\Pi(t_i)]\Phi(0,t_i),\\
\mathbf{E}[\Lambda(t_i)\Pi(t_{i-1})] &= \mathbf{E}^{Q(t_i)}[\Pi(t_{i-1})]\Phi(0,t_i),\\
\mathbf{E}[\Lambda(t_i)] &= \Phi(0,t_i),\\
\mathbf{E}^{Q(s)}[\Pi(t)] &= \sum_{l=1}^{L-1}x_l\sum_{u_j>t}^{N_l}c_l\Xi(t,u_j;s),\\
\mathbf{E}^{Q(s)}[h_l(t)\Pi(t)] &= \sum_{m=1}^{L-1}x_m\sum_{u_j>t}^{N_m}c_mJ_{lm}(t,u_j),
\end{aligned}
$$

are readily computed by lemma 2. Thus we arrive at the proposition.

Proposition 2
The LIBOR spread is expressed as

$$
\alpha = 1 + c - \frac{1}{I_\Phi}\{V_{rec} + I_L - I_1^b + I_2^b\}, \quad (72)
$$

where

$$
I_\Phi = \sum_{i=1}^{n_c}\Phi(0,t_i), \quad (73)
$$

$$
\begin{aligned}
I_L &= \sum_{i=1}^{n_c} \mathbf{E}[\Lambda(t_i)\frac{1}{P_0(t_{i-1},t_i)}], \\
I_1^b &= \sum_{i=1}^{n_c} \mathbf{E}[\Lambda(t_i)\Pi(t_i)], \\
I_2^b &= \sum_{i=1}^{n_c} \mathbf{E}[\Lambda(t_i)\Pi(t_{i-1})].
\end{aligned}
$$

Substituting these quantities calculated with the help of lemmas 1,2 into Eq.(72), we can obtain the fair rate of LIBOR spread.

4 Summary and Concluding Remarks

This is a companion paper to the preceding one, Nakamura(1997)[15], which, according to the structural approach of modelling credit quality, has explored the pricing of defaultable securities, contracts, and their derivatives, and provided a valuation methodology using generalized low-discrepancy sequences highlighted recently from the standpoint of computational efficiency. In contrast, this paper follows another approach, so-called *reduced-form approach*, that is what Duffie-Singleton(1996) [10] has developed recently.

We have first studied the sufficient conditions that the general HJM model becomes Markovian framework, paying attention to a sequential structure of volatility functions which is characterized by the certain ordinary differential equations, and second modelled the hazard rate processes by the multi-dimensional Markovian HJM framework. Of course, there are other modellings of those processes, for instance, exponential affine, quadratic Gaussian stochastic processes, as demonstrated in Nakamura(1998)[16].

Usually the financial industry practice requires the theoretical term structure of spread curves to be completely adjusted to those observed in markets. The remarkable feature of HJM modelling is to easily meet such conditions, while the other modellings seem to be difficult to calibrate to the market spread curves. In this respect our modelling makes sense and can, especially, provide analytically tractable or closed-form expressions of credit derivatives, when specifying the Markovian HJM framework to the minimal one.

As for the American option provision of credit derivative contracts such as callability of default swaps, Nakamura(1998)[16] attempts to value that credit-risky contingent claim, using quadratic Gaussian modelling of default-free interest rate and hazard rate processes and employing as a valuation methodology the stochastic mesh method (developed recently in Broadie-Glasserman(1997)[3]) enhanced highly by some generalized low-discrepancy-sequences. Our Markovian framework is also convenient to implement in those backward induction calculus.

When one employs other Markovian modelling instead of the minimal one, it is no longer easier to derive any closed- or nearly closed-form expressions, and so it would be indispensable to invoke to the Monte-Carlo simulation, as conducted in Inui-Kijima(1998)[12] for the pricing of a simple discount bond option. It seems likely that our LDS-enhanced Markovian pricing methodology works even in such cases and reduces the computational burdens as the required demension increases. Those await the future research.

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