The foundation of Mathematical Finance Historical Tour in Stochastic Analysis

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1 One period Model

In this section, we remind "no arbitrage", the basic idea of mathematical finance. First, we think of the simplest model, one period model. Let us think of the following setting.

There are only two dates, date 0 (the present date) and date 1 (the future).

- There are only two securities, a risk free bond and a stock.
- The price of the bond at date 0 is B_0, and the price of the bond at date 1 is B_1. So the interest rate R is $(B_1 - B_0)/B_0$.
- The price of the stock at the present date is S_0.

Let us assume that there will be only two possibilities on the price of the stock at date 1.

◇ Scenario 1 (ω_1) The price of the stock at date 1 is S_1.
◇ Scenario 2 (ω_2) The price of the stock at date 1 is S_2.

Then the rate of return of the Stock will be $Q_1 = (S_1 - S_0)/S_0$ at Scenario 1 and will be $Q_2 = (S_2 - S_0)/S_0$ at Scenario 2. We assume that $Q_1 < R < Q_2$.

Now let us think of the following derivative. That is, the derivative holder will take Z_1 yen at date 1 if the Scenario 1 takes place, and will take Z_2 yen at date 2 if the Scenario 2 takes place. Then the payoff Z is a function of Scenarios, i.e.

$$Z(\omega_1) = Z_1, \qquad Z(\omega_2) = Z_2$$

We may regard the set of scenarios as a set of events mathematically. So we may regard Z as a random variable. Our main problem is to price this derivative.

Now suppose that we take the following portfolio at date 0 such that we hold the bond by amount of x yen and hold the stock by amount of y yen. Here we assume that there is no restriction on short sale. Therefore x or y can be negative. The cost to take this portfolio is $x + y$ yen, of course. The return will be

$(1 + R)x + (1 + Q_1)y$, if Scenario 1 takes place, and

$(1 + R)x + (1 + Q_2)y$, if Scenario 2 takes place.

Now let us think of the following linear equation.

$$\begin{aligned} (1+R)x + (1+Q_1)y &= Z_1 \\ (1+R)x + (1+Q_2)y &= Z_2 \end{aligned} \tag{1}$$

This equation can be rewritten as follows:

$$\begin{aligned} x + (1+R)^{-1}(1+Q_1)y &= (1+R)^{-1}Z_1 \\ x + (1+R)^{-1}(1+Q_2)y &= (1+R)^{-1}Z_2 \end{aligned} \tag{2}$$

One can easily see that there exists a unique solution (x, y) to Equation (1). Therefore we see that if we pay $x + y$ yen at date 0 and take a portfolio strategy (x, y), we can replicate the same payoff at date 1 as same as the derivative given by Z. This cost $x + y$ is called the replication cost of the derivative. By "no free lunch" argument, we may conclude that the price of derivative is equal to the replication cost $x + y$.

Moreover, we have the following. Let π_1, π_2 be given by

$$\pi_1 = \frac{Q_2 - R}{Q_2 - Q_1}, \qquad \pi_2 = \frac{R - Q_1}{Q_2 - Q_1}.$$

Then we easily obtain

$$x + y = (1+R)^{-1}Z_1\pi_1 + (1+R)^{-1}Z_2\pi_2 \tag{3}$$

Obviously we have

$$\pi_1 + \pi_2 = 1, \qquad \pi_1, \pi_2 > 0$$

and also we have

$$(1+R)^{-1}S_1\pi_1 + (1+R)^{-1}S_2\pi_2 = S_0 \tag{4}$$

One may think that π_1, and π_2 are probability of Events ω_1 and ω_2, respectively. This probability is called risk neutral probability. Equation (3) shows that the price of the derivative at date 0 is given by the expectation of discounted payoff at date 1 under risk neutral probability. Equation (4) shows that the expectation of discounted price of the stock at date 1 under risk neutral probability is equal to the price of the stock at date 0. One should note that the risk neutral probability is different from subjective probability.

Let us think of a little more general model. In the previous model, we think of only two scenarios. What does happen if we think of three scenarios?

We think of the following setting. The interest rate is R. The rate of return of the Stock will be Q_i, $i = 1, 2, 3$, if Scenario i takes place. The payoff of the derivative at date 1 is Z_i, $i = 1, 2, 3$, if Scenario i takes place. Then we have the following linear equation.

$$\begin{aligned}(1+R)x + (1+Q_1)y &= Z_1 \\ (1+R)x + (1+Q_2)y &= Z_2 \\ (1+R)x + (1+Q_3)y &= Z_3\end{aligned}$$

This equation does not have a solution in general, and so we cannot determine the price of the derivative.

Since we want to think of various scenarios, we have to think of a multi-period model or a continuous time model. By using a stochastic differential equation, one can generate an infinitely many scenarios.

2 Continuous time model: Black-Sholes model

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\{W_t\}_{t\in[0,\infty)}$ be a 1-dimensional Brownian motion starting from the origin, and let us think of a filtration $\mathcal{F}_t = \sigma\{W_s;\ s \le t\}$, $t \ge 0$. Also, let $r > 0, \sigma > 0, \mu \in \mathbf{R}$. We assume that there are two securities, Bond and Stock and the information up to time t is given by σ-algebra $\mathcal{F}_t$.

Let S_t be the price of the Stock at time t and let B_t be the price of the Bond at time t. We assume that the price processes S_t and B_t are adapted and they satisfy the following SDE.

$$\begin{aligned}dB_t &= B_t r dt, \\ dS_t &= S_t(\sigma dW_t + \mu dt).\end{aligned}$$

Then we see that

$$\begin{aligned}B_t &= B_0 \exp(rt), \\ S_t &= S_0 \exp(\sigma B_t + (\mu - \frac{\sigma^2}{2})t).\end{aligned}$$

So the interest rate per unit time is r. We think of a continuous trading portfolio strategy such that we hold the Bond by amount of $\eta_t B_t$ yen and hold the Stock by amount of $\xi_t S_t$ yen at time t. The portfolio processe(η_t, ξ_t) has to be adapted. If the strategy is self-financing, we have the following equation.

$$\exp(-rt)(\eta_t B_t + \xi_t S_t) = (\eta_0 B_0 + \xi_0 S_0) + \int_0^t \xi_s d\tilde{S}_s. \tag{5}$$

Here $\tilde{S}_t = \exp(-rt)S_t$ (discounted stock price). The integral appeared here is Ito integral.

Now let us think of an European type derivative such that the payoff at the maturity T is $Z(\omega)$, $\omega \in \Omega$, if the state is ω. Then the random variable Z should be $\mathcal{F}_T$ measurable. For example, an European call option of the exercise price a is given by $Z = \max\{S_T - a, 0\}$.

We need the following theorem.

Theorem 1 (Ito's representation theorem) *If Z is a good random variable, then there are $c \in \mathbf{R}$ and a good adapted process $\{\xi_t\}_{t\in[0,T]}$ such that*

$$\exp(-rT)Z = c + \int_0^T \xi_t d\tilde{S}_t \tag{6}$$

By virtue of this theorem, we see that there is a good self-financing trading strategy (η_t, ξ_t) such that

$$\eta_0 B_0 + \xi_0 S_0 = c$$

and

$$Z = \eta_T B_T + \xi_T S_T = e^{rT}(c + \int_0^T \xi_t d\tilde{S}_t).$$

These imply that one can replicate the derivative Z with an initial cost c.

The following theorem is important to compute the replication cost c.

Theorem 2 (Cameron-Martin-Maruyama-Girsanov) *There exists a probability meaure Q in $(\Omega, \mathcal{F})$ equivalent to the probability measure P such that $\tilde{S}_t = \exp(-rt)S_t$, $t \in [0,T]$, is a martingale under Q.*

By the property of stochastic integral, we see that

$$E^Q[\int_0^T \xi_t d\tilde{S}_t] = 0.$$

So we have

$$c = \exp(-rT)E^Q[Z].$$

Suppose that the payoff Z is given by $Z = f(S_T)$ for some continuous function f of polynomial order growth. If a function u is a nice function defined in $[0,T] \times \mathbf{R}$, by Ito's lemma we have

$$\exp(-rT)u(T, S_T) = u(0, S_0) + \int_0^T \frac{\partial u}{\partial x}(t, S_t)d\tilde{S}_t + \int_0^T \exp(-rt)(\frac{\partial u}{\partial t}(t, S_t) + Lu(t, S_t))dt,$$

where

$$Lu(t,x) = \frac{\sigma^2}{2}x^2\frac{\partial^2 u}{\partial x^2}(t,x) + r\frac{\partial u}{\partial x}(t,x) - ru(t,x)$$

So if u is a solution to the PDE

$$\frac{\partial u}{\partial t}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 u}{\partial x^2}(t,x) + r\frac{\partial u}{\partial x}(t,x) - ru(t,x) = 0 \qquad (t,x) \in [0,T] \times \mathbf{R},$$

$$u(T,x) = f(x),$$

we see that

$$\exp(-rT)u(T,S_T) = u(0,S_0) + \int_0^T \frac{\partial u}{\partial x}(t,S_t)d\tilde{S}_t$$

and so we have

$$u(0,S_0) = E^Q[\exp(-rT)f(S_T)]$$

$$= \exp(-rT)\int_{-\infty}^{\infty} f(S_0\exp(\sigma z + (r - \frac{\sigma^2}{2})T))(\frac{1}{2\pi T})^{\frac{1}{2}} e^{-\frac{z^2}{2T}}dz.$$

Moreover, we see that the hedging strategy is given by $\frac{\partial u}{\partial x}(t,S_t)$ (Delta hedge).

In the case of European call option we have

$$c = \exp(-rT)E^Q[\max\{S_T - a, 0\}]$$

$$= \exp(-rT)\int_{-\infty}^{\infty} \max\{S_0\exp(\sigma z + (r - \frac{\sigma^2}{2})T) - a, 0\}(\frac{1}{2\pi T})^{\frac{1}{2}} e^{-\frac{z^2}{2T}}dz$$

$$= S_0^1 N(d_1) - Ke^{-rT}N(d_2).$$

Here

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy$$

and

$$d_1 = \frac{\log(S_0/a) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

This is the Black-Scholes formula.

3 Historical Remark on Stochastic Analysis

We see that Stochastic Analysis plays very important role in Mathematical Finance. In particular, the following are basic tools.

(1) Brownian motion and additive processes (noise and innovation)

(2) Stochastic Integral, Stochastic Differential Equation and Ito's lemma

(3) Martingale Representation Theorem

(4) Transformation of Measure (Girsanov Transformation)

We will review the following historical papers, the origin of Stochastic Analysis.

(1) Brownian motion etc.

Bachelier, M.L., Théorie de la Spéculation, Ann. de l'École norm. 17(1900) 21-86

Wiener, N., Differential space, J.Math. and Physics 2(1923), 131-174

Lévy, P., Théorie de l'addition des variables aléatoires, Gauthier-Villoars, Paris, 1937

(2) SDE etc.

Kolmogorov, Über die analytishen Methoden in der Wahrscheinlichkeitsrechnung, Math. Ann. 104(1931), 415-458

Itô, K.(伊藤清), Markoff 過程ヲ定メル微分方程式, 全国紙上数学談話会 1077(1942) Differential equations determining a Markoff process, Zenkoku Sizyo Sugaku Danwakaisi

1077(1942)
(English translation is in 'Kiyosi Itô Selected papers', Springer 1987)

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Itô, K., Stochastic integral, Proc. Imp. Acad. Tokyo 20(1944), 519-522

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Itô, K., On stochastic differential equations, Mem. Amer. Math. Soc. 4(1951), 1-51

Lévy, P., Le mouvement brownien plan, Amer. J.Math., 62(1940), 487-550

Doob, J.L., Stochastic Process, John Wiley and Sons, New York, 1953

Kunita, H., and Watanabe, S., On square integrable martingales, Nagoya Math. J. 30(1967),209-245

(3) Martingale Representation

N.Wiener, The homogeneous chaos, Amer.J.Math. 60(1938), 897-936

Itô, K., Multiple Wiener integral, J. Math. Soc. Japan 3(1951), 157-169

(4) Girsanov Transformation

Cameron, R.H., and Martin, W.T., Transformation of Wiener integrals under translations, Ann. Math. 45(1944), 386-396

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Maruyama, G., On the Transition Probability Functions of the Markov Process, Natural Sci. Rep. Ochanomizu Univ. 5(1954), 10-20

Girsanov, I.V., On transforming a certain class of stochastic processes by absolutely continuous substitution of measures, Theory Prob. Appl. 5(1960), 285-301

THÉORIE

DE

LA SPÉCULATION,

PAR M. L. BACHELIER.

INTRODUCTION.

Les influences qui déterminent les mouvements de la Bourse sont innombrables, des événements passés, actuels ou même escomptables, ne présentant souvent aucun rapport apparent avec ses variations, se répercutent sur son cours.

À côté des causes en quelque sorte naturelles des variations, interviennent aussi des causes factices : la Bourse agit sur elle-même et le mouvement actuel est fonction, non seulement des mouvements antérieurs, mais aussi de la position de place.

La détermination de ces mouvements se subordonne à un nombre infini de facteurs : il est dès lors impossible d'en espérer la prévision mathématique. Les opinions contradictoires relatives à ces variations se partagent si bien qu'au même instant les acheteurs croient à la hausse et les vendeurs à la baisse.

Le Calcul des probabilités ne pourra sans doute jamais s'appliquer aux mouvements de la cote et la dynamique de la Bourse ne sera jamais une science exacte.

Mais il est possible d'étudier mathématiquement l'état statique du marché à un instant donné, c'est-à-dire d'établir la loi de probabilité des variations de cours qu'admet à cet instant le marché. Si le marché, en effet, ne prévoit pas les mouvements, il les considère comme étant

DIFFERENTIAL-SPACE

BY NORBERT WIENER

§1. **Introduction.** The notion of a function or a curve as an element in a space of an infinitude of dimensions is familiar to all mathematicians, and has been since the early work of Volterra on functions of lines. It is worthy of note, however, that the physicist is equally concerned with systems the dimensionality of which, if not infinite, is so large that it invites the use of limit-processes in which it is treated as infinite. These systems are the systems of statistical mechanics, and the fact that we treat their dimensionality as infinite is witnessed by our continual employment of such asymptotic formulae as that of Stirling or the Gaussian probability-distribution.

The physicist has often occasion to consider quantities which are of the nature of functions with arguments ranging over such a space of infinitely many dimensions. The density of a gas, or one of its velocity-components at a point, considered as depending on the coördinates and velocities of its molecules, are cases in point. He therefore is implicitly, if not explicitly, studying the theory of functionals. Moreover, he generally replaces any of these functionals by some kind of average value, which is essen-

Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung.

Von

A. Kolmogoroff in Moskau.

Zusammenfassung.

Ein physikalischer Prozeß (die Änderung eines physikalischen Systems) heißt stochastisch-definit, wenn aus der Kenntnis des Zustandes X_0 des Systems in einem gewissen Zeitmoment t_0 die Kenntnis der Verteilungsfunktion der Wahrscheinlichkeiten für die möglichen Zustände X des Systems in einem Zeitmoment $t > t_0$ folgt.

Der Verfasser betrachtet systematisch die einfachsten Fälle der stochastisch-definiten Prozesse und in erster Linie solche, die nach der Zeit stetig sind (darin besteht die wesentliche Neuheit der Methode: Bis jetzt betrachtete man gewöhnlich einen stochastischen Prozeß als eine Reihe von diskreten „Ereignissen").

Wenn die Menge $\mathfrak{A}$ der möglichen verschiedenen Zustände des Systems endlich ist, so läßt sich der stochastisch-definite Prozeß durch gewöhnliche lineare Differentialgleichungen charakterisieren (Kap. II). Wenn der Zustand des Systems durch einen oder mehrere stetige Parameter definiert ist, so wird der analytische Apparat durch parabolische partielle Differentialgleichungen gegeben (Kap. IV). Man kommt dabei zu verschiedenen Verteilungsfunktionen, unter denen die Laplacesche Normalverteilung als natürlicher einfachster Fall erscheint.

Einleitung.

I.

Wenn man Natur- oder Sozialereignisse mathematisch behandeln will, muß man zuerst diese Ereignisse *schematisieren*; man kann nämlich die mathematische Analysis zur Betrachtung eines Änderungsprozesses eines Systems nur dann anwenden, wenn man voraussetzt, daß jeder mögliche Zustand dieses Systems sich mit Hilfe eines bestimmten mathematischen Apparats vollständig beschreiben läßt, z. B. durch die Werte einer bestimmten Anzahl von Parametern; ein solches mathematisch-definier-

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FASCICULE I

THÉORIE DE L'ADDITION

DES

VARIABLES ALÉATOIRES

PAR

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Quai des Grands-Augustins, 55

1937

1077. Markoff過程ヲ定メル微分方程式

伊藤　清（内閣統計局）

ハシガキ

(I) 有限個ノ可能ナ場合 $a_1, a_2, \cdots a_m$ ヲ有シ，自然数ヲ径数トスル simple Markoff process $x_1, x_2, \cdots$ ニ関シテ多クノ遷移確率ヲ考ヘルコトが出来ル。例ヘバ $x_k = a_i$ ナル條件ノ下ニ於ケル $x_l = a_j$ ノ確率、或ハ $x_1 = a_{i_1}, x_2 = a_{i_2}, \cdots, x_n = a_{i_n}$ ナル條件ノ下ニ於ケル $x_{n+1} = a_{i_{n+1}}$ トナル確率等々。シカシ乍ラソレ等ハ結局 $x_k = a_i$ ノ時ノ $x_{k+1} = a_j$ ナル確率 $p_{ij}^{(k)}$ ($k = 1, 2, \cdots, i, j = 1, 2, \cdots m$) ニ帰着セラレル。コレハ Kolmogoroff ノ本(*1)ニモ書イテアル。以後コレヲ基本的ナ遷移確率ト呼バウ。

更ニ可能ナ場合が有限デナクトモ、例ヘバ実数ヲ以テ標識ヅケラレル時ニハ、同ジコトがイヘルノハ云フマデモナイ。

併シナガラ係数が自然数デナクテ実数ノ場合即チ continuous parameter ニ依存スル Markoff process ニ於テ、上ノコトハ如何ニナルカトイフコトハソレ程簡単デハナイ。(*2)

更ニ一般ニ可能ナ場合が実数ニヨリ標識付ケラレ、且ツ continuous parameter ニ依存スル simple Markoff