Ambiguity Aversion, Generalized Esscher Transform, And Catastrophe Risk Pricing

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ABSTRACT

Motivated by the observation that the spread premium of CAT loss bonds is very high relative to the expected loss of the bond principal, and that the premium spread is much more pronounced for CAT bonds with low probability that a contingent loss payment to the bond issuers will be triggered, we extend the traditional Esscher transform to a generalized framework and treat it as a stochastic discount factor to price the CAT risk. The generalized Esscher transform is derived from a modified equilibrium model by allowing a representative agent to act in a robust control framework against model misspecification with respect to rare events in the sense of Anderson, Hansen and Sargent (2000). The model is explicitly solved and the derived pricing kernel is shown to be exactly the generalized Esscher Transform. Using catastrophe bonds data, we examine the empirical implication of our model.

JEL Classification: G12, G13

Key Words: Ambiguity Aversion; Esscher Transform; Catastrophe Risk Pricing; CAT Bonds

1. Introduction

In recent years, the pricing of catastrophe (CAT) risks has been of major interest among insurance and actuarial professionals. This type of research interest is mainly due to the rise of the magnitude of disaster losses and their resulting effects on insurance and reinsurance industry in the past decades. These enormous increases in disaster losses have challenged the ability of private insurance and reinsurance industry as a mechanism to provide coverage against catastrophe risks. Two types of solutions to the insurance capacity gap have been proposed and put into practice. One is mandatory public provision of insurance, which relies on government to spread losses across citizens. The other is through CAT risk securitization.

Since the inception of CAT risk securitization in 1992 with the introduction of index-linked catastrophe loss futures by the Chicago Board of Trade (CBOT), the market of insurance-linked securities has evolved into a business of more than US$ 10 billion issuances. A major segment of the CAT securities market has been catastrophe-linked bonds (CAT bonds) with their earliest introduction by Winterthur Re, USAA, and Swiss Re in 1997. Table 1 displays some CAT bonds issued from 1997 through 2000 reported by Goldman-Sachs.

Please insert Table 1 about here

There are two important points to be made from Table 1. First, the spread premium of CAT loss securities is very high relative to the expected loss of bonds principal. To see this, note that the ratio of the spread premium to the expected loss of bond principal ranges from 2.28 to 50, with the average of 9.09.

Since historical data suggests that catastrophe risk can usually be looked as uncorrelated with the capital market, or more exactly, amounts to a small fraction of the total wealth in the economy, they ought to be priced at close to the risk-free interest rate. In other words, the CAT bond premium should equal actuarial fair losses covered by the contract. The fact that the CAT bond spread is far above the expected loss of bond principal is contradictory to any standard capital market theory such as CAPM theory or Arbitrage Pricing Theory.

There is a second, more subtle point to be taken from Table 1. It appears that the premium spread is much more pronounced for CAT bonds with low probability that a contingent loss payment to the bond issuers will be triggered. This may generate a kind of “smirk” pattern in the cross-sectional plot of the premium to E[loss] ratio against the probability of contingent loss (see Figure 1). As a comparison, there is no apparent relation between the ratio and the expected loss percentage conditional on a loss occurrence, which indicates that the premium implicit in CAT bonds is mainly sensitive to the rareness of the catastrophe events.

With the above two empirical facts in mind, it is worth considerable interest to explore an economic model explaining why spreads in the CAT bond market are so high and why, moving from the CAT bonds with large probabilities of loss occurrence to lower ones, the premium spreads become more pronounced. Based on the economic model, it will as well be interesting to develop a methodology for pricing the CAT linked securities and reinsurance contracts.

Please insert Figure 1 about here

Previous literature on CAT bonds and other related catastrophe contracts could be roughly divided into two major groups:

The first group of articles concentrated on the possible economic explanations for high spread puzzle (See, e.g., Bantwal and Kunreuther (2000), Froot (1999)). For example, Bantwal and Kunreuther suggested that myopic loss aversion and prospect theory, ambiguity aversion, selection bias and threshold behavior, impact of worry and fixed cost of education may account for the puzzle. The second group devotes to describe the pricing formula of CAT-linked contracts. See, for example, Cummins and Geman (1995), Geman and Yor (1997), and Cummins, Lewis and Phillips (1999).

One common problem with these researches is that they failed to explain the second empirical fact mentioned above, i.e., they did not try to explain why premium spreads become less pronounced when we move from CAT bonds with low probabilities of loss occurrence to larger ones. Another problem is that most CAT risk pricing formulas are still based on a rather perfect framework and did not account for the imperfect properties of agent behavior as listed in Bantwal and Kunreuther (2000) into their models.

In this paper, we will introduce a kind of generalized Esscher transform for the pricing of CAT risk. The generalized transform will be supported by an economic model describing the agent behavior of ambiguity aversion in the sense of Anderson, Hansen and Sargent (2000). To explain the second empirical fact mentioned above, we examine the implications of varying ambiguity aversion toward rareness of catastrophe events. Without considering ambiguity aversion, our generalized transform will be reduced to the Esscher transform introduced in Gerber and Shiu (1994). From this perspective, this paper can be viewed as an addition to the list of many actuarial and financial researches contributing to the generalization of the original Gerber-Shiu framework.

The rest of the paper is organized as follows. Section 2 proposes a kind of generalized Esscher transform for compound Poisson process risk pricing and sets up an economic equilibrium model to examine the economic implication of the generalized transform. Section 3 demonstrates how to use the transform to price the catastrophe-linked contracts. This section also presents an estimation of the model using the 1997-2000 CAT bonds data and evaluates the model efficiency by examining 2000-2003 CAT bonds data. Section 4 concludes. Technical details are collected in the Appendix.

2. Generalized Esscher Transform and Economic Implications

2.1 Stochastic Discount Factor and Generalized Esscher Transform

Over the past several decades, there appears a tendency towards a unification of financial economics and actuarial theories. Both insurance and finance are interested in the fair pricing of financial products and the theoretical and empirical developments for asset pricing in both fields currently become emphasizing the concept of “stochastic discount factor” that relates payoffs for contingent claims to market prices in an equilibrium market.

The basic equation for the stochastic discount factor can be written as follows:

$$C_t = \mathrm{E}_t[\eta(t,T)Z_T], \tag{2.1}$$

where C_t is the time-t price of a contingent claim with random payoff Z_T at time T; E_t is the conditional expectation operator conditioning on the information available up to time t, and $\eta(t,T)$ is the so-called stochastic discount factor, or SDF.

If markets are complete, then the stochastic discount factor is unique. But complete case is rare in insurance and as a consequence, there will be infinitely many such stochastic state prices so a natural question to come in this case is which stochastic discount factor should be applied. A particularly tractable specification for the stochastic discount factor is

$$\eta(t,T) = \exp(-\delta(T-t))\zeta(T)/\zeta(t),$$

and $\zeta(t)$ has the form

$$\zeta(t) = \zeta(t;\alpha) = \exp(\alpha X_t)/\ \mathrm{E}[\exp(\alpha X_t)],$$

where X_t is a specified risk process; δ is the risk-free continuously compounded interest rate. α is a real number parameter and E is the expectation operator. This form of SDF is called Esscher transform, which can be dated back to the Swedish actuary F. Esscher. Gerber and Shiu (1994) pioneered the use of Esscher transform as a kind of SDF and applied it in pricing stock options.

Esscher transform specified in Gerber and Shiu (1994) involves only one free parameter, which is related with the risk aversion coefficient, and is thus still within the subjective expected utility framework. As explained in the introduction, the traditional economic theory based on subjective expected utility function has difficulty explaining the high premium spread of CAT bonds and spread discrepancy of bonds with different loss probabilities. As a consequence, the traditional Esscher transform seems not proper as a pricing principle assigning premium to catastrophe risk. Thus in this paper we develop a generalized Esscher transform to price CAT risk.

Since CAT losses is usually described as a compound Poisson process, the following argument will be applied to a general risk Y_t which follows a compound Poisson process; that is,

$$Y_t = \sum_{j=1}^{N_t} L_j \ , \qquad (2.2)$$

where N_t is a Poisson process with intensity $\lambda>0$ and L_j (j=1,2,⋯), independent of each other and N_t, denotes the random loss amount. For convenience, we will assume the random loss variable L_j can be described by identical distribution function, $\Pr(L_j \le x) = F_L(x)$, where $F_L(x)$ denotes the distribution function of a random variable L.

For the risk process $\{Y_t\}$ specified above, we define $\zeta(t)$ to have the following generalized form:

$$\zeta(t) = \zeta(t;\alpha,\beta) = \exp(\alpha Y_t + \beta N_t) / \mathrm{E}[\exp(\alpha Y_t + \beta N_t)] \, . \qquad (2.3)$$

Let $F(x;t)$ be the distribution function of Y_t, i.e., $F(x;t) = \Pr(Y_t \le x)$, the generalized Esscher transform of Y_t is thus defined as a random variable having the cumulative distribution function $F(x,t;\alpha,\beta) = \mathrm{E}[I(Y_t \le x)\zeta(t;\alpha,\beta)]$, where $I(\cdot)$ denotes an event indicator function.

In other words, for generalized Esscher tansform besides applying Esscher transform to the original CAT loss process to represent risk aversion, we augment an Esscher transform applying only to Poisson process N_t to represent ambiguity aversion of agents. In next section, it is shown that this form of SDF is supported by an equilibrium economy with agents who are averse not only to risk but also to uncertainty with respect to loss occurrence in a robust control framework.

Similar to the derivation in Gerber and Shiu (1994), the corresponding moment-generating function of the random variable Y_t under the generalized Esscher transform can then be calculated as

$$M_Y(z,t;\alpha,\beta) = \mathrm{E}[\exp((\alpha + z)Y_t + \beta N_t) / \mathrm{E}[\exp(\alpha Y_t + \beta N_t)]]$$

$$= \exp[\lambda(M_L(\alpha+z)e^{\beta}-1)t]/\exp[\lambda(M_L(\alpha)e^{\beta}-1)t]$$

$$= \exp[\lambda M_L(\alpha)e^{\beta}(\frac{M_L(\alpha+z)}{M_L(\alpha)}-1)t]. \qquad (2.4)$$

where M_L denotes the moment generating function of the random variable L.

Hence the generalized Esscher transform of the compound Poisson process Y_t is again a compound Poisson process, with modified Poisson parameter $\lambda M_L(\alpha)e^{\beta}$ and loss amount becomes a random variable whose moment-generating function is $\frac{M_L(\alpha+z)}{M_L(\alpha)}$.

Example 1: Gamma case

Let $G(x;a,b)$ denote the Gamma distribution with shape parameter a and scale parameter b,

$$G(x;a,b) = \frac{b^a}{\Gamma(a)}\int_0^x y^{a-1}e^{-by}dy, \quad x \geq 0.$$

If loss amount follows Gamma distribution, the moment generating function of the Y_t then becomes

$$M_Y(z,t) = \exp(\lambda[(\frac{b}{b-z})^a - 1]t).$$

Hence the corresponding moment generating function with the modified probability distribution is

$$M_Y(z,t;\alpha,\beta) = \exp(\lambda e^{\beta}(\frac{b}{b-\alpha})^a[(\frac{b-\alpha}{b-\alpha-z})^a - 1]t),$$

which shows that the transformed process is of the same type, with parameters (λ, a, b) replaced by ($\lambda e^{\beta}(\frac{b}{b-\alpha})^b, a, b-\alpha$).

2.2 The Economic Implications

In this section we establish an equilibrium model and discuss the underlying economic implication to the generalized Esscher transform introduced above.

We assume there exists an insurance market where the insurance risk Y_t is traded. A representative agent starts with an initial wealth w at the initial time 0 and besides investing in the risk-free bond;

the agent will also invest in CAT risk contracts. The risk-free bond value is accumulated at risk-free continuously compounded interest rate δ . We further assume at time 0 the agent underwrites a part of the total risk Y_T with T denotes a terminal, prespecified time. The price of insurance covering Y_T is c;the proportion the agent underwrites in the full insurance is m so the total premium the agent receives is mc. We assume all loss payments occur at time T. The agent's asset process W_t $(0\leqslant t<T)$ hence follows $W_t = W_0\, e^{\delta t}$ with $W_0=w+mc$.

As introduced in the introduction, in this paper, we deviate from the standard approach by considering the representative agent who, in addition to being risk averse, also exhibits uncertainty to the insurance risk model in a robust control framework in the sense of Anderson, Hansen and Sargent (2000).

In the robust control settings, the agent is assumed to deal with the risk model as follows. First, having noticed the unreliable aspects of the model estimation based on existing information, he evaluates alternative model description. Second, acknowledging the fact that the reference model is indeed the best statistical characterization of the available information, he penalizes the choice of alternative model by a distance function measuring how far it deviates from the reference model.

Anderson, Hansen and Sargent (2000) measure discrepancy between alternative model and reference model by 'relative entropy', defined as the expected value of a log-likelihood ratio. More exactly, letting $P=(P_t)$ be the probability measure associated with the reference model, the alternate model be described by a probability measure $\tilde{P}=(\tilde{P}_t)\in \mathcal{P}$, in which $\mathcal{P}$ denotes the set of all alternative probability measures that may be chosen. Denotes $\xi_t = d\tilde{P}_t / dP_t$ as the Radon-Nikodym derivative of $\tilde{P}_t$ with respect to P_t, the relative entropy is defined by $I_t = \lim_{\Delta\to 0}\frac{1}{\Delta}\tilde{\mathrm{E}}_t[\log(\frac{\xi_{t+\Delta}}{\xi_t})]$, where $\tilde{\mathrm{E}}_t$ is the conditional expectation operator conditioning on the information available up to time t that is evaluated with respect to the density associated with the alternative or twisted model (i.e., not the reference model).

We now turn to the optimization problem according to the spirit of robust control theory (See, e.g., Liu, Pan and Wang (2005)), we assume the agent seeks to maximize the robust-control utility function U, which is defined as the solution to the following stochastic integral equation:

$$U = U(W,t,y,m) = \inf_{\tilde{P}\in P}\{\tilde{\mathrm{E}}_t[u(W_T - mY_T) - \int_t^T \frac{\gamma \lambda_s U(W_s, s, Y_{s-}, m)}{\phi} ds \big| W_t = W, Y_t = y]\}, \tag{2.5}$$

with boundary condition $U(W,T,y,m) = u(W - my)$, where $\tilde{\mathrm{E}}_t$ is the conditional expectation operator with respect to an alternative probability measure in P; u is the exponential utility function expressed as $u(x) = -(1/\gamma)e^{-\gamma x}$, and γ >0 is the risk coefficient; $\int_t^T \frac{\gamma \lambda_s U(W_s, s, Y_{s-}, m)}{\phi} ds$ represents a penalty function controlling the deviation from the reference model, in which ϕ is a parameter measuring the relative importance of the reference and alternative models (In other words, an agent with higher ϕ exhibits higher aversion to model uncertainty).

Since there seems no apparent relation between spread premium and loss severity for CAT securities, we restrict here that the uncertainty aversion of the agent only applies to the likelihood component of the loss arrival. That is, we effectively assume the agent only has doubt about the occurrence probability of loss events, while is comfortable with the loss magnitude aspect of the model. The Radon-Nikodym derivative ξ_t is thus defined by the following stochastic differential equation.

$$d\xi_t = (e^{h_t} - 1)\xi_{t-} dN_t - (e^{h_t} - 1)\lambda \xi_t dt$$

where h_t is a time dependent function controlling the model distortion magnitude, and where $\xi_0 = 1$. By construction, the process $\{\xi_t, 0 \le t \le T\}$ is a martingale of mean one. The measure $\tilde{P} = (\tilde{P}_t)$ thus defined is indeed a probability measure, and let P be the entire collection of such probability measures.

Given the alternative model specification defined by ξ_t, the distance measure I_t can be calculated as:

$$I_t = \lambda(e^{h_t} h_t - e^{h_t} + 1). \tag{2.7}$$

See Appendix B in Liu, Pan and Wang (2005) for the proof of a more general formula than (2.7).

The equation (2.5) for utility function U then becomes

$$U = U(W,t,y,m) = \inf_{\{h_s\}} \{\tilde{\mathrm{E}}_t[u(W_T - mY_T)$$
$$- \int_t^T \frac{\lambda\gamma}{\phi}[1+(h_s-1)e^{h_s}]U(W_s,s,Y_{s-},m)ds \big| W_t = W, Y_t = y]\}$$

The corresponding HJB equation for U is then as follows:

$$U_t + \delta W U_W + \inf_{h_t} \{\lambda e^{h_t}[\mathrm{E}U(W,t,y+L,m)$$
$$-U(W,t,y,m)] - \frac{\lambda\gamma}{\phi}[1+(h_t-1)e^{h_t}]U\} = 0, \tag{2.8}$$

where U_t is the derivative of U with respect to t, U_W, U_{WW} are its first and second derivatives with respect to W, and the terminal condition is $U(W,T,y,m) = -\frac{1}{\gamma}\exp(-\gamma W)$.

We conjecture that the indirect function U is of the form

$$U(W,t,y,m) = V(W,t)e^{\gamma m y} f(t,m) \tag{2.9}$$

where $V(W,t)$ is defined as $V(W,t) = -\frac{1}{\gamma}\exp(-\gamma W e^{\delta(T-t)})$, and $f(t,m)$ is a time-dependent function.

Insert the form of U in (2.9) into the above HJB equation and canceling the factor of $e^{\gamma m y}$, we can then rewrite (2.8) as

$$V_t f + V f_t + \delta W V_W f$$

$$+ \inf_{h_t} \{\lambda e^{h_t} (\mathrm{E}[\exp(m\gamma L)] V f - V f) - \frac{\lambda\gamma}{\phi}[1+(h_t - 1)e^{h_t}] V f\} = 0. \qquad (2.10)$$

The first and third terms cancel and canceling V<0 from the remaining terms obtains the following ordinary differential equation for f

$$f_t + \sup_{h_t}\{\lambda e^{h_t}[M_L(\gamma m) - 1] f - \frac{\lambda\gamma}{\phi}[1+(h_t - 1)e^{h_t}] f\} = 0, \qquad (2.11)$$

with boundary condition $f(T,m)=1$.

The first order condition for h_t gives the following equation

$$[M_L(\gamma m) - 1] - \frac{\gamma}{\phi} h_t = 0. \qquad (2.12)$$

Notice its solution $h_t^* = \frac{\phi}{\gamma}[M_L(\gamma m) - 1]$ is a constant independent of time t and substituting it and the corresponding solution of (2.11) into the indirect utility function form (2.9), the function U at time 0 is then given by

$$U(w + mc, 0, 0, m) = -\frac{1}{\gamma}\exp\{-\gamma(w + mc)e^{\delta(T-t)}$$

$$+ \int_0^T \lambda[e^{h^*}(M_L(\gamma m) - 1) - \frac{\gamma}{\phi}(1 + (h^* - 1)e^{h^*})] dt\}. \qquad (2.13)$$

Finally the first order condition for m in equation (2.13) gives the following equation for the optimal insurance proportion m^*:

$$c = e^{-\delta T}\int_0^T \lambda e^{h^*}\mathrm{E}[L e^{\gamma m^* L}] dt = e^{-\delta T} T \lambda e^{\frac{\phi}{\gamma}(M_L(\gamma) - 1)}\mathrm{E}[L e^{\gamma m^* L}]. \qquad (2.14)$$

In equilibrium, the representative agent accepts full risk in the insurance market so $m^*=1$, the solution to market equilibrium and the pricing kernel can then be summarized by the following proposition:

Proposition 1: *In equilibrium, the price of CAT risk is given by*

$$c = e^{-\delta T} T \lambda e^{\frac{\phi}{\gamma}(M_L(\gamma)-1)} \mathrm{E}[L e^{\gamma L}]. \tag{2.15}$$

The equilibrium SDF is then given by a generalized Esscher transform to have the form

$$\zeta(t) = \zeta(t;\alpha,\beta) = \exp(\alpha Y_t + \beta N_t) / \mathrm{E}[\exp(\alpha Y_t + \beta N_t)],$$

with $\alpha = \gamma$ *and* $\beta = \frac{\phi}{\gamma}(M_L(\gamma)-1)$.

See **Appendix 1** for proof of the proposition.

Remark 1: Esscher transforms as candidates of SDF are usually supported by risk exchanges equilibrium models. See, for example, Kallsen & Shiryaev (2002) for economic meanings of Esscher transform introduced in Gerber & Shiu (1994). In this paper, we have shown that the specified class of generalized Esscher transform for CAT risk pricing can also be supported by an equilibrium framework. The difference from the previous work is that, this time the involved agent is averse to uncertainty as well as to risk. The economic underpinnings of Esscher transform make it unique as an insurance premium principle and it is also in this sense that Esscher transform can be looked as a bridge pulling traditional actuarial science and modern financial economics together (Gerber and Shiu (1996)).

Remark 2: The above discussion can be generalized to include stock in the financial market. Although we have assumed the insurance risk is traded at the initial time, the above argument for insurance risk pricing can be applied to any time and the derived SDF is also given by a generalized Esscher transform. Notice that once the insurance risk has been transferred at the initial time, the insurance price at later time will be adjusted to keep the insurance fully underwritten by the representative agent.

3. Catastrophe Bond Pricing and Empirical Analysis

3.1. Catastrophe Bond Pricing

In this section, we apply the SDF corresponding to the generalized Esscher transform to price CAT bond. We assume for convenience of discussion that the form of CAT bond is described as follows:

The CAT bond is priced at K, which denotes the bond principal. If the loss for any single CAT event in the period $(0, T)$ is less than a loss trigger A, the agent will get back his principal K, plus risk-free interest $K(e^{\delta T}-1)$, plus spread premium $K\tilde{l}$ at the maturity time T, in which $\tilde{l}$ denotes the spread premium rate. Once a CAT loss exceeds the trigger, the agent will forfeit some or all the principal at time T. We assume the spread premium plus the risk-free interest is guaranteed no matter whether any principal loss occurs.

The aggregate CAT losses exceeding A in the time period $(0, T)$ is assumed to follow a compound Poisson distribution described as follows,

$$Y_T=\sum_{j=1}^{N_T} L_j\ , \qquad (2.2)$$

where CAT (larger than A) occurrences number N_T is Poisson with intensity $\lambda T>0$ and L_j (j=1,2,$\cdots$), independent of each other and N_T, denotes the random loss amount and can be described by identical distribution function, $\Pr(L_j \le x)=F_L(x)$, where $F_L(x)$ denotes the distribution function of a random variable L which is larger than A.

The loss fraction f_{AB} of the principal is in proportion to the first CAT loss L_1 in the range between the trigger (A) and a cap (B), and is given by

$$\begin{aligned} f_{AB} &= Max[0, Min(L_1-A, B-A)]/(B-A) \\ &= (Max[0, L_1-A]-Max[0, L_1-B])/(B-A) \\ &= (L_1-A-(L_1-B)_+)/(B-A) \qquad (3.1) \end{aligned}$$

In other words, the cash flow the agent get back at time T can be described as a random variable

$K \cdot (e^{\delta T} + \tilde{l} - I(N_T > 0) f_{AB})$, where I is the event indicator function. Therefore the random principal loss fraction at time T is $I(N_T > 0) f_{AB}$.

The probability that at least one catastrophe loss occurs in the period (0, T) is given by $\mathrm{E}[I(N_T > 0)] = (1 - e^{-\lambda T})$. The average loss fraction conditional on the catastrophe occurrence can be calculated as $\mathrm{E}(f_{AB}) = \frac{\int_A^B [1 - F_L(x)]dx}{B - A}$. The mean of the bond principal loss proportion is thus given by

$$l = \mathrm{E}[I(N_T > 0) f_{AB}] = \mathrm{E}[I(N_T > 0)]\mathrm{E}(f_{AB}).$$

$$= \frac{(1 - e^{-\lambda T})\int_A^B [1 - F_L(x)]dx}{B - A}. \tag{3.2}$$

Now we apply the stochastic discount factor $\eta(0,T)$ derived from the generalized Esscher transform specified in Section 2.1 to price the above CAT bond. The basic equation can be written as:

$$K = \mathrm{E}[\eta(0,T) K \cdot (e^{\delta T} + \tilde{l} - I(N_T > 0) f_{AB})],$$

from which we can get the formula to calculate the spread premium rate $\tilde{l}$ as follows,

$$\tilde{l} = \mathrm{E}[\exp(\alpha Y_T + \beta N_T) I(N_T > 0) f_{AB}] / \mathrm{E}[\exp(\alpha Y_T + \beta N_T)]. \tag{3.3}$$

If we further assume the catastrophe risk is nonsystematic, that is, not correlated with the market portfolio of securities, the absolute risk aversion parameter of the representative investor thus goes to zero, i.e., $\alpha \to 0$. In this case we can only focus on the effect of the ambiguity aversion.

Equation (3.3) then becomes

$$\tilde{l} = \mathrm{E}[\exp(\beta N_T) I(N_T > 0) f_{AB}] / \mathrm{E}[\exp(\beta N_T)]$$

$$= \mathrm{E}(f_{AB}) \cdot \mathrm{E}[\exp(\beta N_T) I(N_T > 0)] / \mathrm{E}[\exp(\beta N_T)]$$

in which $\mathrm{E}[\exp(\beta N_T) I(N_T > 0)] / \mathrm{E}[\exp(\beta N_T)]$ is the modified probability that at least one catastrophe loss occurs and can be calculated to be $1 - e^{-\lambda T e^{\beta}}$. Therefore we have $\tilde{l} = \dfrac{(1 - e^{-\lambda T e^{\beta}}) \int_A^B [1 - F_L(x)] dx}{B - A}$, and the ratio of the spread premium to the mean principal loss can be given by the following proposition:

Proposition 2: *Assuming the representative agent is risk neutral to the catastrophe risk Y_T, the ratio of spread premium rate to the expected principle loss proportion (i.e., the ratio of the modified mean principal loss to the expected loss) of the related CAT bonds is given by*

$$\frac{\tilde{l}}{l} = \frac{1 - e^{-\lambda T e^{\beta}}}{1 - e^{-\lambda T}} \approx \frac{\lambda T e^{\beta}}{\lambda T} = e^{\beta}, \tag{3.4}$$

in which the second approximation equation holds when λ is very small.

In Proposition 2, we involve a parameter β to denote the ambiguity aversion. Since the uncertainty results partly from the scarcity of historical statistical information, it seems natural to assume $\beta = \beta(l)$ is a decreasing function of the mean principle loss proportion l.

3.2. In-sample Fitting and Out-of-Sample Performance

In this section we examine the empirical implication of our model using empirical catastrophe bonds data. We first calibrate our model to match the empirical data listed in Table 1. Since we have ignored the premium loadings for expenses in the previous discussion while the catastrophe securities as financial instruments, usually have high transaction costs, we should eliminate the expenses effects in the estimation of the multiple of premium relative to actuarially expected losses. Froot (1999) has assumed in his explanation of high price of catastrophe reinsurance that the brokerage and underwriting expenses come to be about 10 percent of premium. The elimination of these expenses will drive down the average ratio of premiums to expected losses from 9.09 to about 8.18.

To calibrate the empirical data to our model specification, we face the problem of which kind of functional form for β be chosen to fit the sample data. An in-sample test shows that the function form for e^{β} corresponding to the power function, that is: $e^{\beta} = b_0 l^{-b_1}$ provides a simple and good fitting and the parameters are estimated to be b_0=0.2163 and b_1=−0.6728. Figure 2 shows the calibrated result estimated by the power function as well as the empirical ratio of expense-adjusted premium to expected loss against the expected CAT loss (based on 1997-2000 data). It can be seen from the graph that the calibrated result provides an excellent fit to the observed data.

Please insert Figure 2 about here

Having specified the form of β and related parameters that best fit the catastrophe bond samples issued from 1997 to 2000. We now turn to examine the model's out-of-sample pricing performance. For this purpose, we rely on the catastrophe securities data collected from 2000 to 2003. The data are obtained from Sigma journal of Swiss Re. Table 2 is a list of CAT securities outstanding as of 31 December 2003.

Please insert Table 2 about here

Figure 3 shows the calculated result based on the model and parameters estimated above, as well as the empirical ratio of expense-adjusted premium to expected loss against the expected CAT loss as listed in Table 2. The figure shows that the estimated result provides an adequate fit to the observed data. Therefore, using the generalized Esscher transform based on the robust control theory seems serving as an excellent model to price the catastrophe risk and CAT linked securities.

Please insert Figure 3 about here

4. Conclusion

Motivated by the observation that the spread premium of CAT loss securities is very high relative to the expected loss of principal of bonds, and the premium spread is much more pronounced for CAT bonds with low probability that a contingent loss payment to the bond issuers will be triggered, we have extended the traditional Esscher Transform to a generalized framework and treated it as a stochastic discount pricing factor to price the CAT risk. The generalized Esscher transform is derived from a modified equilibrium model by allowing the representative agent to act in a robust control framework against model misspecification with respect to rare events in the sense of Anderson, Hansen and Sargent (2000). The model is explicitly solved and the derived pricing kernel is shown to be exactly the generalized Esscher Transform. Using catastrophe bonds data, we examine the empirical implication of our model.

There are several extensions we may do to our current investigation. First, we have considered the issue of uncertainty aversion in a robust control framework and thus envision the model misspecification as a permanent psychological characteristic of a decision maker. On the other hand, an agent may learn the model through successive approximations. It would be an important extension to incorporate forms of learning, i.e., the credibility theory, into our framework. Second, we restrict the CAT loss process here to be a compound Poisson process. Since there is usually seasonality in catastrophe occurrence, it will be interesting to describe the CAT loss by a time-changed Levy process and in the most general setting, the process written as a semimartingale. Discussion for the corresponding change of measure in these cases can be found in Carr & Wu (2004), and Bühlmann et al (1998). Finally, just as Froot pointed out (1999), there is typically a shift of coverage window in reinsurance market after a large CAT event occurs. To explain this time series empirical fact, it might be helpful to model how the prior performance of CAT events might affect the magnitude of uncertainty aversion of decision makers in an insurance market. For this purpose, we may simulate the approach on similar topic in the framework of wealth-based prospect theory (see, e.g., the seminal contribution of Barberis, Huang and Santos (2001)). For all these issues we leave for future research.

Appendix 1

Proof of Proposition 1: According to equation (2.4), the moment generating function under the generalized Esscher transform specified in Proposition 1 can be calculated as

$$M_Y(z,t;\alpha,\beta) = \exp[\lambda M_L(\alpha)e^{\beta}(\frac{M_L(\alpha+z)}{M_L(\alpha)}-1)t],$$

with $\alpha=\gamma$ and $\beta=\frac{\phi}{\gamma}(M_L(\gamma)-1)$.

Hence the generalized Esscher transform of Y_t is again a compound Poisson transform, with modified Poisson parameter $\lambda M_L(\gamma)e^{\frac{\phi}{\gamma}(M_L(\gamma)-1)}$ and loss amount becomes a random variable whose moment-generating function is $\frac{M_L(\gamma+z)}{M_L(\gamma)}$. The modified mean of loss amount can then be calculated as

$$\frac{d}{dz}(\frac{M_L(\gamma+z)}{M_L(\gamma)})\bigg|_{z=1} = \mathrm{E}[L\frac{e^{\gamma zL}}{M_L(z\gamma)}]\bigg|_{z=1} = \mathrm{E}[L\frac{e^{\gamma L}}{M_L(\gamma)}].$$

Therefore the price of CAT risk Y_t in period $(t,\ T)$ is given by

$$c = e^{-\delta T}\int_0^T \lambda e^{\frac{\phi}{\gamma}(M_L(\gamma)-1)} M_L(\gamma)\mathrm{E}[L\frac{e^{\gamma L}}{M_L(\gamma)}]dt$$

$$= e^{-\delta T}T\lambda e^{\frac{\phi}{\gamma}(M_L(\gamma)-1)}\mathrm{E}[Le^{\gamma L}].$$

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Table 1: Catastrophe bond issues (1997~2000 bonds data)

The Spread Premium is the annual coupon rate above one-year LIBOR. The Prob of First Loss is the probability that a contingent payment will be trigged under the bond. The E[L | L>0] is the expected principal payment to the issuing insurer, conditional on the occurrence of a loss that triggers payment under the bond, expressed as a percentage of the principle of the bond. The Expected Loss is the product of the probability of first loss and E[L | L>0]. Prem to E[loss] is the ratio of the spread premium to the expected loss of principle of the bond.

Date	Transaction Sponsor	Spread Premium (%)	Prob of First Loss (%)	E[L | L>0] (%)	Expected Loss (%)	Prem to E[Loss]
March-00	SCOR	2.70	0.19	57.89	0.11	24.55
March-00	SCOR	3.70	0.29	79.31	0.23	16.09
March-00	SCOR	14.00	5.47	59.23	3.24	4.32
March-00	Lehman Re	4.50	1.13	64.60	0.73	6.16
November-99	American Re	2.95	0.17	100.00	0.17	17.35
November-99	American Re	5.40	0.78	80.77	0.63	8.57
November-99	American Re	8.50	0.17	100.00	0.17	50.00
November-99	Gerling	4.50	1.00	75.00	0.75	6.00
June-99	Gerling	5.20	0.60	75.00	0.45	11.56
June-99	USAA	3.66	0.76	57.89	0.44	8.32
July-99	Sorema	4.50	0.84	53.57	0.45	10.00
July-98	Yasuda	3.70	1.00	94.00	0.94	3.94
March-99	Kemper	3.69	0.58	86.21	0.50	7.38
March-99	Kemper	4.50	0.62	96.77	0.60	7.50
May-99	Oriental Land	3.10	0.64	66.04	0.42	7.35
February-99	St. Paul/F&G Re	4.00	1.15	36.52	0.42	9.52
February-99	St. Paul/F&G Re	8.25	5.25	54.10	2.84	2.90
December-98	Center Solutions	4.17	1.20	64.17	0.77	5.42
December-98	Allianz	8.22	6.40	56.41	3.61	2.28
August-98	XL/MidOcean Re	4.12	0.61	63.93	0.39	10.56
August-98	XL/MidOcean Re	5.90	1.50	70.00	1.05	5.62
July-98	St. Paul/F&G Re	4.44	1.21	42.98	0.52	8.54
July-98	St. Paul/F&G Re	8.27	4.40	59.09	2.60	3.18
June-98	USAA	4.16	0.87	65.52	0.57	7.30
March-98	Center Solutions	3.67	1.53	54.25	0.83	4.42
December-97	Tokio Marine & Fire	2.09	1.02	34.71	0.35	5.90
December-97	Tokio Marine & Fire	4.36	1.02	68.63	0.70	6.23
July-97	USAA	5.76	1.00	62.00	0.62	9.29
August-97	Swiss Re	2.55	1.00	45.60	0.46	5.59
August-97	Swiss Re	2.80	1.00	46.00	0.46	6.09
August-97	Swiss Re	4.75	1.00	76.00	0.76	6.25
August-97	Swiss Re	6.25	2.40	100.00	2.40	2.60
					Average	9.09
					Median	6.77

Source: Cummins, Lalonde and Phillips (2004)

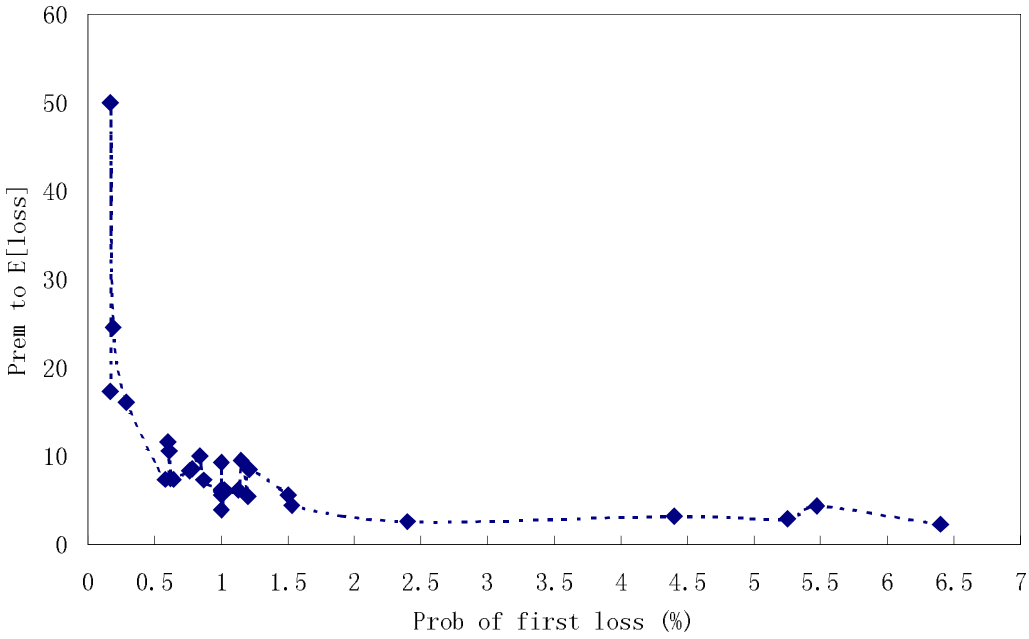

Figure 1: The “smirk” curve of empirical premium spreads against loss probabilities

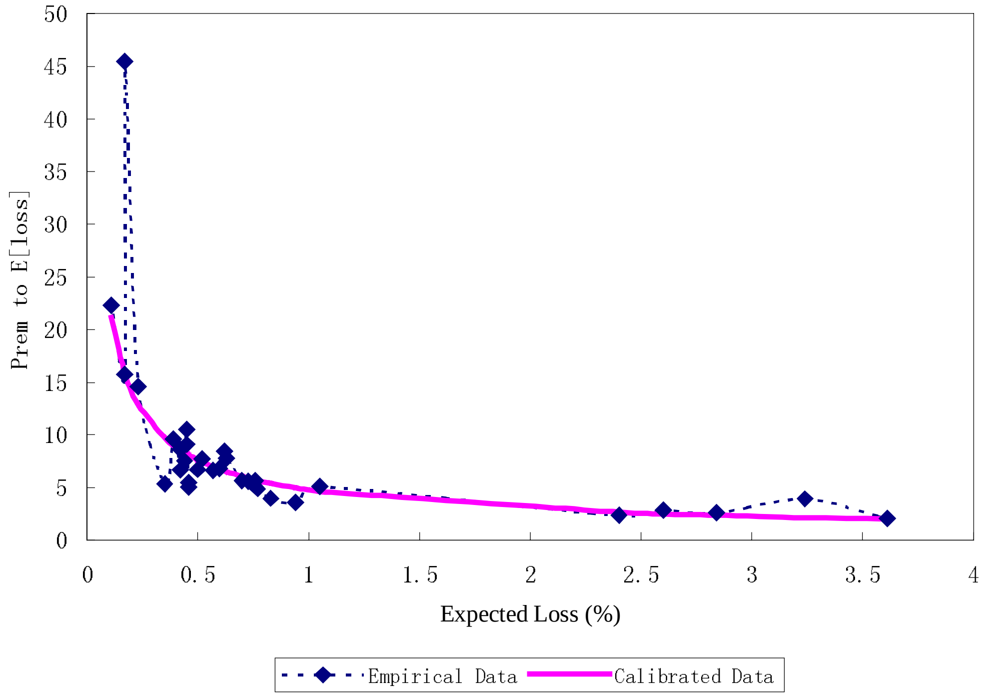

Figure 2 The calibrated and empirical ratios of expense-adjusted premium spreads to expected loss against expected CAT losses (In-sample fitting to 1997-2000 CAT bonds data)

Table2 CAT securities outstanding as of 31 December 2003 (2000~2003 bonds data)

CAT bond sponsor	Scheduled maturity	Spread at issuance (%)	Expected loss (%)	Spread to expected loss
Swiss Re	2006-06	15.5	4.86	3.19
Swiss Re	2006-06	15.25	4.86	3.14
Swiss Re	2006-12	15	4.86	3.09
Swiss Re	2006-06	1	0.01	100.00
SCOR	2005-01	2.38	0.05	47.60
SCOR	2005-01	6.75	0.9	7.50
Oriental Land	2004-05	3.1	0.41	7.56
TREIP	2006-06	3.45	0.53	6.51
Nissa Dowa	2005-05	4	0.67	5.97
AGF	2005-11	2.6	0.22	11.82
AGF	2005-11	5.85	1.16	5.04
Swiss Re	2007-06	4.75	1.27	3.74
Swiss Re	2007-06	5.75	1.28	4.49
Swiss Re	2005-12	5	1.28	3.91
Tokio Marine & Fire	2007-11	4.3	0.7	6.14
Zenkyoren	2006-06	2.45	0.22	11.14
Zenkyoren	2006-06	3.5	0.49	7.14
Swiss Re	2006-06	6	1.28	4.69
Swiss Re	2006-06	5	1.27	3.94
Swiss Re	2006-06	1.75	0.22	7.95
Swiss Re	2006-06	4.25	1.29	3.29
Swiss Re	2006-06	7.5	1.31	5.73
EDF	2008-12	1.5	0.02	75.00
EDF	2008-12	3.9	0.54	7.22
Swiss Re	2006-01	3.85	0.52	7.40
Swiss Re	2006-01	2.3	0.22	10.45
USAA	2004-06	4.99	0.68	7.34
USAA	2005-06	4.9	0.67	7.31
USAA	2006-06	4.95	0.48	10.31
Swiss Re	2007-06	4.5	1.29	3.49
Swiss Re	2007-06	5.75	1.28	4.49
Swiss Re	2005-05	5.25	0.68	7.72
Swiss Re	2005-05	5.75	0.76	7.57
Synidicate 33	2005-04	6.75	1.14	5.92
Zurich Re/Converium	2004-06	8	1.11	7.21
Zurich Re/Converium	2004-06	4	0.67	5.97
Swiss Re	2007-01	1.35	0.02	67.50

Source: Sigma of Swiss Re, No.1, 2004

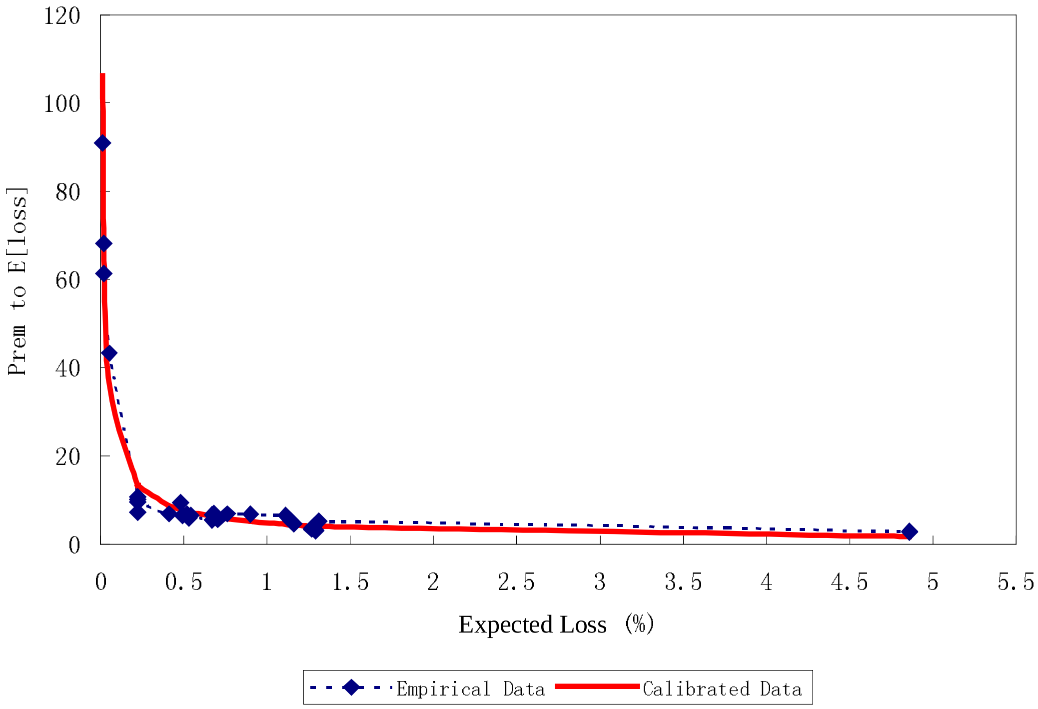

Figure 3 The calibrated and empirical ratios of expense-adjusted premium spreads to expected loss against expected CAT losses (Out of sample fitting to 2000~2003 CAT bonds data using 1997-2000 bonds data based estimation)