

A multifactor, equilibrium model for the term structure and inflation

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Summary

This paper describes the development of a stochastic model for the combined term structure of:

- rates of interest on fixed-interest bonds;
- rates of interest on index-linked bonds;
- consumer price inflation.

Much of the detailed development in the paper concentrates on the fixed-interest model. This makes use of the new framework developed in recent years by Flesaker & Hughston (1996) (FH), Rutkowski (1997) and Rogers (1997). These papers are similar to the work of Heath, Jarrow & Morton (1992) (HJM) in that they describe a general framework for the development of arbitrage-free term-structure models. However, the new framework provides a relatively simple means of ensuring that nominal interest rates always remain positive. (This is not the case with HJM.) Here we develop a specific equilibrium model within the FH framework which is driven by a multifactor Ornstein-Uhlenbeck (OU) process. We exploit the normality of the distribution of these driving factors to derive prices for zero-coupon bonds and interest rates. One of the OU factors is designed to create long-term cycles in the term structure mimicking observed behaviour in the UK and elsewhere over the last 100 years.

The paper then describes a two-factor model for real rates of interest as might be inferred from index-linked bond prices. This model is a generalisation of the Vasicek (1977) model. A model for consumer price inflation is developed in which the rate of price inflation is equal to the different between nominal and real short-term rates of interest adjusted for a inflation risk premium (to reflect a market preference for index-linked assets) and then subject to a zero mean error.

The paper finishes with a discussion of the calibration of the fixed-interest part of the model.

Keywords: multifactor; arbitrage-free; positive interest; Ornstein-Uhlenbeck; equilibrium; nominal rates; real rates; inflation.

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0 Note on notation

For $i = 1, 2, 3$ let $Z_i(t)$ be n_i dimensional Brownian motion under the real-world measure P . Corresponding processes under the equivalent risk-neutral measure (Q) and terminal measure (P_∞) (where this is required) are denoted by $\tilde{Z}_i(t)$ and $\hat{Z}_i(t)$ respectively. The filtration generated by $Z_i(t)$ is denoted by \mathcal{F}_t^i . The filtration generated by $\{Z_1(t), \dots, Z_i(t)\}$ is denoted by $\mathcal{F}_t^{(i)}$. $\tilde{Z}_i(t)$ and $Z_i(t)$ are linked by the market price of risk $\lambda_i(t)$:

$$d\tilde{Z}_i(t) = dZ_i(t) + \lambda_i(t)dt$$

1 A model for fixed-interest bond prices

1.1 Background

Following the approach developed by Flesaker & Hughston (1996), Rutkowski (1997) and Rogers (1997) we propose that zero-coupon bond prices $P(t, T)$ are modelled according to the following stochastic process.

Let $M(t, s)$ for $0 \leq t \leq s < \infty$ and for $0 \leq t \leq s$ be a family of strictly positive stochastic processes over the index s which are martingales with respect to t under some probability measure P_∞ . That is, given s , for $t < u < s$, $E_{P_\infty}[M(u, s) | \mathcal{F}_t] = M(t, s)$. Furthermore, we define $M(0, s) = 1$ for all s and assume that for each s , $M(t, s)$ is a diffusion process adapted to a finite (say n_1) dimensional Brownian motion, $\hat{Z}_1(t)$ (under P_∞).

Zero-coupon bond prices are defined as

$$P(t, T) = \frac{\int_t^\infty M(t, s)\phi(s)ds}{\int_t^\infty M(t, s)\phi(s)ds}$$

[This form, proposed by Flesaker & Hughston (1996), was generalised by Rutkowski (1997) and Rogers (1997). Rutkowski (1997) defines

$$P(t, T) = \frac{E_{P_\infty}[A_T | \mathcal{F}_t]}{A_t}$$

where A_t is a strictly-positive supermartingale under the measure P_∞ . The Flesaker & Hughston (1996) form which we will use in this paper is a special case of this framework where $A_t = \int_t^\infty M(t, s)\phi(s)ds$ where $M(t, s)$ is a positive martingale under P_∞ and $\phi(s) > 0$ for all s . Rutkowski (1997) and Rogers (1997) demonstrate that models of this type are arbitrage free.]

Since $M(0, s) = 1$ for all s we may infer that

$$\phi(s) = \frac{\partial}{\partial s} P(0, s)$$

up to a constant, non-zero scaling factor.

Instantaneous forward rates are then

$$\begin{aligned} f(t, T) &= -\frac{\partial}{\partial T} \log P(t, T) \\ &= \frac{M(t, T)\phi(T)}{\int_T^\infty M(t, s)\phi(s)ds} \\ \Rightarrow r(t) &= f(t, t) \\ &= \frac{M(t, t)\phi(t)}{\int_t^\infty M(t, s)\phi(s)ds} \end{aligned}$$

Although we can write down an expression for the short rate, $r(t)$, in this way it is not possible, in general, to express the dynamics of $r(t)$ in any simple fashion (for example, like we can with the Vasicek, 1977, model).

We can also write down expressions for bond volatilities which enables us to link the model into the framework of Heath, Jarrow & Morton (1992). Since $M(t, s)$ is a martingale under P_∞ we can write $dM(t, T) = M(t, T)\sigma_1^T(t, T)d\hat{Z}_1(t)$. Define

$$V(t, T) = \frac{\int_T^\infty M(t, s)\phi(s)\sigma_1(t, s)ds}{\int_T^\infty M(t, s)\phi(s)ds}$$

Note that since $\sigma_1(t, s)$ is an $n_1 \times 1$ vector, $V(t, T)$ is also $n_1 \times 1$. We will write $V(t, T) = (V_1(t, T), \dots, V_{n_1}(t, T))^T$.

The dynamics of the zero-coupon bond prices can then be expressed in the form

$$\begin{aligned} \frac{dP(t, T)}{P(t, T)} &= r(t)dt + S_P(t, T)^T (d\hat{Z}_1(t) - V(t, t)dt) \\ \text{where } S_P(t, T) &= V(t, T) - V(t, t) \end{aligned}$$

It follows that $S_P(t, T)$ is the price volatility function with each of the n_1 components defining the volatility of the price of a particular bond with respect to each of the n_1 sources of uncertainty.

Since we have expressed the price dynamics in the way given above we can immediately see that if

$$\tilde{Z}_1(t) = \hat{Z}_1(t) - \int_0^t V(s, s) ds$$

then $dP(t, T) = P(t, T) \left(r(t) dt + S_P(t, T)^T d\tilde{Z}_1(t) \right)$.

Suppose that the function $\sigma_1(t, T)$ has been defined in such a way that

$$E_{P_\infty} \left[\exp \left(\frac{1}{2} \int_0^t V_i(s, s)^2 ds \right) \right] < \infty \quad \text{for each } i$$

(for example, see Baxter & Rennie, 1996). Then, by the Cameron-Martin-Girsanov (CMG) Theorem, there exists a measure Q equivalent to P_∞ under which $\tilde{Z}_1(t)$ is an n_1 -dimensional Brownian motion. Given the form of $dP(t, T)$ we can see that Q is the risk-neutral measure. Provided each $\sigma_{1i}(t, T)$ for all $t, T > t$, is bounded, then $V_i(s, s)$ must be bounded, so the CMG condition is satisfied.

We are then free to make a further change of measure from Q to the real-world measure P .

1.2 A specific multifactor equilibrium model

We have already expressed $M(t, T)$ in the following way

$$\begin{aligned} M(0, T) &= 1 \quad \text{for all } T \\ dM(t, T) &= M(t, T) \sigma_1(t, T)^T dZ_1(t) \\ &= M(t, T) \sum_{i=1}^{n_1} \sigma_{1i}(t, T) d\hat{Z}_{1i}(t) \end{aligned}$$

where $\hat{Z}_{11}(t), \dots, \hat{Z}_{1n_1}(t)$ are n_1 independent Brownian motions under P_∞ .

Suppose now that $\sigma_{1i}(t, T) = \sigma_{1i} \exp[-\alpha_{1i}(T - t)]$. We have

$$\begin{aligned} M(t, T) &= \exp \left[\sum_{i=1}^{n_1} \left\{ \int_0^t \sigma_{1i}(u, T) d\hat{Z}_{1i}(t) - \frac{1}{2} \int_0^t \sigma_{1i}(u, T)^2 du \right\} \right] \\ &= \exp \left[\sum_{i=1}^{n_1} \left\{ \sigma_{1i} \int_0^t e^{-\alpha_{1i}(T-u)} d\tilde{Z}_{1i}(u) - \frac{\sigma_{1i}^2}{4\alpha_{1i}} e^{-2\alpha_{1i}(T-t)} (1 - e^{-2\alpha_{1i}(t)}) \right\} \right] \\ &= \exp \left[\sum_{i=1}^{n_1} \left\{ \sigma_{1i} e^{-\alpha_{1i}(T-t)} X_{1i}(t) - \frac{\sigma_{1i}^2}{4\alpha_{1i}} e^{-2\alpha_{1i}(T-t)} (1 - e^{-2\alpha_{1i}(t)}) \right\} \right] \end{aligned}$$

where $X_{1i}(t) = \int_0^t e^{-\alpha_{1i}(t-u)} d\hat{Z}_{1i}(u)$

Now we recognise the $X_{1i}(t)$ as standard Ornstein-Uhlenbeck processes (for example, see Øksendal, 1998): that is, $X_{1i}(t)$ is the solution to the stochastic differential equation $X_{1i}(0) = 0$, $dX_{1i}(t) = -\alpha_{1i}X_{1i}(t)dt + d\tilde{Z}_{1i}(t)$. Further details are given in Appendix A.

Now suppose that

$$\phi(s) = \phi \exp \left[-\beta s + \sum_{i=1}^{n_1} \left\{ \sigma_{1i} e^{-\alpha_{1i}s} \tilde{X}_{1i}(0) - \frac{\sigma_{1i}^2}{4\alpha_{1i}} e^{-2\alpha_{1i}s} \right\} \right]$$

for some ϕ , β and $\tilde{X}_{1i}(0)$. Then

$$\int_T^\infty \phi(s) M(t, s) ds = \phi e^{-\beta s} \int_T^\infty \exp \left[-\beta u + \sum_{i=1}^{n_1} \left\{ \sigma_{1i} e^{-\alpha_{1i}u} \tilde{X}_{1i}(t) - \frac{\sigma_{1i}^2}{4\alpha_{1i}} e^{-2\alpha_{1i}u} \right\} \right] du$$

where $\tilde{X}_{1i}(t) = X_{1i}(t) + e^{-\alpha_{1i}t} \tilde{X}_{1i}(0)$ is also a standard Ornstein-Uhlenbeck process but starting away from zero. Then we have

$$P(t, T) = \frac{\int_T^\infty H(u, \tilde{X}_1(t)) du}{\int_0^\infty H(u, \tilde{X}_1(t)) du}$$

where $H(u, x) = \exp \left[-\beta u + \sum_{i=1}^{n_1} \left\{ \sigma_{1i} e^{-\alpha_{1i}u} x_i - \frac{\sigma_{1i}^2}{4\alpha_{1i}} e^{-2\alpha_{1i}u} \right\} \right]$

1.3 The risk-free rate and forward rates

Applying the general formula in Section 1.1 we see that the forward-rate curve is

$$\begin{aligned} f(t, T) &= \frac{H(T-t, \tilde{X}_1(t))}{\int_T^\infty H(u, \tilde{X}_1(t)) du} \\ &= \left\{ \int_T^\infty \exp \left[-\beta \{u - (T-t)\} + \sum_{i=1}^{n_1} \left(\sigma_{1i} \tilde{X}_{1i}(t) \left[e^{-\alpha_{1i}u} - e^{-\alpha_{1i}(T-t)} \right] \right. \right. \right. \\ &\quad \left. \left. \left. - \frac{\sigma_{1i}^2}{4\alpha_{1i}} \left[e^{-2\alpha_{1i}u} - e^{-2\alpha_{1i}(T-t)} \right] \right) \right] du \right\}^{-1} \\ &= \left\{ \int_0^\infty \exp \left[-\beta v + \sum_{i=1}^{n_1} \left\{ \sigma_{1i} \tilde{X}_{1i}(t) e^{-\alpha_{1i}(T-t)} \left(e^{-\alpha_{1i}v} - 1 \right) \right. \right. \right. \\ &\quad \left. \left. \left. - \frac{\sigma_{1i}^2}{4\alpha_{1i}} e^{-2\alpha_{1i}(T-t)} \left(e^{-2\alpha_{1i}v} - 1 \right) \right\} \right] dv \right\}^{-1} \\ \Rightarrow r(t) &= \left\{ \int_0^\infty \exp \left[-\beta v + \sum_{i=1}^{n_1} \left\{ \sigma_{1i} \tilde{X}_{1i}(t) \left(e^{-\alpha_{1i}v} - 1 \right) - \frac{\sigma_{1i}^2}{4\alpha_{1i}} \left(e^{-2\alpha_{1i}v} - 1 \right) \right\} \right] dv \right\}^{-1} \end{aligned}$$

If we look more closely at the formula for $f(t, T)$ we can see that as T tends to infinity, $f(t, T)$ tends to β : that is, β is the constant long-term forward rate. (Note that Dybvig, Ingersoll & Ross, 1994, established that under the assumption of no arbitrage a model for the term-structure of interest rates must have a non-decreasing long-term *spot* rate.)

The par yield on irredeemable bonds (assuming continuous payment of coupons) is

$$\rho(t) = \left[\int_0^\infty P(t, t+s) ds \right]^{-1} = \frac{\int_0^\infty H(u, \tilde{X}_{1i}(t)) du}{\int_0^\infty u H(u, \tilde{X}_{1i}(t)) du}$$

and this can be developed in the same way as $f(t, T)$ and $r(t)$ above.

1.4 Equivalence of P_∞ and Q

Recall that $\tilde{Z}_{1i}(t) = \hat{Z}_{1i}(t) - V_i(t, t)dt$ where the $\tilde{Z}_{1i}(t)$ and $\hat{Z}_{1i}(t)$ are Brownian Motions under the risk-neutral measure Q and the terminal measure P_∞ respectively. Here

$$V_i(t, t) = \frac{\int_0^\infty H(u, \tilde{X}_1(t)) \sigma_{1i} e^{-\alpha_{1i} u} du}{\int_0^\infty H(u, \tilde{X}_1(t)) du}.$$

Since $H(u, x) > 0$ for all $u > 0, x$, we have $V_i(t, t) < \sigma_{1i}$, for all t . For equivalence between P_∞ and Q we require $E_{P_\infty} \left[\exp \left(\frac{1}{2} \int_0^t V_i(s, s) ds \right) \right] < \infty$ for each i (the CMG condition). Since $V_i(t, t)$ is bounded this condition is satisfied.

1.5 Equilibrium

From the form of $H(u, x)$, and $\tilde{X}_1(t) = (\tilde{X}_{11}(t), \dots, \tilde{X}_{1n_1}(t))^T$ we can see that the $P(t, T)$ are Markov and time homogeneous. The model should, therefore, be described as an equilibrium model rather than a no-arbitrage model. With the former the model plus knowledge of $\tilde{X}_1(t)$ gives us a set of theoretical prices which may differ from those observed. With the latter initial observed prices form part of the input (hence the earlier note that $\phi(s) = \partial P(0, s) / \partial s$) but this results in the loss of time homogeneity. Both approaches have their own merits. Here the intention is that the number of factors, n_1 , should be large enough to ensure that once $\tilde{X}_1(t)$ has been estimated there is a close correspondence (but not exact) between theoretical and observed prices for all t . It can then be argued that frictions in the market such as transaction costs and buying and selling spreads prevent exploitation of the price errors.

1.6 Practical considerations

The structure of this model is such that only a limited number of random factors ($\tilde{X}_{1i}(t)$ for $i = 1, \dots, n_1$) need to be recorded in order for us to be able to reconstruct the evolution of the term structure through time, calculate prices, returns on assets and so on. This is in contrast to some no-arbitrage models based upon the Heath-Jarrow-Morton (1992) framework which require a record of the entire forward rate curve at all times.

In this the first stage of the development of this model it was not considered necessary to allow for correlation between the $\tilde{X}_{1i}(t)$. This is because the correlations did not enrich, in any obvious way, the range of shapes of forward-rate curves *etc.* which could be generated. This means that it would be unlikely that the more complex model would fit historical data significantly better than the model without correlations.

The $\tilde{X}_{1i}(t)$, for $i = 1, \dots, n_1$, follow a standard Ornstein-Uhlenbeck process under P_∞ . The processes are therefore particularly simple to simulate accurately under this measure given that, for $s > t$, $\tilde{X}_{1i}(s)$ is normally distributed. The nature of the changes of measure for each of the $\tilde{X}_{1i}(t)$ means that $\tilde{X}_{1i}(s)$ given $\tilde{X}_{1i}(t)$ is no longer normally distributed under Q . For simulation purposes, we are interested in the real-world measure P which has not really been discussed so far. If a constant market price of risk is employed relative to Q then the same problem exists (that $\tilde{X}_{1i}(s)$ is not normally distributed exists – although over a one-month period the normal approximation is reasonable). As an alternative we can employ a constant change of measure between P_∞ and P . This ensures that the $\tilde{X}_{1i}(t)$ still follow an Ornstein-Uhlenbeck process under P but with non-zero means. A less desirable consequence of this, though, is that this does occasionally allow risk-premia to become negative from time to time. The frequency of this clearly depends upon the parametrisation of the model and the size of the change of measure with a low frequency being tolerable for the sake of ease of simulation of the $\tilde{X}_{1i}(t)$.

It is necessary to carry out numerical integration in order to compute bond prices and interest rates on a given date and given $\tilde{X}_1(t)$. However, this step can be done in a straightforward and accurate way, since it only involves one-dimensional integration. Furthermore, with only a little extra work we can use numerical integration to calculate the distribution of many quantities.

The calibration of this part of the model is discussed in Section 4.

2 Real rates of interest, index-linked bonds and inflation

2.1 Basic principles

We start here by developing some general ideas before proposing a specific model.

Let $C(t)$ be the consumer price index at time t for a particular currency. Let $Q(t, T)$ be the price at time t for a payment of $C(T)/C(t)$ at time T (that is, the $Q(t, T)$ represent index-linked, zero-coupon prices). We assume for simplicity that $C(t)$ is becomes known at time t and that index-linked bonds index payments without a time lag. Real rates of interest can be derived from $Q(t, T)$ in the following simple ways:

$$\begin{aligned} \text{Spot rates: } R_Q(t, T) &= -\frac{1}{T-t} \log Q(t, T) \\ \text{Forward rates: } f_Q(t, T) &= -\frac{\partial}{\partial T} \log Q(t, T) \\ \text{Real risk-free rate: } r_Q(t) &= R_Q(t, t) = f_Q(t, t) \end{aligned}$$

In the index-linked bond market there is no real risk-free asset (that is, an asset which returns over the interval t to $t + dt$, for small dt , the increase in $C(t)$ plus the real risk-free rate $r_Q(t)$ with no volatility). In practice, the real forward and spot-rate curves are inferred from a limited number of index-linked coupon bonds. From either curve we can infer what $r_Q(t)$ would be if it existed.

Here we propose an equilibrium derivation of the $Q(t, T)$. We start with a diffusion model for $r_Q(t)$ under the risk-neutral measure Q (which we extend from coverage of the fixed-interest market to the index-linked market) and calculate prices according to the following formula:

$$Q(t, T) = E_Q \left[\exp \left(- \int_t^T r_Q(s) ds \right) \middle| \mathcal{F}_t^2 \right]$$

Under this model we have

$$\begin{aligned} dQ(t, T) &= Q(t, T) \left(r_Q(t) dt + S_Q(t, T)^T d\tilde{Z}_2(t) \right) \\ &= Q(t, T) \left\{ \left(r_Q(t) + \lambda_2(t)^T S_Q(t, T) \right) dt + S_Q(t, T)^T dZ_2(t) \right\} \end{aligned}$$

where $\tilde{Z}_2(t)$ is n_2 -dimensional Brownian motion under Q , $\tilde{Z}_2(t)$ is independent of $\tilde{Z}_1(t)$ (this assumption can be relaxed easily) and $\mathcal{F}_t^2 = \sigma \left(\{ \tilde{Z}_2(s) \mid 0 \leq s \leq t \} \right)$.

We will assume that $r_Q(t)$ is independent of $r(t)$ the (nominal) risk-free rate and of the consumer prices index, $C(t)$.

The consumer prices index is a positive process which is declared on a monthly basis. Here we will model it as a continuous-time diffusion process:

$$dC(t) = C(t) [\mu_C(t)dt + \sigma_3(t)dZ_3(t)]$$

where $Z_3(t)$ (which is $n_3 = 1$ -dimensional Brownian motion under the real world probability measure P) is independent of $Z_1(t)$ and $Z_2(t)$.

We will assume that $\mu_C(t)$ is measurable with respect to the filtration $\mathcal{F}^{(3)}$ the filtration generated by $Z_1(t)$ and $Z_2(t)$: that is, the drift of the price index is determined by bond prices. (We will see below that under Q the drift is equal to $r(t) - r_Q(t)$ under Q .) Furthermore we will assume that $\sigma_3(t)$ is small but non-zero. This reflects the assumption that $C(t)$ is, essentially, a commodities index and that at least some of the individual commodities making up the index have prices which change in an unpredictable way. The requirement that $\sigma_3(t) > 0$ also enables inflation to have different drifts under the real-world and risk-neutral measures.

Now the $Q(t, T)$ do not give the prices of tradeable assets in the sense that the definition of the payment at T is continually being changed. On the other hand, if we define $L(t, T) = C(t)Q(t, T)$, then the $L(t, T)$ do represent the prices of tradeable assets since the payment at T is always $C(T)$. Then we have

$$\begin{aligned} dL(t, T) &= Q(t, T)dC(t) + C(t)dQ(t, T) + dC(t)dQ(t, T) \\ &= L(t, T) (\mu_C(t)dt + \sigma_3(t)dZ_3(t) \\ &\quad + L(t, T) \left\{ (r_Q(t) + \lambda_2(t)^T S_Q(t, T)) dt + S_Q(t, T)^T dZ_2(t) \right\} + 0 \\ &= L(t, T) \left((\mu_C(t) + r_Q(t) + \lambda_2(t)^T S_Q(t, T))dt + S_Q(t, T)^T dZ_2(t) + \sigma_3(t)dZ_3(t) \right) \end{aligned}$$

(Independence of $C(t)$ and the $Q(t, T)$ means that $dC(t).dQ(t, T) = 0$.)

Since $L(t, T)$ is a tradeable asset we can also write its dynamics as

$$\begin{aligned} dL(t, T) &= L(t, T) \left(r(t)dt + S_L(t, T)^T d\tilde{Z}'(t) \right) \\ \text{where } d\tilde{Z}'(t) &= \begin{pmatrix} d\tilde{Z}_2(t) \\ d\tilde{Z}_3(t) \end{pmatrix} \end{aligned}$$

It follows, by recalling that $d\tilde{Z}_i(t) = dZ_i(t) + \lambda_i(t)dt$, that

$$\begin{aligned} r(t) &= \mu_C(t) + r_Q(t) - \lambda_3(t)\sigma_C(t) \\ \Rightarrow \mu_C(t) &= r(t) - r_Q(t) + \lambda_3(t)\sigma_3(t) \end{aligned}$$

from which we see that $dC(t) = C(t) \left[(r(t) - r_Q(t)) dt + \sigma_3(t) d\tilde{Z}_3(t) \right]$.

It is reasonable to discuss the choice of $\lambda_3(t)$ here. Investors are generally more interested in real returns rather than nominal returns, especially where inflation is a relatively uncertain process. We can conjecture, therefore, that if a real risk-free asset existed then it would have a lower expected return than a nominal risk-free asset. As a consequence $\mu_C(t)$ should be less than $r(t) - r_Q(t)$. This means that the market price of inflation risk, $\lambda_3(t)$, should be less than zero (if $\sigma_3(t) > 0$). An appropriate choice of $\lambda_2(t)$ will ensure that longer-dated, risky, index-linked bonds give the right level of risk premium over nominal cash and relative to fixed-interest bonds.

2.2 A specific model for index-linked bond prices

We consider here a two-factor equilibrium model for index-linked prices which is a generalisation of the Vasicek (1977) model. A similar approach to fixed interest prices has been taken, for example, by Langetieg (1980).

Suppose that

$$\begin{aligned} dr_Q(t) &= -\alpha_{21} (r_Q(t) - m_Q(t)) dt + \sigma_{21} d\tilde{Z}_{21}(t) \\ \text{where } dm_Q(t) &= -\alpha_{22} (m_Q(t) - \mu_Q) dt + \sigma_{22} d\tilde{Z}_{22}(t) \\ \text{or } dy_Q(t) &= -A_Q (y_Q(t) - \tilde{\mu}_Q) + S_Q d\tilde{Z}_2(t) \\ \text{where } y_Q(t) &= (r_Q(t), m_Q(t))^T \\ \tilde{\mu}_Q &= (\mu_Q, \mu_Q)^T \\ A_Q &= \begin{pmatrix} \alpha_{21} & -\alpha_{21} \\ 0 & \alpha_{22} \end{pmatrix} \\ S_Q &= \begin{pmatrix} \sigma_{21} & 0 \\ 0 & \sigma_{22} \end{pmatrix} \end{aligned}$$

The process $m_Q(t)$ is a standard Ornstein-Uhlenbeck process with mean-reversion level μ_Q . This process should itself be interpreted as the local mean reversion level of the main process $r_Q(t)$. Superficially this model looks like the Hull & White (1990) model, but in their development $m_Q(t)$ is a deterministic function which is determined by the initial set of prices $Q(0, T)$.

Under this model we find that

$$\begin{aligned} Q(t, T) &= \exp [A_Q(T-t) - B_{Q1}(T-t)r_Q(t) - B_{Q2}(T-t)m_Q(t)] \\ \text{where } B_{Q1}(s) &= \frac{1 - e^{-\alpha_{21}s}}{\alpha_{21}} \\ B_{Q2}(s) &= \frac{\alpha_{21}}{\alpha_{21} - \alpha_{22}} \left[\frac{1 - e^{-\alpha_{22}s}}{\alpha_{22}} - \frac{1 - e^{-\alpha_{21}s}}{\alpha_{21}} \right] \end{aligned}$$

$$\begin{aligned}
A_Q(s) = & (B_{Q1}(s) - s) \left(\mu_Q - \frac{\sigma_{21}^2}{2\alpha_{21}^2} \right) + B_{Q2}(s)\mu_Q - \frac{\sigma_{21}^2 B_{Q1}(s)^2}{4\alpha_{21}} \\
& + \frac{\sigma_{22}^2}{2} \left[\frac{s}{\alpha_{22}^2} - 2 \frac{(B_{Q2}(s) + B_{Q1}(s))}{\alpha_{22}^2} + \frac{1}{(\alpha_{21} - \alpha_{22})^2} \frac{(1 - e^{-2\alpha_{21}s})}{2\alpha_{21}} \right. \\
& \left. - \frac{2\alpha_{21}}{\alpha_{22}(\alpha_{21} - \alpha_{22})^2} \frac{(1 - e^{-(\alpha_{21} + \alpha_{22})s})}{(\alpha_{21} + \alpha_{22})} \right. \\
& \left. + \frac{\alpha_{21}^2}{\alpha_{22}^2(\alpha_{21} - \alpha_{22})^2} \frac{(1 - e^{-2\alpha_{22}s})}{2\alpha_{22}} \right]
\end{aligned}$$

When we choose the market prices of risk, $\lambda_2(t)$ these should be consistent with the $\lambda_1(t)$ and our views on the relationship between fixed-interest and index-linked bonds. For example, we have already noted that short-dated index-linked bonds should return less than short-dated fixed-interest bonds through the market-price of inflation risk, $\lambda_3(t)$. This will automatically affect returns on longer-dated bonds in the same way, but longer-term inflation risks may mean that we wish to have a larger or a smaller difference between long-dated index-linked and fixed-interest bonds.

3 Structure of the term-structure model

In describing this model we can note two things.

First, the model is Markov. Other approaches (for example, Vector ARMA models) include non-Markov elements. For example, some variables in the model might depend upon price inflation over the last two years rather than just the current rate of price inflation. However, such models can be given a state-space representation, which records at time t all necessary variables to allow simulation of the observations at time $t+1$. Such extended models are Markov in structure even though some elements are not directly observable in the natural sense. In an arbitrage-free framework it makes sense that current prices take account of all available information: past as well as present. For example, if the future nominal risk-free rate of interest depends upon rates over the last two years then expected future rates will incorporate that relevant information. When we look at forward rates of interest we should find that these should reflect the same structure, otherwise incorporation of the additional past information would give an advantage to an intelligent investor.

Second, the model defines nominal and real interest rates and then inflation is driven by the difference between the short rates. A more natural construction might start with inflation and build a term-structure model on top of that (for example, Wilkie, 1995). The present model is Markov and so inflation could be described at the top level and bond prices below that. However, the resulting formulation of the model would look much less compact than it is in its present form.

A third point which can be mentioned relates to the use of other actuarial models. In some cases such models may provide good forecasts of the medium and long term while supporting characteristics which make short-term predictions which differ substantially from inferences which we can gain from current market and economic data (for example, government policy). Under such circumstances, it is common practice to modify short-term dynamics of the model while keeping the longer term version of the model unchanged. As an example, it is often the case with the Wilkie (1995) model that inflation is modelled in the short term using lower volatility and with inflation expectations modified to reflect current market information. With the current model we feel that such alterations should not be necessary since the inclusion of a full term-structure (of interest rates and inflation) means that the model immediately reflects fully an accurate view of future expectations of interest rates and inflation.

4 Calibration of the fixed-interest model: discussion

The parameters for this model can be approached in a number of different ways. We will describe two here.

In the light of the Dybvig, Ingersoll & Ross (1994) result (that the infinite maturity spot rate must be constant or increasing) it was a challenge to mimic the volatility of the yields to redemption on long-dated coupon bonds observed on markets such as the UK. This can be done with an appropriate choice of values for the α_{1i} and the σ_{1i} .

4.1 Historical data

Historical data may be comprised of a mixture of coupon-bond prices, treasury bill rates and, more recently, prices of zero-coupon bonds, and bond futures and options.

It should be borne in mind that the model was devised with the intention that one or more of the α_{1i} (say α_{1n_1}) should be relatively low. This allows for long-term fluctuations in the general level of interest rates (particularly those on long-dated coupon bonds). A consequence of this is that estimated values of the process $\tilde{X}_{1n_1}(t)$ will be insufficient to get a reliable estimate of α_{1n_1} . On the other hand, the shapes of the fitted yield curves on individual dates do depend upon the α_{1i} , so that additional data about α_{1n_1} can be gained from the range of bond price data.

Full statistical analysis includes modelling how the prices of individual bonds vary relative to their theoretical prices over time. Here we propose a different approach which should (although this is not tested) produce almost as effective answers.

Let $\theta = (\beta, \alpha_1, \sigma_1)$ be the fixed parameter set and π_t be the set of prices on a given date. The error structure of the π_t given θ and $\tilde{X}_1(t)$ is the subject of studies by Cairns (1998, 1999) (although this paper uses a simpler forward-rate curve model). Thus,

given θ and π_t we can estimate $\hat{X}_1(t)$ for each t independent of all other dates (so-called *descriptive* modelling). This gives rise to a likelihood function $L_2(\theta; \hat{x}_1(\theta))$ where $\hat{x}_1(\theta) = \{\hat{X}_1(t)(\theta) : t \in \mathcal{T}\}$, \mathcal{T} is the reference index and $\hat{X}_1(t)(\theta)$ is the estimate of $\hat{X}_1(t)$ given θ and π_t . This can be combined with the likelihood function

$$L_1(\theta; x_1(\theta); \pi) = \prod_{t \in \mathcal{T}} L_1(\theta; \hat{X}_1(t); \pi_t)$$

However, one can infer from Cairns (1998, 1999) that the quality of fit on all dates can be relatively insensitive to changes in the choices of some components of θ . Maximisation of $L_2(\theta; \hat{x}_1(\theta))$ or $L_1(\theta; x_1(\theta); \pi)L_2(\theta; \hat{x}_1(\theta))$ is essentially the same as maximum likelihood for traditional time series even though the observations $x_1(t)$ are here replaced by $\hat{X}_1(t)(\theta)$.

If the choice of model and, in particular, the number of factors n_1 is acceptable then we should find the following:

- The quality of fit on any particular date, t , given $\hat{\theta}$ and $\hat{X}_1(t)(\hat{\theta})$ should not vary substantially over time. For example, the model should be able to mimic the full range of yield curves which we observe in the market. Furthermore, the differences between estimated and observed prices should be sufficiently small to avoid exploitation of apparent arbitrage opportunities.
- The estimated $\hat{X}_1(t)(\hat{\theta})$ should evolve through time in a way which is consistent with the underlying model. Should it be found that the processes $\hat{X}_{1,i}(t)(\hat{\theta})$ for $i = 1, \dots, n_1$ are correlated then the model should be adjusted accordingly although this will result in rather more parameters than we might prefer.

4.2 Qualitative methods

Alternative methods to calibration involve making prior statements about *target* quantities such as:

- the mean value of specific interest rates (for example, the 3-month treasury-bill rate, and 5 and 25-year par yields);
- the level of short-term variability of specified rates;
- the level of long-term variability of specified rates;
- the degree of influence of the various driving factors on different interest rates in the short and long term.

Parameter values can then be found by analytical methods or by simulation to match as closely as possible the targets.

The prior statements are based upon historical experience but take subjective views on the relative importance of certain periods of data. Subjective judgements are especially important for the treatment of factors which are subject to long cycles (that is, for low α_{1i}) where there is insufficient data.

Consider, for example, par yields on long-dated bonds and irredeemable bonds in the UK (for example, see Wilkie, 1995, Figure 6.1). Over a period of 100 years or so, these yields have ranged from about 2.5% up to 15%. The target for 25-year par yields should, therefore, cover this range with reasonable probabilities of attaining both 2.5% and 15% over, say, a one-hundred-year period. It is specifically this requirement which dictates the need for one of the α_{1i} to be relatively low, although the mean-reversionary behaviour of other rates suggests this as well.

The non-linear dependence of prices and interest rates upon the $\tilde{X}_{1i}(t)$ complicates matters somewhat. However, we can make some *crude* approximations which turn out to be quite effective. Thus,

$$\begin{aligned} f(t, T) &= f(t, T)(\tilde{X}_{1i}(t)) \\ &\approx f(t, T)(0) \left(1 + \sum_{i=1}^{n_1} \sqrt{2\alpha_{1i}} d_i(T-t) \tilde{X}_{1i}(t) \right) \\ &\approx \beta \left(1 + \sum_{i=1}^{n_1} \sqrt{2\alpha_{1i}} d_i(T-t) \tilde{X}_{1i}(t) \right) \end{aligned}$$

where $d_i(u) = \frac{\sigma_{1i} e^{-\alpha_{1i} u}}{(\beta + \alpha_{1i})} \sqrt{\frac{\alpha_{1i}}{2}}$

Note that $Var(\sqrt{2\alpha_{1i}} \tilde{X}_{1i}(t)) \rightarrow 1$ as $t \rightarrow \infty$. Thus the long-term effect of $\tilde{X}_{1i}(t)$ on $f(t, t+s)$ is $\beta d_i(s)$ in contrast to the short-term effect of local volatility in $\tilde{X}_{1i}(t)$ which is $\beta \sqrt{2\alpha_{1i}} d_i(s)$.

The par yield on irredeemable bonds is

$$\begin{aligned} \rho(t) = \rho(t)(\tilde{X}_{1i}(t)) &= \left[\int_0^\infty P(t, t+s) ds \right]^1 \\ &= \frac{\int_0^\infty H(u, \tilde{X}_{1i}(t)) du}{\int_0^\infty u H(u, \tilde{X}_{1i}(t)) du} \\ &\approx \beta \left[1 + \sum_{i=1}^{n_1} \frac{\sigma_{1i} \beta}{(\beta + \alpha_{1i})^2} \sqrt{\frac{\alpha_{1i}}{2}} \tilde{X}_{1i}(t) \right] \end{aligned}$$

Long-term Standard deviations (%)	Individual S.D. due to			Total S.D.	
	X_{11}	X_{12}	X_{13}	predicted	sample
$f(t, t + \frac{1}{4})$	3.15	1.90	3.16	5.07	5.05
$f(t, t + 5)$	0.47	0.70	2.46	2.60	3.10
$f(t, t + 25)$	0.00	0.01	0.91	0.91	1.11
$\rho(t)$	0.39	0.38	1.58	1.67	2.70(*)
Short-term Standard deviations (%)					
$f(t, t + \frac{1}{4})$	2.82	1.14	0.99	3.19	4.04
$f(t, t + 5)$	0.42	0.44	0.78	0.99	1.11
$f(t, t + 25)$	0.00	0.01	0.29	0.29	0.33
$\rho(t)$	0.35	0.24	0.50	0.65	1.25(*)

Table 1: Standard deviations driven by individual factors and overall standard deviations for selected rates of interest. Predicted standard deviations estimated using the linearisation. (*) The sample standard deviation for $\rho(t)$ is, in fact, the sample standard deviation of the 25-year par yield.

Actual variances turn out to be a bit higher than those predicted by these formulae because of the non-linear dependence of $f(t, T)$ on $\tilde{X}_1(t)$. However, we are provided with a useful starting point. Furthermore, we can get a useful guide to the relative importance of each of the $\tilde{X}_{1i}(t)$ on various interest rates.

4.3 Example

The following parameter values (with $n_1 = 3$) were chosen partly based on trial and error and partly using the linearisation.

$\beta = 0.05$ (this needs to be sufficiently low to get some 25-year par yields as low as 3% with reasonable frequency)

$\alpha_1 = (0.4, 0.2, 0.05)$

$\sigma_1 = (0.7, 0.3, 0.4)$

The dependencies quoted in Table 1 can be seen graphically in Figures 1, 2 and 3.

In Figure 1 we have plotted the 25-year par yield over a typical 100-year period, in the sense that there are periods of stable low rates interspersed with bursts of high rates. Figure 1 also plots $\tilde{X}_{1i}(t)$ for $i = 1, 2, 3$ over the same period. This allows us to see that certain features in the dynamics of the 25-year par yield can be explained by variations in each of the three driving factors. For example, the broad level of the yield is driven by $\tilde{X}_{13}(t)$ while more local peaks and dips are caused by local peaks and dips in $\tilde{X}_{11}(t)$ and $\tilde{X}_{12}(t)$. These short-term dependencies concur (in as far as we can assess this visually)

with the final row of Table 1.

In Figure 2 we plot a longer run of data and give scatter plots which allow us to get a better picture of the dependency of 25-year par yields on the $\tilde{X}_{1i}(t)$. In particular we can see the strong dependency noted in Table 1 (and that this is non-linear) on $\tilde{X}_{13}(t)$ and the weak long-term dependency on $\tilde{X}_{11}(t)$ and $\tilde{X}_{12}(t)$.

Figure 3 gives equivalent plots for 3-month spot rates. As expected there is a greater degree of short-term volatility and of long-term variation compared to the 25-year par yields. We can also see how long-term variability depends in more equal terms on $\tilde{X}_{11}(t)$ and $\tilde{X}_{13}(t)$, and (to a lesser extent) $\tilde{X}_{12}(t)$. A closer analysis of simulated 3-month rates shows that short-term rates of interest sometimes experience long periods of low stable rates and other periods of considerable volatility. Such behaviour is entirely consistent with that observable in many developed countries. For example, in Germany rates have been low and stable for many years having gone through a period of considerable turmoil earlier in the century. Overall 3-month rates could be seen to range from 0.1% (as we have seen in Japan) up to 45% (consistent with, for example, some East European countries).

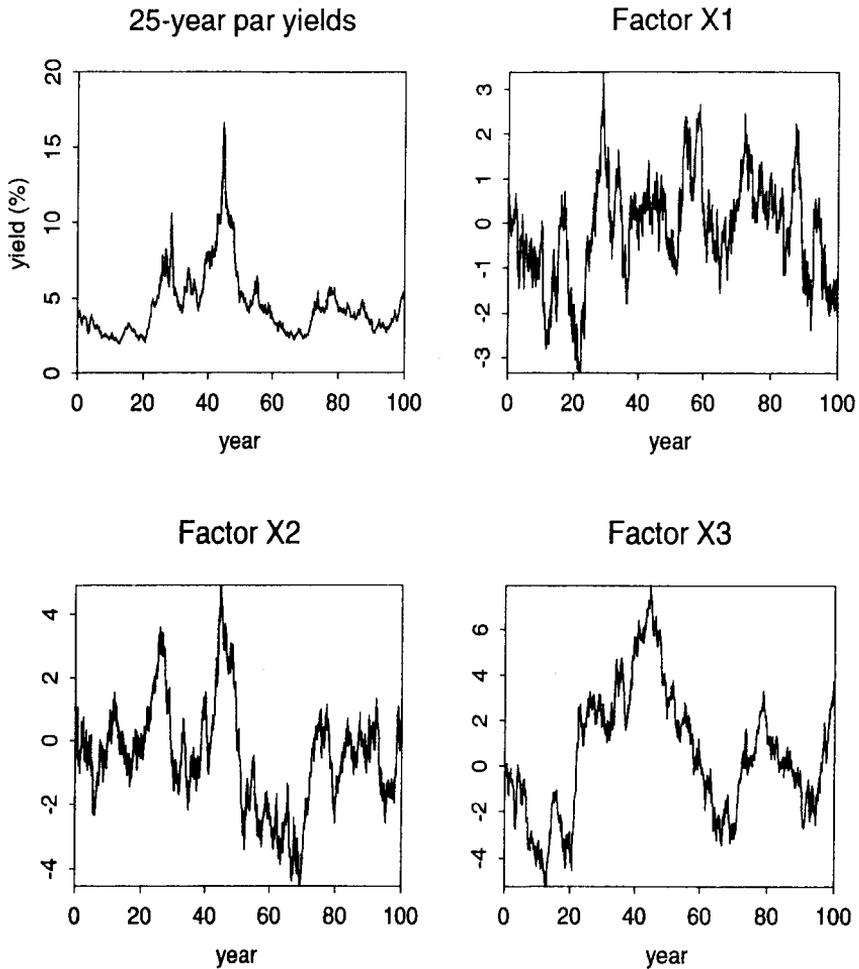


Figure 1: 25-year par yields. Top left: variation over a 100-year period. Top right and bottom left/right: variation of $\tilde{X}_{1i}(t)$ for $i = 1, 2, 3$.

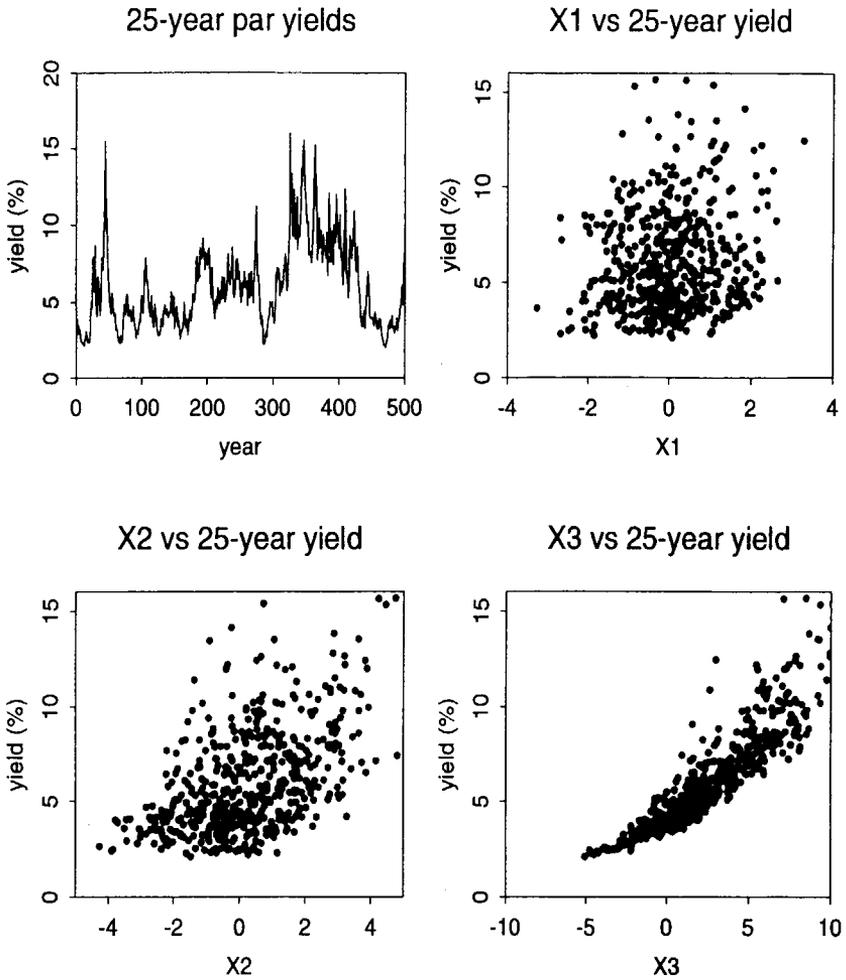


Figure 2: 25-year par yields. Top left: variation over a 500-year period. Top right and bottom left/right: Scatter plots of 25-year par yields against $\tilde{X}_{1i}(t)$ for $i = 1, 2, 3$.

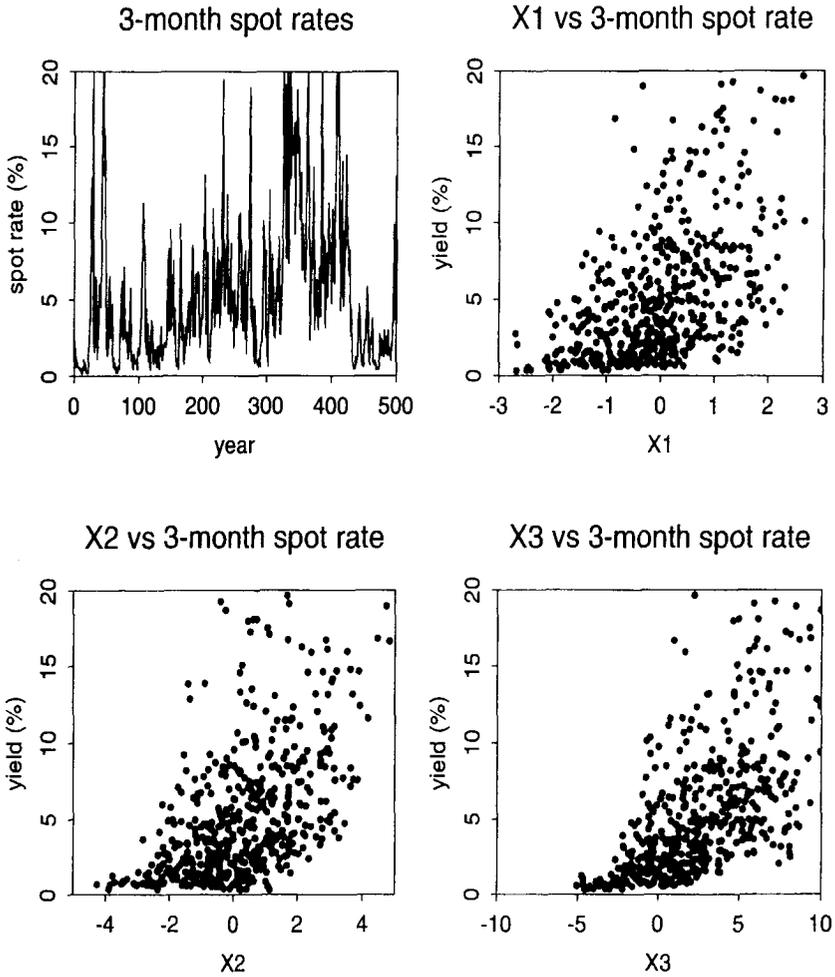


Figure 3: 3-month spot rates. Top left: variation over a 500-year period. (Some values ranging from 20% up to 45% have been cut off to allow us to make out more of the main detail.) Top right and bottom left/right: Scatter plots of 25-year par yields against $\tilde{X}_{1i}(t)$ for $i = 1, 2, 3$.

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Appendix A: The Ornstein-Uhlenbeck Process

Suppose $dX(t) = -\alpha X(t)dt + \sigma dZ(t)$.

Let $Y(t) = \exp(\alpha t)X(t)$. Then, by Ito's formula:

$$\begin{aligned}dY(t) &= \alpha e^{\alpha t} X(t) dt + e^{\alpha t} dX(t) \\ &= \sigma e^{\alpha t} dZ(t) \\ \Rightarrow Y(t) &= Y(0) + \sigma \int_0^t e^{\alpha u} dZ(u) \\ \Rightarrow X(t) &= e^{-\alpha t} X(0) + \sigma \int_0^t e^{-\alpha(t-u)} dZ(u)\end{aligned}$$

It follows that, for $t < s$, $X(s)$ given \mathcal{F}_t is normally distributed with

$$\begin{aligned}E[X(s) | \mathcal{F}_t] &= e^{-\alpha(s-t)} X(t) \\ \text{Var}[X(s) | \mathcal{F}_t] &= \sigma^2 \frac{(1 - e^{-2\alpha(s-t)})}{2\alpha}\end{aligned}$$

