

Market Value of Insurance Liability

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ABSTRACT

This paper suggests a method to calculate insurance liability as market value to be applied in the situation where management concern arises. Each building block is explained to calculate the market value of the insurance liability. Some key issues are specified.

1. INTRODUCTION

Recently, there has been much written about the need to define market value of insurance liability. It would be useful to have some yardstick for early warning in regard to the level of reserve adequacy before something unexpected happens. Insurance ALM has been discussed mainly from asset side, however, it is critical for ALM to also have a control from liability side too. This article will introduce a method to effectively measure the appropriate value or market value of insurance liability.

2. MODELLING MARKET VALUE OF LIABILITY

In general, market value of liability is calculated directly or indirectly. Direct method is by looking future liability cash flow and discounting them at appropriate rate, while indirect method is achieved by first estimating market value of asset and capital, then subtracting the latter from the former. Both approaches have appeal.

It is also important to know for what purpose market value of liability is needed because it is not uniquely pinpointed without assumptions.

Here, assuming management of the insurance company has concern for future profitability on the particular block of business, combination of direct and indirect methods are taken.

3. DESCRIBING EACH RISK PROFILE

We denote market value of liability at time t as $V(t)$. Although in the field of financial engineering it is common to describe the underlying -asset as stochastic differential equation, such description as

$$\begin{aligned} V(t) &= V(0) \cdot \exp X(t) \\ dX(t) &= \mu(t) dt + \sigma(t) dW_t \\ \text{where } \{ W_t \} &\text{ is standard brownian motion} \end{aligned}$$

suggests difficulties in figuring out risk parameters $\mu(t)$ and $\sigma(t)$ in evaluating $V(t)$.

It is natural to define $V(t)$ as the sum of the contingency claim prices where each contingency arises from the uncertainty of the cash flow at each node. Hence, we define $V(t)$ as the expectation of corresponding random variable ;

$$V(t) := E_t^* [\sum_{s>t} LCF_s \cdot \exp (- \int_t^s R^*(u) du)]$$

where LCF_s is the liability cash flow at time s , $R^*(u)$ is spot interest rate adjusted for no arbitrage as well as for insurance risk, and E_t^* is conditional expectation operator at time t after risk adjustment.

Consider the insurance liability below for illustrative purposes.

“Insurer pays the annuity if such “event of incidence” happens as death, sickness or retirement during the contract period specified from the date that the event happens until the date that such “event of termination”

happens as lapse of certain years, recovery or death.”
 We call this obligation “annuity claim liability”.

Note: We assume for simplicity that each insured would submit such annuity claim at most once in the period.

To determine risk characteristics of the annuity claim liability, consider two risk profiles ;

- ① Incidence Rate Process
- ② Survival Function Process

Note: Although we assume no withdrawal option during the period, we could evaluate the price of those options using derivative pricing method once we determine $V(t)$.

Also, we assume closed model for simplicity. It is possible to think open model in a similar way.

Denote the contract period T and divide T into n intervals

$$t_{i+1} - t_i = T/n = h \quad (i = 0, 1, \dots, n-1)$$

Denote number of insureds “active” at time t (active means no event of incidence has happened to the insured until time t) $N_a(t)$.

Assume

$$N_a(0) = N$$

Incidence rate process $\{ icR(t) \}$ is defined as Jump Diffusion Process

$$icR(t) = \tau_A(t) \cdot icR^A(t) + \tau_B(t) \cdot icR^B(t)$$

$$\tau_A(t) + \tau_B(t) = 1, \quad \tau_A(t), \tau_B(t) \geq 0$$

$$icR^A(t) = \begin{cases} k \cdot \text{Bin} (N_a(t), \Delta q^A) / N_a(t), & t \in Q' \\ 0, & t \notin Q' \end{cases}$$

Q' is collection of t_i' s, $k, \Delta q^A > 0$, and $\text{Bin}(N,p)$ represents binomial distribution with parameter N and p .

$$\text{icR}^B(t) = \exp \left[\int_0^t \mu^B_{\text{LicR}}(s) ds + \int_0^t \sigma^B_{\text{LicR}}(s) dW_s^B \right]$$

$\mu^B_{\text{LicR}}(s), \sigma^B_{\text{LicR}}(s)$ are smooth functions of s determined by the information available up to s .

Figure 1. shows 10 sample paths generated from $\{\text{icR}(t)\}$ for $k = .1, \Delta q^A = 1, \mu^B_{\text{LicR}}(s) = .05, \sigma^B_{\text{LicR}}(s) = 2$ along with $T = 2(\text{years}), h = 1/12$.

Similarly, Survival function process $\{S(t)\}$ is defined as Jump Diffusion Process

$$S(t) = \tau_\alpha(t) \cdot S^\alpha(t) + \tau_\beta(t) \cdot S^\beta(t)$$

$$\tau_\alpha(t) + \tau_\beta(t) = 1, \quad \tau_\alpha(t), \tau_\beta(t) \geq 0$$

$$S^\alpha(t) = \left\{ \begin{array}{ll} S^\alpha(t-0) \cdot [1 - 1 \cdot \text{Bin}(\Phi_0 \cdot S^\alpha(t-0), \Delta q^\alpha)] / (\Phi_0 \cdot S^\alpha(t-0)) & t \in Q' \\ S^\alpha(t-0) & t \notin Q' \end{array} \right.$$

$\Phi_0, 1, \Delta q^\alpha > 0$

$$S^\beta(t) = \text{Min}[S^{\beta'}(t), 1]$$

$$S^{\beta'}(t) = \exp \left[- \left(\int_0^t \mu^{\beta}_{\text{LS}}(s) ds + \int_0^t \sigma^{\beta}_{\text{LS}}(s) dW_s^\beta \right) \right]$$

$$S^\alpha(0) = S^\beta(0) = 1$$

$\mu^{\beta}_{\text{LS}}(s), \sigma^{\beta}_{\text{LS}}(s)$ are smooth functions of s determined by the information available up to s .

Thus, $S(t)$ describes the probability that event of termination has not occurred for t years since the event of incidence.

Figure 2. shows 10 sample paths generated from $\{S(t)\}$ for $l = .2, \Delta q^\alpha = .007, \mu^{\beta}_{\text{LS}}(s) = 3(0 < s < 6), .5(6 \leq s < 12), .2, (12 \leq s < 18), .02(18 \leq s), \sigma^{\beta}_{\text{LS}}(s) = 4$.

4. DEFINITION OF INTEREST RATE RISK

Next, we need model interest rate risk. One way to do so is simply using spot interest rate process such as

$$dR(t) = \mu_R(t) dt + \sigma_R(t) dW_t^R$$

We can estimate market price of risk $\lambda(R(t), t)$ at time t ($0 < t < \tau$) for relative riskiness of zero coupon bond with different maturities looking at the initial yield curve.

$$\lambda(R(t), t) := \frac{\mu_P(R(t), t) - R(t)}{\sigma_P(R(t), t)}$$

$\mu_P(R(t), t)$, $\sigma_P(R(t), t)$ are estimates for expectation and standard deviation of the price of a "representative" zero coupon bond at time t respectively.

Define

$$\begin{aligned} \beta &= -\lambda(R(t), t) \\ \psi(t) &= \exp \left[\int_0^t \beta(s) dW_s^R - (1/2) \cdot \int_0^t \beta^2(s) ds \right] \end{aligned}$$

Under the probability measure P^* defined by

$$P^*(A) := E[\chi_A \cdot \psi(\tau)], \quad 0 < t < \tau$$

$$\chi_A = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \notin A \end{cases}$$

the process

$$W_t^{R^*} := W_t^R - \int_0^t \beta(s) ds$$

is standard brownian motion (Girsanov's theorem), and $\{R^*(t)\}$ is

represented under P^* as

$$dR^*(t) = [\mu_{R^*}(t) - \sigma_{R^*}(t) \cdot \lambda(R^*(t), t)] dt + \sigma_{R^*}(t) dW^{R^*}_t$$

Because relative price of zero coupon bond

$$\left\{ \frac{P(t, \tau)}{B(t)} \right\}$$

$P(t, \tau)$ is price of zero coupon bond with maturity τ at time t ,

$$B(t) = B(0) \cdot \exp\left(-\int_0^t R(s) ds\right)$$

is P^* Martingale, assuming no arbitrage, prices of zero coupon bonds are represented as

$$P(t, \tau) = E^*_t \left[\exp\left(-\int_t^\tau R^*(s) ds\right) \right]$$

E^*_t is conditional expectation operator at time t under P^*

Note: Unlike zero coupon bond, price of insurance liability $V(t)$ is concerned with uncertain cash flow due to the risk traded in incomplete market. Assuming that we know beforehand the distribution of the cash flow, we can compose risk-minimizing portfolio.

We use as illustrative purpose Visicek model

$$dR(t) = \theta(R' - R(t)) dt + \sigma_R dW^{R^*}_t, \quad R' > 0$$

Of course we can in stead use other models such as HJM model. Figure 3. shows 10 sample paths generated from $\{R^*(t)\}$ for $\theta = .1$, $R' = .05$, $\sigma_R = .02$ and $\lambda = .01$ with minimum spot interest rate assumed as .5%.

5. KEY ISSUES TO BE CONSIDERED

So far, we have prepared necessary tools to price $V(t)$. We can adopt different treatment from here to get $V(t)$. In any case, a lot of actuarial and/or financial judgement is required before we are able to automatically calculate $V(t)$ using numerical method.

Some key issues are as follows.

① Modeling Asset Allocation and Investment Strategy

The idea is to combine asset cash flow with liability cash flow to get net cash flow and optimize investment portfolio considering risk return trade-off. Required cost of capital is then translated into the liability evaluation model.

② Modeling Reinsurance Strategy

Use of reinsurance could increase capacity of the insurer and reduce required capital for the existing block of business. Optimized risk portfolio for liability is achieved and evaluated through liability evaluation model.

③ Required Risk Based Capital and Solvency Margin Concern

Risk based capital must be considered as necessary equity level to protect from business interruption due to solvency concern. This is relevant to the issue of deciding required cost of capital.

④ Modeling Dividend Policy

From distributable earnings stream for each future path, the insurer can forecast and look at the effect of dividend policy on solvency and pressure on profitability. Equity growth is indirectly tied with liability evaluation.

⑤ Modeling Distributable Earnings at each future node

Earnings after required capital charge are calculated at each node in the future and discounted by the appropriate internal rate of return (IRR) to get the enterprise's equity price benchmark. The purpose of this

calculation is to figure out non interest rate risk component in the discount rate by comparing IRR to no arbitrage interest rate $R^*(t)$. This component might be regarded as “project finance risk after interest rate adjusted”.

6. NEW PROPOSAL TO DEFINE MARKET VALUE OF LIABILITY

If we knew the appropriate discount rate of liability cash flow for each future node, we can theoretically calculate market value of insurance liability. There could be many ways to get the rate but the rate is not necessarily uniquely calculated. One of the approaches is to focus the value of getting the sense for

- ① Appropriateness of reserve assumption at valuation date
- ② Adequacy of GAAP accounting reserve at valuation date
- ③ Management chosen scenario based appropriateness of reserve assumption at future node
- ④ Management chosen scenario based adequacy of GAAP accounting reserve at future node
- ⑤ Consistency with the company's ALM policy and flexibility to change in ALM policy
- ⑥ Actual to forecast reserve gap
- ⑦ Likelihood of achieving the projected IRR

One of the difficulties of pricing insurance liability with stochastic risk model is the difficulty in explaining to management that the appropriateness of reserve adequacy level could change dramatically once new information is available at one node ahead.

Figure 4. shows two different incidence rate sample paths which anticipate different future paths.

Even if we have explained well the new sea level of liability, we would like to add information on how serious the effect of the change in sea level will be on long term profitability.

- To go backwards, let's pick a current projected IRR satisfying the relation stochastically between future distributable earnings flow and current project equity.

- Use that project equity as benchmark index to evaluate stochastically discount net cash flow to estimate option adjusted spread (OAS*) built in over no arbitrage interest rate (R*).

- Discount liability cash flow at R* + OAS* to get market value of insurance liability.

For numerical purposes, at valuation time t_0 , we can calculate market value of insurance liability as following manner.

- Set OAS at time t_0 ; OAS_{t_0} arbitrarily and denote

$$R^{**}(u) = R^*(u) + OAS_{t_0}$$

- Calculate $V(t)$ for $t \geq t_0$ using liability cash flow at time s ; LCF_s as

$$V(t) := E_t^* [\sum_{s>t} LCF_s \cdot \exp (- \int_t^s R^{**}(u) du)]$$

- Similarly using asset cash flow at time s ; ACF_s , calculate

$$A(t) := E_t^* [\sum_{s>t} ACF_s \cdot \exp (- \int_t^s R^{**}(u) du)]$$

which is market price of asset at time t .

- Calculate net cash flow as

$$NCF_t = ACF_t - LCF_t$$

- Discount future net cash flow at the rate $R^{**}(t)$ and take expectation operator $E_{t_0}^*$ as

$$Eqty_{t_0} = E_{t_0}^* [\sum_{t>t_0} NCF_t \cdot \exp (- \int_{t_0}^t R^{**}(s) ds)]$$

• Compare $Eqty_{t_0}$ to the enterprise's equity price index* at time t_0 ; $Eqty_{t_0}^*$

• Adjust OAS_{t_0} and iterate the above steps until

$$| Eqty_{t_0}^* - Eqty_{t_0} | < \varepsilon$$

for small $\varepsilon > 0$ with appropriate $OAS_{t_0}^*$

• Calculate $V(t_0)$ as

$$V(t_0) := E_{t_0}^* [\sum_{t>t_0} LCF_t \cdot \exp (- \int_{t_0}^t R^{*OAS}(s) ds)]$$

$$R^{*OAS}(u) = R^*(u) + OAS_{t_0}^*$$

* Enterprise's equity price index might be calculated at time t_0 discounting future distributable earnings (plus appropriate goodwill value) DE_t with project finance based hurdle rate ; $IRR_{t_0}^*$ which is set in consistent with management's intention on required cost of capital as

$$Eqty_{t_0}^* = E_{t_0}^* [\sum_{t>t_0} DE_t / (1 + IRR_{t_0}^*)^{t-t_0}]$$

Note: While OAS is based on stochastic net cash flow simulation, IRR^* is mainly from management policy. If $R^* + OAS$ is different by far from IRR^* , some action might be necessary with regard to current financial plan.

7.APPLICATION

Consider the annuity claim liability defined in section 3. Assume at time 0,

we have a financial projection with distributable earnings shown in Table 1. It suggests that although we have IRR=6.32%, if we discount future distributable earnings at hurdle rate 9% and combine the result with current goodwill value 6.48, we get net present value 0, that is, IRR = 9%.

Instead of finding OAS on book equity 41.5, we will use equity price index defined as

$$\begin{array}{rclclcl} \text{Equity price index} & = & \text{Book Equity} & + & \text{Goodwill} \\ 47.98 & = & 41.5 & + & 6.48 \end{array}$$

to find OAS.

Note: It is possible to generate distributable earnings for each future period stochastically using book reserve table and find a representative hurdle rate which makes current net present value equal zero.

Table 2. shows a simplified simulation snapshot in the process of finding optimal OAS(denoted as OAS^*) which produces matching total discount net cash flow with equity price index. Ten sample paths are created with trial $OAS = 5\%$. Table 3. shows monthly cash flow for 10th sample path in Table 2. Looking at bottom line of Table 2., the average of total discount asset cash flow is 290.2, and the average of total discount liability cash flow is 210.8, that is the average of total discount net cash flow is 79.3. It suggests that OAS^* will be bigger than 5% because equity price index is 47.98. It also might be that looking at Table 1., book asset value is 409 and book liability value is 367.5 both of which are conservative figures compared to those corresponding values in Table2. If OAS^* is 6% after simulation, hurdle rate 9% with goodwill equal 6.48 is a little aggressive assumption with no arbitrage interest rate 1% to 2% range. If hurdle rate is reduced to 8% with goodwill value 4.02, then equity price index becomes 45.52 and OAS^* will be 6.5% this time, which might be a reasonable level.

8.CONCLUSION

In this paper, tools were described to express typical insurance risk profile and calculate cash flow based price of insurance liability. Using risk neutral approach, a method was introduced on how to decompose discount risk premium into an interest rate component and an incomplete insurance risk component. Then equity price index was defined instead of book equity to reflect goodwill value and finally optimal OAS(Option Adjusted Spread) was introduced to match total discount net cash flow with equity price index. If optimal OAS is considered to be consistent with hurdle rate, then optimal OAS is used to discount total liability cash flow to get market-value of insurance liability. Some examples of simulation were added.

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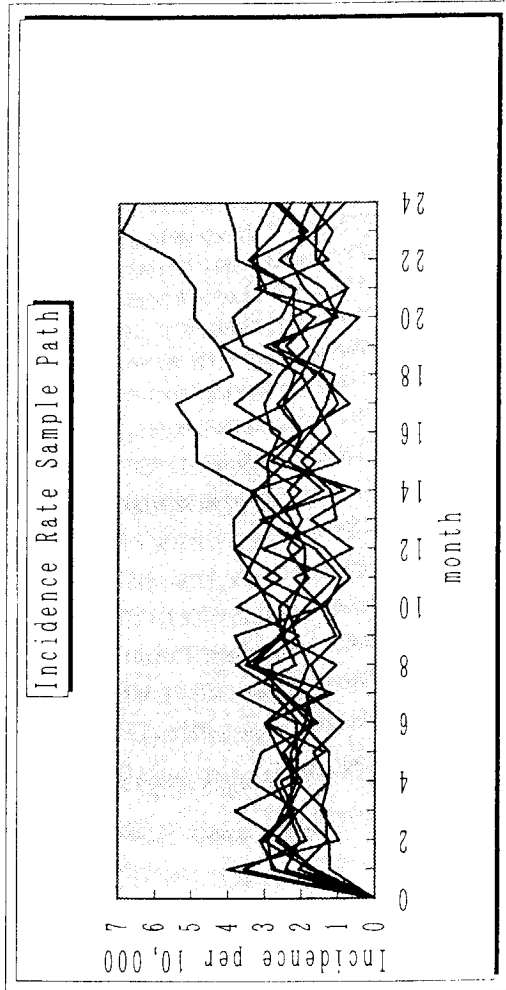


Figure 1.

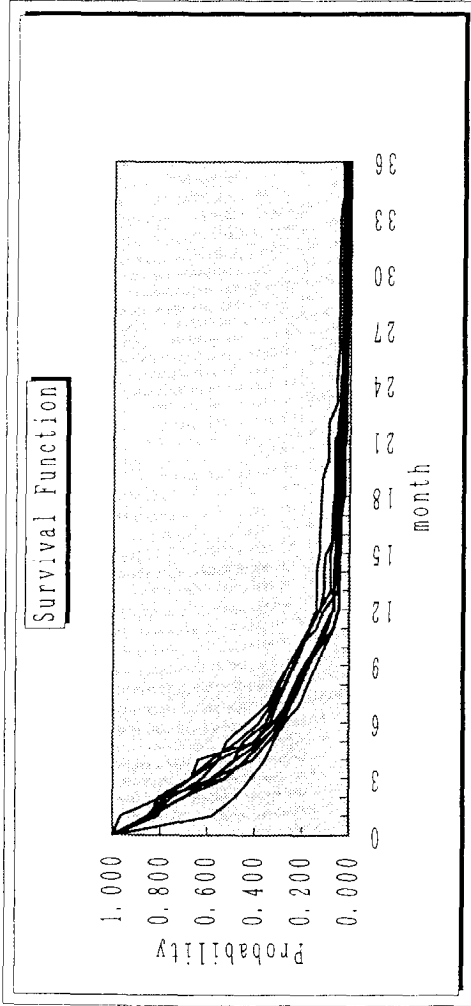


Figure 2.

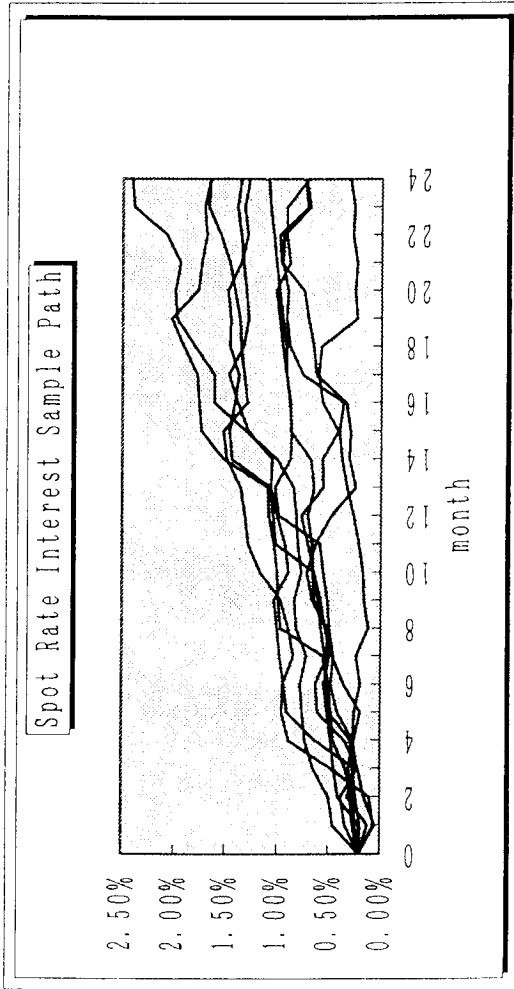


Figure 3.

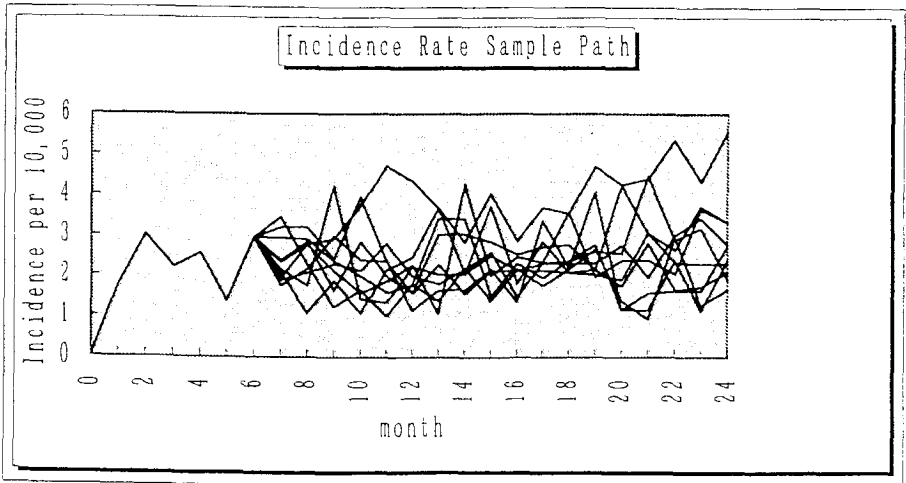
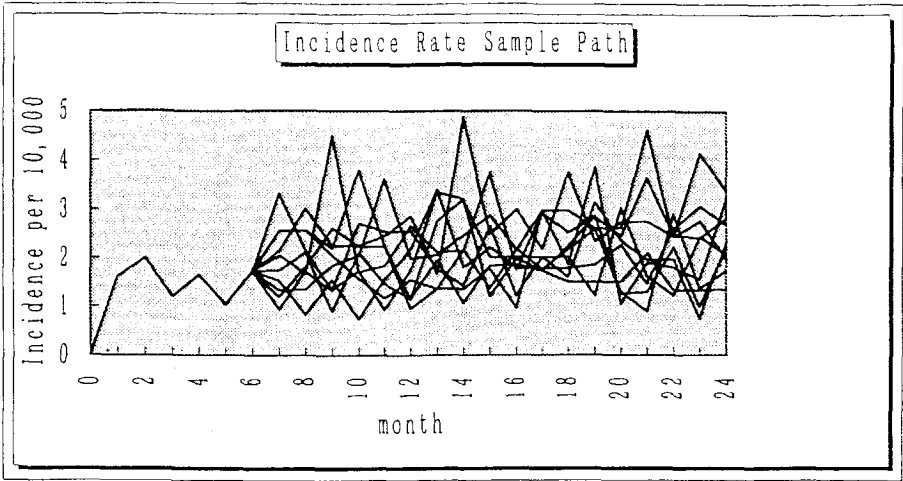


Figure 4.

Distributable Earnings

Policy year	Premium Cashflow	Expenses	Benefit Cashflow	Net Inv. Income	Book Reserve	Book Asset	Distrib. Earnings	Discount IRR* 6.32%	Required Cap. per V 20%
0			0.00	0.00	367.50	409.00		0	0.00
1		10.00	50.75	11.37	318.50	359.62	-64.08	0.9406	63.70
2		8.00	168.00	4.79	210.00	188.41	-41.01	0.8847	42.00
3		7.00	113.75	1.45	52.50	69.11	69.70	0.8321	10.50
4		0.00	42.00	0.57	0.00	27.68	49.26	0.7827	0.00
5		0.00	0.00	0.83	0.00	0.00	0.00	0.7362	0.00

NPV: 0.00

Book Equity at t=0: 41.50

Distributable Earnings adjusted for Goodwill

Policy year	Premium Cashflow	Expenses	Benefit Cashflow	Net Inv. Income	Book Reserve	Goodwill+ Asset	Distrib. Earnings	Discount h* 9.00%	Required Cap. per V 20%
0					367.50	415.48		0	0.00
1		10.00	50.75	11.37	318.50	366.10	-64.08	0.9174	63.70
2		8.00	168.00	4.79	210.00	194.89	-41.01	0.8417	42.00
3		7.00	113.75	1.45	52.50	75.59	69.70	0.7722	10.50
4		0.00	42.00	0.57	0.00	34.16	55.74	0.7084	0.00
5		0.00	0.00	0.83	0.00	0.00	0.00	0.6499	0.00

NPV: 0.00

Equity Price Index at t=0: 47.98

Table 1.

Equity Price Index = 47.98

OAS = 5%

Sample#	Total Discount Asset Casf Flow	Total Discount Liab. Casf Flow	Total Discount Net Casf Flow
1	281.0	244.2	36.8
2	299.6	144.2	155.4
3	306.1	171.2	134.9
4	295.0	162.6	132.4
5	290.7	243.6	47.1
6	287.3	157.4	129.9
7	285.7	329.2	-43.5
8	286.7	186.3	100.4
9	284.1	259.1	25.0
10	285.4	210.6	74.8
Average	290.2	210.8	79.3

Table 2.

Valuation t= 0

Patb ;	icR	S	R	AR	Comb.
	1	3	2	2	10

OAS = 5.0%

Lapse	Liab. Cash Flow	Discount Liab. Cash Flow	Asset Cash Flow	Discount Asset Cash Flow	Net Cash Flow	Discount Net Cash Flow	Asset Balance
0	0.00	0.00	0.00	0.00	0.00	0.00	409.00
1	1.54	0.00	0.13	0.13	0.13	0.13	383.13
2	2.83	1.52	0.14	0.14	-1.40	-1.39	381.73
3	3.77	2.79	0.11	0.11	-2.72	-2.69	379.01
4	4.25	3.71	0.11	0.11	-3.66	-3.59	375.35
5	4.70	4.15	0.15	0.15	-4.09	-4.00	371.26
6	5.44	4.57	0.03	0.03	-4.67	-4.54	366.59
7	6.05	5.27	0.07	0.06	-5.37	-5.21	361.21
8	6.51	5.83	0.08	0.08	-5.97	-5.76	355.24
9	6.28	6.25	0.17	0.16	-6.35	-6.09	348.90
10	6.06	5.99	0.17	0.16	-6.11	-5.83	342.78
11	6.83	5.76	0.17	0.17	-5.89	-5.59	336.90
12	7.01	6.46	0.04	0.04	-6.79	-6.42	330.10
13	7.38	6.59	0.05	0.04	-6.96	-6.54	323.14
14	7.12	6.90	0.05	0.05	-7.33	-6.86	315.82
15	7.17	6.63	0.06	0.05	-7.06	-6.58	308.75
16	7.55	6.64	0.14	0.13	-7.04	-6.52	301.72
17	7.88	6.96	0.05	0.04	-7.50	-6.91	294.21
18	7.85	7.23	0.04	0.04	-7.84	-7.19	286.37
19	8.37	7.15	0.23	0.21	-7.62	-6.95	278.75
20	9.48	7.59	0.19	0.17	-8.19	-7.43	270.57
21	9.27	8.55	0.18	0.17	-9.29	-8.38	261.28
22	9.61	8.32	0.17	0.15	-9.10	-8.17	252.18
23	9.12	8.58	0.22	0.19	-9.39	-8.38	242.78
24	8.74	8.10	0.21	0.18	-8.91	-7.91	233.87
25	7.38	7.72	0.20	0.18	-8.54	-7.54	225.33
26	6.27	6.49	0.26	0.23	-7.12	-6.26	218.21
27	5.43	5.48	0.25	0.22	-6.02	-5.26	212.19
28	4.76	4.72	0.22	0.19	-5.22	-4.53	206.97
29	4.18	4.12	0.21	0.19	-4.55	-3.93	202.42
30	3.65	3.59	0.22	0.19	-3.96	-3.40	198.46
31	3.23	3.12	0.14	0.12	-3.51	-3.00	194.95
32	2.92	2.75	0.14	0.12	-3.09	-2.63	191.86
33	2.70	2.47	0.21	0.18	-2.72	-2.29	189.15
34	2.54	2.27	0.17	0.14	-2.53	-2.13	186.61
35	2.42	2.12	0.25	0.21	-2.28	-1.91	184.33
36	2.35	2.01	0.28	0.23	-2.14	-1.78	182.19
37	2.25	1.94	0.27	0.22	-2.08	-1.72	180.11
38	2.15	1.84	0.24	0.20	-2.00	-1.64	178.11
39	2.04	1.75	0.25	0.20	-1.90	-1.55	176.20
40	1.94	1.66	0.24	0.20	-1.80	-1.46	174.40
41	1.84	1.57	0.24	0.20	-1.70	-1.37	172.71
42	1.74	1.47	0.25	0.20	-1.59	-1.28	171.12
43	1.65	1.39	0.26	0.20	-1.48	-1.18	169.63
44	1.54	1.31	0.21	0.16	-1.45	-1.15	168.19
45	1.45	1.22	0.21	0.16	-1.34	-1.05	166.85
46	1.36	1.14	0.20	0.16	-1.25	-0.98	165.60
47	1.24	1.06	0.21	0.17	-1.14	-0.89	164.46
48	1.15	0.96	0.26	0.20	-0.98	-0.76	163.48
49	1.03	0.88	0.26	0.20	-0.89	-0.68	162.59
50	0.93	0.79	0.26	0.20	-0.77	-0.59	161.82
51	0.83	0.71	0.26	0.20	-0.67	-0.51	161.15
52	0.72	0.63	0.30	0.23	-0.53	-0.40	160.62
53	0.58	0.54	0.30	0.23	-0.42	-0.31	160.20
54	0.48	0.44	0.26	0.20	-0.32	-0.24	159.88
55	0.36	0.35	0.25	0.19	-0.22	-0.17	159.66
56	0.23	0.26	0.27	0.20	-0.08	-0.06	159.58
57	0.15	0.17	0.28	0.20	0.05	0.03	159.63
58	0.07	0.11	0.30	0.22	0.16	0.11	159.78
59	0.03	0.05	0.34	0.25	0.27	0.20	160.06
60	0.00	0.00	383.13	275.86	383.09	275.83	0.00
Total	234.41	210.64	394.57	285.39	160.15	74.75	0.00

Table 3.

