The Maintenance Properties of nth Stop-Loss Order

CHENG YU

Co-operators Life Insurance Company

Regina, Saskatchewan, Canada

JEFFREY S. PAI

The Warren Centre for Actuarial Studies and Research

Faculty of Management, University of Manitoba

Winnipeg, Canada R3T 5V4

Telephone: 1 204 474 6412

Facsimile: 1 204 474 7545

E-mail: jpai@cc.umanitoba.ca

Abstract: This paper generalized the concept of stop-loss transforms to the nth stop-loss transforms. Some useful properties of the nth stop-loss transforms were discovered and a recursion formula for the nth stop-loss transforms was established. Also, the maintenance properties of the nth stop-loss order under convolution and compound operations were proved.

Keywords: classical risk model, homogeneous Poisson processes, stop-loss transform, stop-loss order.

1 Stop-Loss Transforms and Recursion Formula

The concept of stop-loss transforms and its properties play an important role in this paper. At first we generalize the concept of stop-loss transforms in [1] as follows.

Definition 1 Suppose random variable X is nonnegative with its distribution function being $F(x)$, its survival function being $\overline{F}(x) = 1 - F(x)$, and $E(X^n) < \infty$. Let

$$\Pi^{(n)}(u) = E[\{(X-u)_+\}^n], \quad u \geq 0, n = 1, 2, \cdots, \tag{1}$$

where

$$(x-u)_+ = \begin{cases} 0, & \text{for } x \leq u, \\ x-u, & \text{for } x > u, \end{cases}$$

$$\Pi^{(0)}(u) = \overline{F}(u) = 1 - F(u). \tag{2}$$

As a function of u, $\Pi^{(n)}(u), n = 1, 2, \cdots$ will have domain $[0, \infty)$. We call function $\Pi^{(n)}(u)$ the nth stop-loss transform of X. It is easy to see that the concept of stop-loss transform in [1] (see page 25, definition 3.1.4 in [1]) is the special case of definition 1 when $n = 1$.

The following corollary then becomes obvious.

Corollary 2 $\Pi^{(n)}(0) = E(X^n),\ n = 1, 2, \cdots$ and $\Pi^{(0)}(0) = 1$.

Example 3 Prove that

$$\Pi^{(1)}(u) = \int_u^{\infty} \overline{F}(x)dx. \tag{3}$$

Proof. Let $n = 1$ in (1) and take integration by parts, we have

$$\begin{aligned}
\Pi^{(1)}(u) &= E[(X-u)_+] \\
&= \int_u^{\infty} (x-u)dF(x) = -\int_u^{\infty} (x-u)d\overline{F}(x) \\
&= -(x-u)\overline{F}(x) \mid_{x=u}^{\infty} + \int_u^{\infty} \overline{F}(x)dx \\
&= \int_u^{\infty} \overline{F}(x)dx.
\end{aligned}$$

Note that in the above proof we used the following equation:

$$\lim_{x\to\infty} (x-u)\overline{F}(x) = 0.$$

When $E(X) < \infty$, the above equation always holds (see proposition 4. Letting $n = 1$ in proposition 4, we get the above equation). For convenience to use later, we prove a more general result as follows:

Proposition 4 If nonnegative random variable X has a finite nth moment, then

$$\lim_{x\to\infty} (x-u)^n \overline{F}(x) = 0, \quad \forall u \geq 0, \tag{4}$$

where $\overline{F}(x)$ is the survival function of X.

Proof. Because the nth moment of X is finite, we have

$$\lim_{x\to\infty}\int_x^\infty y^n dF(y) = 0.$$

Hence

$$\lim_{x\to\infty}(x-u)^n\overline{F}(x) \le \lim_{x\to\infty} x^n\overline{F}(x) \le \lim_{x\to\infty}\int_x^\infty y^n dF(y) = 0.$$

The proposition is proved.

Example 5 Suppose $E(X^2) < \infty$, then

$$E(X^2) = 2\int_0^\infty \Pi_X^{(1)}(u)du. \tag{5}$$

Proof.

$$E(X^2) = \int_0^\infty x^2 dF_X(x) = -\int_0^\infty x^2 d\overline{F}_X(x).$$

By using integration by parts and then (4) (let $u = 0$ in (4)), we have

$$\begin{aligned} E(X^2) &= 2\int_0^\infty x\overline{F}_X(x)dx = 2\int_0^\infty [\overline{F}_X(x)\int_0^x dy]dx \\ &= 2\int_0^\infty\int_y^\infty \overline{F}_X(x)dxdy = 2\int_0^\infty \Pi_X^{(1)}(y)dy. \end{aligned}$$

In the above proof we interchange the order of integration and use our results from Example 3 to complete the proof.

The following recursion formula for the nth stop-loss transforms is significant for some later results.

Theorem 6

$$\Pi^{(n)}(u) = n\int_u^\infty \Pi^{(n-1)}(x)dx, \quad n = 1, 2, \cdots. \tag{6}$$

To prove this theorem we need the following lemma which has its own meaning and can be used in other occasions.

Lemma 7 Suppose $F(x)$ is a distribution function. If function $f(x, y)$ satisfies the following conditions:

(a) $\frac{\partial f(x,y)}{\partial y}$ exists,

(b) When Δu is in some neighborhood of 0, say $(-\alpha, \alpha)$, we have

$$\mid \frac{f(x, u+\Delta) - f(x,u)}{\Delta u} \mid \leq g(x),$$

where the nonnegative function $g(x)$ is Stieltjes integrable on $[0, u]$ with respect to the distribution function $F(x)$, i.e.

$$\int_0^u g(x)dF(x) < \infty.$$

(c) $\lim_{|x-y|\to 0} \mid \frac{f(x,y)}{x-y} \mid = 0.$

Then

$$\frac{d}{du}[\int_0^u f(x,u)dF(x)] = \int_0^u \frac{\partial f(x,u)}{\partial u} dF(x). \tag{7}$$

That is, we could place the derivative of the left side of (7) into the integration of which the upper limit is variable u.

Proof. According to the definition of derivative we have

$$\begin{aligned}
&\frac{d}{du}[\int_0^u f(x,u)dF(x)] \\
= &\lim_{\Delta u\to 0}\frac{1}{\Delta u}\{\int_0^{u+\Delta u} f(x,u+\Delta u)dF(x) - \int_0^u f(x,u)dF(x)\} \\
= &\lim_{\Delta u\to 0}\int_0^u \frac{f(x,u+\Delta u)-f(x,u)}{\Delta u}dF(x) + \lim_{\Delta u\to 0}\int_u^{u+\Delta u}\frac{f(x,u+\Delta u)}{\Delta u}dF(x) \\
= &A+B,
\end{aligned}$$

where A and B express the first and the second limit above, respectively. From condition (b) we know that the integrand in A satisfies the condition of Lebesgue's convergence theorem, so the limit can be taken into the integration, that is

$$A = \int_0^u \frac{\partial f(x,u)}{\partial u}dF(x).$$

From condition (c) we have

$$\begin{aligned}
\mid B \mid &= \mid \lim_{\Delta u\to 0}\int_u^{u+\Delta u}\frac{f(x,u+\Delta u)}{\Delta u}dF(x) \mid \\
&\le \lim_{\Delta u\to 0}\int_u^{u+\Delta u} \mid \frac{f(x,u+\Delta u)}{\Delta u} \mid dF(x) \\
&= \lim_{\Delta u\to 0}\int_u^{u+\Delta u} \mid \frac{f(x,u+\Delta u)}{u+\Delta u-x} \mid\mid \frac{u+\Delta u-x}{\Delta u} \mid dF(x) \\
&\le \lim_{\Delta u\to 0}\int_u^{u+\Delta u} \mid \frac{f(x,u+\Delta u)}{u+\Delta u-x} \mid dF(x) = 0.
\end{aligned}$$

Condition (c) assures the final equation is true. This completes the proof.

Now we prove theorem 6.

Because of $E(X^n) < \infty$, we know

$$\lim_{u\to\infty}\int_u^\infty (x-u)^n dF(x) \le \lim_{u\to\infty}\int_u^\infty x^n dF(x) = 0.$$

So,

$$\Pi^{(n)}(\infty) = \lim_{u\to\infty}\int_u^\infty (x-u)^n dF(x) = 0. \tag{8}$$

If the following equation holds,

$$\frac{d}{du}[\Pi^{(n)}(u)] = -n\Pi^{(n-1)}(u), \tag{9}$$

by taking integration from u to ∞ at the both sides of (9), we would then have

$$\int_u^\infty \frac{d}{dx}\Pi^{(n)}(x)dx = -n\int_u^\infty \Pi^{(n-1)}(x)dx.$$

That is,

$$\Pi^{(n)}(\infty) - \Pi^{(n)}(u) = -n\int_u^\infty \Pi^{(n-1)}(x)dx.$$

By (8) we have

$$\Pi^{(n)}(u) = n\int_u^\infty \Pi^{(n-1)}(x)dx.$$

We need only then to prove (9) true.

At first we prove (9) for $n > 1$. We set $f(x, y) = (x - y)^n$ in lemma 7. Then the condition (a) of lemma 7 holds. Furthermore,

$$| \frac{f(x, u + \Delta u) - f(x, u)}{\Delta u} | = | \frac{(x - u - \Delta u)^n - (x - u)^n}{\Delta u} |$$

$$= \ | \frac{1}{\Delta u}[-C_n^1(x-u)^{n-1}\Delta u + C_n^2(x-u)^{(n-2)}(\Delta u)^2 + \cdots$$

$$+(-1)^k C_n^k (x-u)^{(n-k)}(\Delta u)^k + \cdots + (-1)^n(\Delta u)^n] \ |$$

$$\leq \ | n(x-u)^{n-1} | + \sum_{k=2}^{n} C_n^k \ | x - u |^{(n-k)} | \Delta u |^{(k-1)},$$

where $C_n^k = \frac{n!}{k!(n-k)!}$.

If x and u both take values in finite intervals, without loss of generality, we suppose the interval is $[0, A]$, and $| \Delta u | \leq 1$, then the right side of above equation is bounded. If we let G denote this bound, then we can take G as $g(x)$ in lemma 7 and the condition (b) of lemma 7 holds. Furthermore,

$$\lim_{|x-y| \to 0} | \frac{f(x, y)}{x - y} | = \lim_{|x-y| \to 0} | x - y |^{n-1} = 0 \quad \text{for } n > 1.$$

So the condition (c) also holds.

In the following we use lemma 7 to prove formula (9) for $n > 1$. In lemma 7, the interval of integration is 0 to u, but now we need the interval of integration to be u to ∞. We begin as follows:

$$\begin{aligned}
\Pi^{(n)}(u) &= \int_u^\infty (x-u)^n dF(x) \\
&= \int_0^\infty (x-u)^n dF(x) - \int_0^u (x-u)^n dF(x) \\
&= \int_0^\infty \sum_{k=0}^{n} (-1)^k C_n^k x^{n-k} u^k dF(x) - \int_0^u (x-u)^n dF(x) \\
&= \sum_{k=0}^{n} (-1)^k C_n^k u^k E(X^{n-k}) - \int_0^u (x-u)^n dF(x) \\
&= I(u) - J(u),
\end{aligned}$$

where $I(u)$ denotes the sum at the right side above and $J(u)$ denotes the integral. Taking derivative of $I(u)$ and $J(u)$ respectively, we have

$$\begin{aligned}
\frac{d}{du} I(u) &= \sum_{k=1}^{n} (-1)^k C_n^k k u^{k-1} E(X^{n-k}) \\
&= -n \sum_{i=0}^{n-1} (-1)^i C_{n-1}^i u^i E(X^{n-1-i}) \\
&= -n \int_0^\infty (x-u)^{n-1} dF(x).
\end{aligned} \tag{10}$$

And by lemma 7,

$$\frac{d}{du} J(u) = \frac{d}{du}[\int_0^u (x-u)^n dF(x)] = -n \int_0^u (x-u)^{n-1} dF(x). \tag{11}$$

Combine (10) and (11) we have

$$\begin{aligned}
\frac{d}{du} \Pi^{(n)}(u) &= -n[\int_0^\infty (x-u)^{n-1} dF(x) - \int_0^u (x-u)^{n-1} dF(x)] \\
&= -n \int_u^\infty (x-u)^{n-1} dF(x) = -n\Pi^{(n-1)}(u).
\end{aligned}$$

That is formula (9) holds for $n = 2, 3, \cdots$. So theorem 6 holds for $n = 2, 3, \cdots$, too. In the following we check theorem 6 directly for $n = 1$. Taking integration by parts,

$$\Pi^{(1)}(u) = \int_u^\infty (x-u)dF(x) = \{-(x-u)\overline{F}(x)\}\mid_{x=u}^{\infty} + \int_u^\infty \overline{F}(x)dx$$
$$= \int_u^\infty \Pi^{(0)}(x)dx.$$

Thus theorem 6 holds for $n=1$. The proof of theorem 6 is complete.

Corollary 8 A distribution function $F(x)$ (or survival function $\overline{F}(x)$) and its nth stop-loss transform (n is an arbitrary nonnegative integer) are determined by each other.

Proof. When $n=0$, $\Pi_F^{(0)}(x) = \overline{F}(x) = 1 - F(x)$. Corollary 8 becomes true. When $n \geq 1$, from (6) we know that $\Pi_F^{(n)}(x)$ is determined by $\Pi_F^{(n-1)}(x)$. And by (9), we have

$$\Pi_F^{(n-1)}(x) = -\frac{1}{n}\frac{d}{dx}\Pi_F^n(x).$$

Then we arrive at our conclusion by induction.

2 Stop-loss Orders and Their Properties

Definition 9. We say that X is less than Y in the meaning of the nth stop-loss order, denoted by $X <_{sl(n)} Y$, if

$$E(X^k) \leq E(Y^k), \quad k = 1, 2, \cdots, n-1. \tag{12}$$

$$\Pi_X^{(n)}(u) \leq \Pi_Y^{(n)}(u), \quad \forall u \geq 0. \tag{13}$$

When $n = 0$, the formula (12) disappears and formula (13) becomes

$$\overline{F}_X(u) \le \overline{F}_Y(u), \quad \forall u \ge 0.$$

When $n = 1$, then formula (12) is trivial and formula (13) becomes

$$\int_u^\infty \overline{F}_X(x)dx \le \int_u^\infty \overline{F}_Y(x)dx, \quad \forall u \ge 0.$$

Now we study a class of functions with certain properites. Suppose function $u(x)$, $-\infty < x < \infty$ satisfies: $u^{(n+1)}(x)$ exists except at a finite number of points, and

$$(-1)^{k-1}u^{(k)}(x) \ge 0, \quad \forall x, k = 1, 2, \cdots, n+1. \tag{14}$$

Let

$$U_n = \{u(x) : u(x) \text{ satisfies } (14)\}, n = 0, 1, 2, \cdots.$$

Obviously, $U_{n+1} \subset U_n$, that is, classes of functions decrease with respect to n, $n = 0, 1, 2, \cdots$.

Inequality (14) implies that

$$u^{(k)}(x) \ge 0, \quad \text{when k is odd,}$$

$$u^{(k)}(x) \le 0, \text{ when k is even.}$$

Let

$$w(x) = -u(-x),\ u \in U_n.$$

Then for arbitrary real number x and nonnegative integer $k \le n+1$, we have

$$w^{(k)}(x) = (-1)^{(k+1)}u^{(k)}(-x) \ge 0. \tag{15}$$

Let

$$W_n = \{w(x) : \ w(x) \text{ satisfies } (15)\}.$$

It is easy to see that if we let $u(x) = -w(-x)$, where $w(x) \in W_n$, then

$$u^{(k)}(x) = (-1)^{k+1} w^{(k)}(-x),$$

$$(-1)^{(k-1)} u^{(k)}(x) = (-1)^{2k} w^{(k)}(-x) \geq 0,$$

so $u(x) \in U_n$. Hence we go to a conclusion that there is an one to one correspondence between the elements of U_n and W_n.

The following theorem and its proof is similar to that of theorem 4.2.1 in [1]. But here we add one sufficient and necessary condition, (17), and the proof becomes more clear than that in [1].

Theorem 10 $X <_{sl(n)} Y$, if and only if

$$E[u(-X)] \geq E[u(-Y)], \quad \forall u \in U_n, \tag{16}$$

if and only if

$$E[w(X)] \leq E[w(Y)], \quad \forall w \in W_n. \tag{17}$$

Proof. At first we prove the equivalence of (16) and (17). Suppose inequality (16) holds, we want to prove (17) holds. Let $u(x) = -w(-x)$, then $u(x) \in U_n$. Hence by (16) we have

$$E[u(-X)] \geq E[u(-Y)].$$

That is

$$E[-w(X)] \geq E[-w(Y)].$$

Thus

$$E[w(X)] \leq E[w(Y)].$$

Hence inequality (17) holds. It is similar to deduce (16) from (17).

In the following we prove $X <_{sl(n)} Y \Longleftrightarrow$ (17).

($\Leftarrow$): Suppose (17) holds. Let

$$w(x) = \{(x-u)_+\}^k,\ u \geq 0, 1 \leq k \leq n.$$

Then $\forall i \leq k, -\infty < x < \infty,$

$$w^{(i)}(x) = \begin{cases} k(k-1)\cdots(k-i+1)(x-u)^{(n-i)}, & \text{for } x > u, \\ 0, \quad \text{for } x < u. \end{cases}$$

and $\forall i > k, -\infty < x < \infty,\ \ w^{(i)}(x) = 0.$

Since $w^{(k)}(x) \geq 0$ for all positive integer k, we have $w(x) \in W_n$. By the assumption of (17) we have

$$E[w(X)] \leq E[w(Y)].$$

That is

$$E[\{(X-u)_+\}^k] \leq E[\{(Y-u)_+\}^k].$$

Let k take value from 1 to $n-1$, and let $u = 0$, we see that the inequalities (12) hold; let $k = n$, we go to (13). So, $X <_{sl(n)} Y$ by definition.

($\Rightarrow$) Suppose $w^{(k)}(x) \geq 0, \forall k = 1, 2, \cdots n+1$, then we have the following expansion of $w(x)$

$$w(x) = \sum_{k=0}^{n} \frac{w^{(k)}(0)}{k!} x^k + \int_0^x \frac{(x-t)^n}{n!} dw^{(n)}(t). \tag{18}$$

We prove formula (18) at first. Taking integration by parts, we have

$$\begin{aligned}
& \int_0^x \frac{(x-t)^n}{n!} dw^{(n)}(t) \\
= & \left[\frac{(x-t)^n}{n!} w^{(n)}(t)\right] \Big|_{t=0}^{x} - \int_0^x w^{(n)}(t) \frac{d}{dt}\left[\frac{(x-t)^n}{n!}\right] \\
= & -\frac{x^n}{n!} w^{(n)}(0) + \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} w^{(n)}(t) dt \\
= & -\frac{x^n}{n!} w^{(n)}(0) + \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} dw^{(n-1)}(t) \\
= & \cdots = -\sum_{k=1}^{n} \frac{w^{(n)}(0)}{k!} x^k + \int_0^x dw(t) \\
= & -\sum_{k=1}^{n} \frac{w^{(n)}}{k!} x^k + w(x) - w(0).
\end{aligned}$$

Removing the terms in the right side except $w(x)$, we go to (18). Now suppose $X <_{sl(n)} Y$ we want to prove $E[w(X)] \leq E[w(Y)]$. By formula (18) we have

$$\begin{aligned}
E[w(X)] &= E\left[\sum_{k=0}^{n} \frac{w^{(n)}(0)}{k!} X^k + \int_0^X \frac{(X-u)^n}{n!} dw^{(n)}(u)\right] \\
&= \sum_{k=0}^{n} \frac{w^{(n)}(0)}{k!} E(X^k) + E\left[\int_0^X \frac{(X-u)_+^n}{n!} dw^{(n)}(u)\right] \\
&= \sum_{k=0}^{n} \frac{w^{(n)}(0)}{k!} E(X^k) + \int_0^\infty \frac{E[\{(X-u)_+\}^n]}{n!} dw^{(n)}(u).
\end{aligned}$$

(In the right side above the upper limit of integration can be expanded from X to ∞, because $(x-u)_+ = 0$ when $u > x$).

By $X <_{sl(n)} Y$, we know that

$$E(X^k) \leq E(Y^k), \quad k = 0, 1, \cdots n,$$

and

$$E[\{(X-u)_+\}^n] \leq E[\{(Y-u)_+\}^n], \quad \forall u \geq 0.$$

So,

$$E[w(X)] \leq \sum_{k=0}^{n} \frac{w^{(n)}(0)}{k!} E(Y^k) + \int_0^\infty \frac{E[\{(Y-u)_+\}^n]}{n!} dw^{(n)}(u).$$

From the above we see that the right side of the final inequality is just $E[w(Y)]$.

We then have

$$E[w(X)] \leq E[w(Y)].$$

The proof is complete.

Proposition 11. Suppose $E(X) = E(Y)$. If $X <_{sl(1)} Y$ then

$$var(X) \leq var(Y).$$

Proof. From (5) we know

$$E(X^2) = 2\int_0^\infty \Pi_X^{(1)}(y)dy \leq 2\int_0^\infty \Pi_Y^{(1)}(y)dy = E(Y^2).$$

Hence, by $E(X) = E(Y)$,

$$var(X) = E(X^2) - [E(X)]^2 \leq E(Y^2) - [E(Y)]^2 = var(Y).$$

The proof is complete.

Theorem 12. If $X <_{sl(n)} Y$, then

$$X <_{sl(m)} Y, \quad \forall m > n.$$

Proof. Because of the decreasing property of U_n with respect to n, when $m > n$, we have $U_m \subset U_n$. By theorem 10 we arrive at our desired conclusion.

Proposition 13. If $E(X) \le E(Y)$ and $\exists\, c \ge 0$ such that

$$F_X(x) \le F_Y(x), \quad \text{for } x \le c, \tag{19}$$

$$F_X(x) \ge F_Y(x), \quad \text{for } x > c. \tag{20}$$

Then $X <_{sl(1)} Y$.

Proof. Let

$$h(x) = \Pi_Y^{(1)}(x) - \Pi_X^{(1)}(x) = \int_x^\infty \overline{F}_Y(u)du - \int_x^\infty \overline{F}_X(u)du,$$

then we have

$$h'(x) = -\overline{F}_Y(x) - [-\overline{F}_X(x)] = F_Y(x) - F_X(x).$$

And by conditions (19) and (20) we have

$$h'(x) \ge 0, \quad \text{for } x \le c,$$

$$h'(x) \le 0, \quad \text{for } x > c,$$

We then have

$$h(0) = \int_0^\infty \overline{F}_Y(u)du - \int_0^\infty \overline{F}_X(u)du = E(Y) - E(X) \geq 0,$$

and

$$h(\infty) = \lim_{x\to\infty} h(x) = \lim_{x\to\infty} \int_x^\infty \overline{F}_Y(u)du - \lim_{x\to\infty} \int_x^\infty \overline{F}_X(u)du = 0.$$

We conclude that $h(x) \geq 0$, $\forall x \geq 0$. Otherwise, if $h(x) < 0$ for some x_1, then there must be an intersection point of $h(x)$ with the x-axis, say, at point x_0, $x_0 < x_1$, and $h'(x) \leq 0$ must hold for $\forall x \geq x_0$, that means $h(\infty) = 0$ can not hold. Now from $h(x) \geq 0$, $\forall x \geq 0$, we have $\Pi_X^{(1)}(x) \leq \Pi_Y^{(1)}(x)$, $\forall x \geq 0$. So we have $X <_{sl(1)} Y$ by definition.

We can interpret proposition 13 more simply by diagram. By conditions (19) and (20) we know that the curves of $\overline{F}_X(x) = 1 - F_X(x)$ and $\overline{F}_Y(x) = 1 - F_Y(x)$ intersect at $x = c$. We know also that $E(X)$ equals the area under the curve of $\overline{F}_X(x)$ and $E(Y)$ equals the area under the curve of $\overline{F}_Y(u)$. Hence, for arbitrary $u \geq 0$, the area on the right side of $x = u$ and under the curve of $\overline{F}_X(x)$ must be less than that under the curve of $\overline{F}_Y(x)$. That is $\Pi_X^{(1)}(u) \leq \Pi_Y^{(1)}(u)$, $\forall u \geq 0$, which is desired for proposition 13.

Proposition 14. If $E(X) \leq E(Y)$, and $\exists\, a, b, 0 \leq a \leq b < \infty$ such that

$$dF_X(x) \leq dF_Y(x), \quad \text{for } x \leq a \text{ or } x \geq b, \tag{21}$$

$$dF_X(x) \geq dF_Y(x), \quad \text{for } a < x < b. \tag{22}$$

Then $X <_{sl(1)} Y$.

Proof. Similar to the proof of proposition 13, we need to show

$$h(x) = \Pi_Y^{(1)}(x) - \Pi_X^{(1)}(x) \geq 0.$$

We have

$$h'(x) = -\overline{F}_Y(x) - [-\overline{F}_X(x)] = F_Y(x) - F_X(x) = \int_0^x [dF_Y(x) - dF_X(x)].$$

By conditions (21) and (22) we know that when $x \leq a$, $h'(x) \geq 0$ and $h'(x)$ monotonously increases; when $a < x < b$, $h'(x)$ monotonously decreases; when $x \geq b$, $h'(x)$ increases again, and

$$\lim_{x \to \infty} h'(x) = \int_0^\infty dF_Y(x) - \int_0^\infty dF_X(x) = 1 - 1 = 0,$$

There must be a point c, such that $a < c < b$, and $h'(x) \geq 0$, $\forall x \leq c$; $h'(x) \leq 0$ $\forall x > c$. Furthermore, as we have seen in the proposition 13, we have

$$h(0) = E(Y) - E(X) \geq 0,$$

and

$$\lim_{x \to \infty} h(x) = 0.$$

The figure of $h(x)$ is the same as that in the proposition 13. Hence we have $X <_{sl(1)} Y$ as in the proposition 13.

When X and Y are both continuous, denoting the distribution density function by $f_X(x)$ and $f_Y(x)$ respectively, then the conditions (21) and (22) are equivalent to:

$$f_X(x) \leq f_Y(x), \quad \text{for } x \leq a \text{ or } x \geq b,$$

and

$$f_X(x) \geq f_Y(x), \quad \text{for } a < x < b.$$

When X and Y both are discrete, assuming their domain is $\{x_i,\ i = 1, 2, \cdots\}$ and their probability functions are $P_X(x_i)$ and $P_Y(x_i)$ respectively, then conditions (21) and (22) are equivalent to

$$P_X(x_i) \leq P_Y(x_i), \quad \text{for } x_i \leq a \text{ or } x_i \geq b,$$

$$P_X(x_i) \geq P_Y(x_i), \quad \text{for } a < x_i < b.$$

Next we show the maintenance properties of the nth stop-loss order.

Theorem 15 The nth stop-loss order is maintained under the summation of independent random variables. That is, if

$$X_i <_{sl(n)} Y_i, \quad i = 1, 2, \cdots k,$$

where k is a positive integer, then

$$\sum_{i=1}^{k} X_i <_{sl(n)} \sum_{i=1}^{k} Y_i, \quad n = 0, 1, 2, \cdots. \tag{23}$$

It was proved in [1] that the 1^{st} stop-loss order is maintained under the summation of independent random variables (see page 30 of [1], theorem 3.2.2.). Theorem 15 is its generalization and the method used here for proving the theorem is completely different from the method in [1].

Proof. We first prove theorem 15 for $k = 2$.

Suppose X_1 and X_2 are independent, Y_1 and Y_2 are independent and

$$X_i <_{sl(n)} Y_i, \quad i = 1, 2, \; n \geq 0.$$

We now use theorem 10 to prove (23). By theorem 10, $\forall\; w(x) \in W_n$, we need only to prove

$$E[w(X_1 + X_2)] \leq E[w(Y_1 + Y_2)].$$

Let

$$w_1(x, t) = w(x + t), \tag{24}$$

where t is a real number. Since $w(x) \in W_n$, from the definition of W_n we have

$$\frac{d^k}{dx^k} w_1(x, t) = w^{(k)}(x + t) \geq 0, \quad k = 1, \cdots, n + 1.$$

Again by the definition of W_n, we know that for a fixed t, $w_1(x, t)$ is a function of x and belongs to W_n. From $X_1 <_{sl(n)} Y_1$, and by theorem 10, we can conclude that

$$\int_0^\infty w(x+t) dF_{X_1}(x) = E[w_1(X_1, t)] \leq E[w_1(Y_1, t)] = \int_0^\infty w(x+t) dF_{Y_1}(x). \tag{25}$$

Further, let

$$w_2(x) = E[w_1(Y_1, x)] = \int_0^\infty w(y+x)dF_{Y_1}(y). \tag{26}$$

Since $w^{(k)}(x) \geq 0$, we have

$$w_2^{(k)}(x) = \int_0^\infty w^{(k)}(y+x)dF_{Y_1}(x) \geq 0, \quad k = 1, 2, \cdots, n+1.$$

Hence $w_2(x) \in W_n$. From this and the condition $X_2 <_{sl(n)} Y_2$ we have

$$\begin{aligned} &\int_0^\infty [\int_0^\infty w(y+x)dF_{Y_1}(y)]dF_{X_2}(x) = \int_0^\infty w_2(x)dF_{X_2}(x) = E[w_2(X_2)] \\ &\leq E[w_2(Y_2)] = \int_0^\infty [\int_0^\infty w(y+x)dF_{Y_1}(y)]dF_{Y_2}(x). \end{aligned} \tag{27}$$

Taking the integration of the both sides of (25) with the distribution function $dF_{X_2}(t)$, we have

$$\int_0^\infty [\int_0^\infty w(y+t)dF_{X_1}(y)]dF_{X_2}(t) \leq \int_0^\infty [\int_0^\infty w(y+t)dF_{Y_1}(y)]dF_{X_2}(t). \tag{28}$$

Combine (28) and (27) to arrive at

$$\int_0^\infty [\int_0^\infty w(y+t)dF_{X_1}(y)]dF_{X_2}(t) \leq \int_0^\infty [\int_0^\infty w(y+x)dF_{Y_1}(y)]dF_{Y_2}(t).$$

This is simply $E[w(X_1 + X_2)] \leq E[w(Y_1 + Y_2)]$. Next by mathematical induction we can conclude that (23) holds. The proof is complete.

Theorem 16. The nth stop-loss order is maintained under a compound operation. That is, suppose $X_1, X_2, \cdots$, $Y_1, Y_2, \cdots$, and integer valued, N_1, N_2

are all independent random variables. In addition, N_1 and N_2 have identical probability distributions. Let

$$S_1 = \sum_{i=1}^{N_1} X_i, \quad S_2 = \sum_{i=1}^{N_2} Y_i.$$

If

$$X_i <_{sl(n)} Y_i, \quad i = 1, 2, \cdots,$$

then

$$S_1 <_{sl(n)} S_2. \tag{29}$$

Proof. According to theorem 10, it is sufficient to prove that

$$\forall\, w \in W_n, \quad E[w(S_1)] \le E[w(S_2)].$$

In fact we have,

$$\begin{aligned} E[w(S_1)] &= E[E[w(S_1) \mid N_1]] \\ &= \textstyle\sum_{n=0}^{\infty} E[w(S_1) \mid N_1 = n] Pr(N_1 = n) \\ &= \textstyle\sum_{n=0}^{\infty} E[w(X_1 + X_2 + \cdots + X_n) \mid N_1 = n] Pr(N_1 = n) \\ &= \textstyle\sum_{n=0}^{\infty} E[w(X_1 + X_2 + \cdots + X_n)] Pr(N_1 = n). \end{aligned}$$

The last equation holds because $X_1, X_2, \cdots, X_n$ and N_1 are independent. Next, using theorem 15 we have

$$E[X_1 + X_2 + \cdots + X_n] \le E[Y_1 + Y_2 + \cdots + Y_n].$$

Notice N_1 and N_2 have identical probability distributions, so we have

$$\begin{aligned} E[w(S_1)] &\leq \sum_{n=0}^{\infty} E[w(Y_1 + Y_2 + \cdots + Y_n)]Pr(N_1 = n) \\ &= \sum_{n=0}^{\infty} E[w(Y_1 + Y_2 + \cdots + Y_n)]Pr(N_2 = n) \\ &= E[w(S_2)]. \end{aligned}$$

The proof is complete.

References

[1] Goovaerts, M.J., Kaas, R., et al. (1990), *Effective Actuarial Methods*, North-Holland.