PROBLEMS OF OPTIMIZATION OF AN INVESTMENT PORTOFOLIO

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Formulations of problems of formation of optimal investment portfolio are resulted in the paper at various definitions of risk of the projects. The interest rates are assumed determined.

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One of the important tasks facing to the potential investors, is the task of formation of a portfolio of the projects. Certainly, this task has many aspects, only mathematical aspects are considered here.

Investment project (see [1,2]) is the vector $C = (c_0, \ldots, c_n)$ which positive components correspond to return of money to the investor, negative to investments of money in the project. Let a family of the projects $C_1 = (c_{10}, \ldots, c_{1n}), \ldots, C_m = (c_{m0}, \ldots, c_{mn})$ is offered to the investor. Let's consider the offers flexible, that is the investor can accept the decision on financing any share x_j of the project C_j ($X_j \in [0,1]$). The bank interest rate for k year we shall sign as i_k, the appropriate discount factors are. $d_k = \prod_{j=0}^{k-1}(1+i_j)^{-1}$. We consider the interest rates to be determined in this paper.

Let investor has a start capital F_0. The first reason for forming a portfolio is nonruin at any moment (we consider a situation when the attraction of additional money is impossible). The condition of nonruin at the moment of time k is $F_0 + \sum_{s=0}^{m}\sum_{j=0}^{k} c_{sj} d_j \geq 0$.

The risk of the investor is that the return of money under the project can stop at any moment. Some approaches to modeling of similar risks and formation of an optimum portfolio of the investments are considered in paper.

1. Function of the purpose.

An important problem is how to formalize system of preferences of the investor. Naturally the main complexity is multicriteria of the problem. Two approaches to this task are usually considered. If all probability characteristics of possible situations are known one can maximize the probability P [Q>T] where Q is discounted profit on a portfolio and T is the a priori given allowable profit level. In this paper this criterion of quality of a portfolio is considered.

If only the basic characteristics of distributions are known it is possible, following H.Markowitz, to minimize variance Var [Q] when the level of the average income E [Q] is given.

2. Markov chains.

As space of condition Ω we consider the set of vectors $(a_1, \ldots, a_m)$, where a_i is an element of the set $\{-1, 0, 1\}$. Substantially if $a_j = -1$ then the return of money on j project was stopped before transition in this condition, if $a_j = 0$ then the return of money stops and the equality $a_j = 1$ shows that the payments are proceeding.

The transition from a condition $A = (a_1, \ldots, a_m)$ to a condition $B = (b_1, \ldots, b_m)$ is possible if the conditions are carried out:

$a_j \in \{0, -1\} \Rightarrow b_j = -1;$

$b_j \in \{0, 1\} \Rightarrow a_j = 1.$

The appropriate probability of transition we sign P(A,B). Transitions on different arches are considered to be independent.

We show the formulation of a task of formation of an optimum portfolio within the framework of such model.

Let portfolio $X = (x_1, \ldots, x_m)$ is generated. Let's consider set of functions φ: $\{1, \ldots, m\} \to \{1, \ldots, n\}$ such, that $c_{s,\varphi(s)} > 0$ (we believe in addition, that $c_{s,n+1} = 0$ for all s). The value $\varphi(s)$ equals to the moment of the termination of payments on the project, thus the condition $c_{s,\varphi(s)} > 0$ means that the termination of return of money is possible only in the moment when the return is predetermined. If $\varphi(s) = n+1$ then the payments on the project do not stop.

When realized the function φ then discounted profit on a portfolio equals

$$Q(X, \varphi) = \sum_{s=1}^{m} \sum_{k=0}^{\varphi(s)-1} x_s c_{sk} d_k$$

The function φ gives some trajectory of Markov process. Let's calculate probability of its realization. In k's moment the termination of financing in accuracy occurs under those projects C_s where $\varphi(s) = k$. Thus there is a transition in a condition

$A(\varphi, k) = (\mathrm{sign}(\varphi(1) - k), \ldots, \mathrm{sign}(\varphi(m) - k)).$

Let's note that $A(j, 0) = (1, \ldots, 1)$; $A(j, n+1) = (-1, \ldots, -1)$. Thus probability of realization of such trajectory equals

$$P[j] = \prod_{k=0}^{n} P(A(\varphi, k), A(\varphi, k+1)).$$

Thus we receive the following extreme problem.

To find numbers x_i satisfying conditions:

1. $x_i \in [0,1]$

2. $F_0 + \sum_{s=0}^{m} \sum_{j=0}^{k} c_{sj} d_j \geq 0$ for k = 1, …, n.

3. The value $\sum_{\varphi} P(\varphi)$ reaches a maximum. Here summation is made on those

functions φ for which $Q(X,\varphi) \geq T$.

3. Independent refusals of the projects.

The task becomes simpler in the assumption of independence of random events – refusals of the projects in those or other moments of time. Thus the conditional probabilities P [j, k] of refusals on the project j in the moment of time k in condition that before payments under the project did not stop should be known. It is natural to believe that the probabilities P [j, k] grow with growth k. It corresponds to growth of uncertainty with removal time. For example it is possible to put P [j, k] = $\frac{k}{r_j + k}$, where r_j is positive number reflecting a degree of risk of the project. The less is r_j the more is risk. Discounted profit on the k project equals $A(j,k) = \sum_{i=0}^{k} c_{js} q_s$ with probability P [j, k].

The task is: to find numbers x_i satisfying to conditions 1 and 2 of item 2 and function φ(j) from item 2, such, that $\sum_{j=1}^{m} x_j A(j,\varphi(j)) \geq T$ and value $\prod_{j=0}^{m} P[j,\varphi(j)]$ has maximum.

4. Indexes of a condition.

The independence of refusals of the projects is a too strong condition. Each of the projects is influenced by a condition of economy as a whole. Let condition of economy as a whole is described by random process index $R_0(k)$. The activity under each project also is characterized by an index $R_j(k)$. Thus we believe that the random processes $R_0, R_1, \ldots, R_m$ are independent in aggregate. Further let for each project C_j the threshold U_j be determined i.e. such number that if $R_0(s) + R_j(s) < U_j$ when $s < k$, $R_0(k) + R_j(k) \geq U_j$ and $C_{jk} > 0$ then the payments under the project stop in the moment k.

Here omitting details we note that it is possible to consider random sizes {Rj (k)} to be independent in aggregate having normal distribution with variance and average increasing linearly when k growths (Bashelje-Samuelson assumption).

In the end of this we can say that these approaches node some important features of investment risks.

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