Term Structure Modeling for Pension Funds: What to do in Practice?*

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Abstract

With the increased emphasis on market valuation in accounting rules and solvency regulation, the proper modeling of interest rate dynamics has become increasingly important for pension funds. A number of pension fund characteristics make these models particularly demanding. First, as the obligations of pension funds stretch far into the future, the model should be reasonable both for short rates and very long term rates. Second, as the value of liabilities increases enormously if interest rates approach zero, especially the probability of very low rates should be modeled correctly. Third, as pension rights are usually indexed, the interaction between interest rates and inflation should be addressed. Fourth, in order to allow for long term analysis, the simulation results should preferably be stationary. Fifth, account has to be taken to possible structural breaks in the inflation and interest rate dynamics, if only to comply with maximum return assumptions of supervisors. In this paper we present a new affine discrete-time, three-factor model of the term structure of interest rates that meets these criteria. The factors are the short term rate, expected inflation and stochastic risk aversion. The model is applied to an unbalanced panel of German/euro area zero-coupon yields for maturities of one to sixty years, and estimated using the extended Kalman filter.

Keywords: Discrete time, no-arbitrage, expected inflation, stochastic risk aversion, stochastic volatility, generalized essentially affine model.

JEL codes: E34, G13.

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1 Introduction

Modeling the term structure of interest rates has a long tradition in finance. Starting with Vasicek (1977) and Cox, Ingersoll, and Ross (1985), a wide variety of models has been introduced to describe the theoretical behavior of interest rates. By specifying a particular functional form for the dynamics of short term interest rates and the reward investors require to take on interest rate risk, these models describe the evolution of yields of all maturities. This characteristic makes them potentially very appealing for defined benefit pension funds or life insurance companies. With the international trend in both accounting rules and regulatory frameworks going towards market valuation, being able to predict possible future developments of yields of all maturities becomes more and more important, as the valuation of their liabilities depends crucially on the term structure of interest rates. Notwithstanding the huge literature developed so far, the current models seem hardly able to fulfill all demands for this particular task. This paper tries to bridge this gap, by formulating a new discrete-time arbitrage-free three factor affine term structure model, and estimating it on European data.[1]

What challenging tasks are we facing in this respect? First of all, since the obligations of pension funds or life insurers stretch far into the future, the model should not only fit the short end of the yield curve well, but also the long end. In the Netherlands, for instance, the supervisory authority for pension funds publishes monthly a nominal term structure for maturities from one to sixty years based on swap market data. Starting from January 2007, pension funds are obliged to use this term structure to estimate the value of their pension

[1]Although many papers are written on the US term structure, empirical work on German / euro area data is much more limited. Some examples are Cassola and Barros Luís (2003), Dewachter, Lyrio, and Maes (2004), Fendel (2005), and Hördahl, Tristani, and Vestin (2006). These papers all use a Gaussian framework, in which variances are assumed to be constant.

liabilities.[2] The vast majority of estimated term structure models focusses on the short end of the yield curve. The longest maturity is often only five years, whereas studies using maturities longer than ten years are almost unavailable. This focus on the short end is primarily due to data availability. Liquid markets for long maturity bonds or swaps hardly existed before the last couple of years.

A second, though related, complicating factor we are facing, is the sensitivity of these long term liabilities to interest rate changes. As the average duration of liabilities is about sixteen years, relatively small yield changes might have substantial effects. Consequently, the volatility and, especially, the probability of extremely low values should be correctly modeled for all yields. A third requirement is related to indexation policy of pension funds. As most pension funds intend to link pension benefits to either price or wage inflation, the interaction between interest rates and inflation should be incorporated. An important aspect here is that the interaction goes both ways. High inflation induces central banks to increase interest rates. They do so exactly because high interest rates after some time depress aggregate demand and thereby reduce inflation. Fourth, in order to be able to usefully apply the model in simulation exercises, the model should produce reasonable (also long term) forecasts. Consequently, (near) unit roots should be avoided if possible. Moreover, when simulating with the model, one should be aware of possible structural breaks in the inflation/interest rate dynamics. With respect to inflation, nowadays there seems to be much more consensus on the benefits of a low inflation environment. This might help in avoiding inflationary spells as we experienced in the seventies. As to interest rates, the increased emphasis on market valuation is likely to permanently reduce long term yields relative to short ones as the demand for long term

[2] In the first couple of years they will be allowed to use a duration approach. The average duration of Dutch pension liabilities is however still about sixteen years.

bonds by pension funds and insurance companies will increase. Long term bonds reduce the duration mismatch between their assets and liabilities, and thereby reduce risk instead of increasing it. In view of these developments, the Dutch supervisory authority prescribes an assumed equilibrium nominal return on fixed income assets of at most 4.5%. Pension funds will be obliged to use this lower than historical return in their asset liability analysis to show the long run perspective of the fund.

So, how are these problems handled in this paper? With respect to the scarcity problem for long maturity data, an unbalanced panel of interest rates is used, with data on zero coupon yields for one to ten years starting in September 1972, but data on 60-year yields starting only in September 2001. In order to increase the impact of long maturities, the variance of the measurement errors is restricted to be the same for all maturities. As a result, the variance is reduced for long maturities, thereby increasing the penalty for pricing errors in this range. The lower bound is preserved by including a level effect in the variance equation. Although this is common practice in finance models since Cox, Ingersoll, and Ross (1985), term structure models including inflation or other macro variables hardly ever include it.[3] Without a level effect, interest rates are symmetric around their mean, which implies that either extremely low rates are predicted too often, or that the volatility is underestimated. The main reason for the absence of a level effect in macro-finance models is probably the trade-off between flexibility in correlations and volatilities of the risk factors (Dai and Singleton 2000), which renders stochastic volatility models unattractive from an economic perspective. The trade-off is however necessary to guarantee that variances can never become negative.

[3] See for instance Campbell and Viceira (2002), Ang and Piazzesi (2003), Ang and Bekaert (2004), Dewachter, Lyrio and Maes (2004, 2006), Fendel (2005), Bernanke, Reinhart, and Sack (2005), Dewachter and Lyrio (2006), Hördahl, Tristani, and Vestin (2006), Wu (2006), and Rudebusch and Wu (2007). An exception is Spencer (2004), who specifies a 10-factor model for the US yield curve, including one heteroscedasticity factor which is a linear combination of several macroeconomic variables.

Whenever the process approaches an area where a variance is zero, it has to bounce back into the right direction (with positive variances). This so-called Feller condition implies among others that factors that determine the variance of the process are not allowed to depend on the other factors. This is particularly troublesome for macro-finance models as inflation and interest rates are clearly interrelated both in their first and second moment. In this paper, we follow Spreij, Veerman, and Vlaar (2007) in extending the class of affine term structure models as classified by Dai and Singleton (2000). The former study allows the correlation restrictions related to the multivariate Feller condition, to be relaxed for factors (in this case short rates and expected inflation) that share the same volatility process, determined by these factors. Whenever a variance becomes negative, it is restricted to zero. The resulting approximation error turns out to be negligible.

One reason for the presence of (near) unit roots in many term structure models is the empirical regularity that the yield curve often experiences parallel shifts over time. As long term rates are determined by expected future short term rates plus the price of interest rate risk, an obvious explanation for these parallel shifts is a (near) unit root in the short rate process. At least for our German data, this assumption is not correct however. In order to disentangle the time series properties from the crosssectional ones (over the maturities), maximal flexibility is given to the specification of the price of risk. A first innovation in this respect is the modeling of stochastic risk aversion. This unobservable factor only affects the price of risk, without affecting the short rate itself.[4] The second innovation concerns a generalization of the essentially affine specification of Duffee (2002). To further enhance the proper specification of the time series dynamics (essential for simulations), a two-step

[4]Stochastic risk appetite might explain the fact that US and European long term yields are strongly correlated, whereas short rates are not.

procedure is followed. In the first step, the dynamics of short term rates and expected inflation is estimated. In the second step, the price of risk is determined, conditional on the time-series parameters of the first step.[5] The final problem for simulations, possible structural breaks, is handled by adjusting the equilibrium values for expected inflation and nominal short rates, as well as an adjustment in the assumed volatility of these driving factors.

The rest of this paper is organized as follows. Section 2 describes the model. First, the general features of a discrete-time essentially affine model are given, then the short-term dynamics is presented after which our specific model is shown. Section 3 shows the estimation results. Section 4 provides the simulation results, both for the estimated parameters and for the calibrated (lower) equilibrium values. Finally, Section 5 concludes, and the appendix provides the data sources.

2 A discrete-time generalized essentially affine yield model

This section sets out the general linear dynamic framework for deriving bond prices under the no-arbitrage assumption. The model is a discrete time generalization of the general affine model developed in a continuous time framework by Duffie and Kan (1996), and extended to the essentially affine class by Duffee (2002).

2.1 General features

Models of the term structure of interest rates describe the dynamics of the relationship between spot rates of zero-coupon bonds and their term to maturity. Let $P_{\tau t}$ be the price at time t of a nominal zero-coupon bond maturing at time $t + \tau$. Under the law of one price

[5] A two-step estimation procedure for macro finance term structure models was previously adopted by Ang, Piazzesi, and Wei (2006).

there exists a nominal stochastic discount factor or pricing kernel M_{t+1} such that

$$P_{\tau t} = E_t[M_{t+1} P_{\tau-1,t+1}], \tag{1}$$

where E_t denotes the expected value conditional on information available at date t. See Cochrane (2001) for a detailed discussion on pricing kernels. Note in particular that the nominal interest rate on a short-term bond is risk-free, and that relation (1) implies that the log nominal short rate (denoted i_t) is equal to the negative of the natural logarithm of the conditional expectation of the pricing kernel:

$$i_t \equiv \log \frac{P_{0,t+1}}{P_{1t}} = -\log E_t[M_{t+1}],$$

where the second equality follows from the fact that the payoff of a nominal bond at maturity is $P_{0,t+1} = 1$. The joint conditional distribution of bond prices and the discount factor is assumed to be multivariate normal. This implies that the log bond price $p_{\tau t} \equiv \log P_{\tau t}$ is given by

$$p_{\tau t} = E_t[m_{t+1} + p_{\tau-1,t+1}] + \frac{1}{2} var_t[m_{t+1} + p_{\tau-1,t+1}], \tag{2}$$

where $m_{t+1} \equiv \log M_{t+1}$ and var_t is the variance conditional on time t information. For the one period bond this implies, $p_{1t} = -i_t = E_t[m_{t+1}] + \frac{1}{2} var_t[m_{t+1}]$. Combining this result with Equation 2 one gets:

$$p_{\tau t} = -i_t + E_t[p_{\tau-1,t+1}] + \frac{1}{2} var_t[p_{\tau-1,t+1}] + cov_t[m_{t+1}, p_{\tau-1,t+1}], \tag{3}$$

By repeated substitution of Equation 3, it follows that long term yields are determined by expected future short term rates, Jensen inequality terms, and a term premium related to the covariance between future pricing kernels and bond prices.

The model proposed in this paper is in the class of affine term structure models. In this class of models, (minus) the log bond price is assumed to be an affine (linear) function of some underlying state variables:

$$-p_{\tau t} = A_\tau + B_\tau x_t \tag{4}$$

where x_t is an n-dimensional vector of underlying factors. In the finance literature, these factors are usually unobservable, but in the macro-finance models these include the macroeconomic variables. The dynamics of the factors is again assumed to be affine:

$$x_t = (I_n - \Phi)\theta + \Phi x_{t-1} + \Sigma V_{t-1}^{1/2} \varepsilon_t \tag{5}$$

where I_n is an n-dimensional identity matrix, θ is an n-dimensional vector with means of the state variables, Φ and Σ are $n \times n$ matrices, $\varepsilon_t \sim N(0, I_n)$ is a vector of i.i.d standard normally distributed error terms, and V_t is a diagonal matrix with the diagonal given by:

$$Diag[V_t] = \Gamma \begin{bmatrix} 1 \\ x_t \end{bmatrix} \tag{6}$$

In order to identify all parameters individually and to assure that the variances in (6) are all nonnegative, restrictions have to be imposed on Φ, Σ, and/or Γ.

A final necessary input to derive the theoretical term structure dynamics is an assumption regarding the stochastic behavior of the pricing kernel. We will use a generalization of the essentially affine specification of Duffee (2002), in which the variance of the pricing kernel is not necessarily affine in the underlying state variables, but which still results in an analytical solution for the term structure parameters:

$$m_t = -i_{t-1} - \frac{1}{2} var_{t-1}[m_t] + [1, x'_{t-1}]\Lambda' V_{t-1}^{-1/2} \varepsilon_t \tag{7}$$

Only if the rows of Λ are proportional to the ones of Γ (in which case the prices of risk are proportional to volatility) the variance of the pricing kernel is an affine functions of the underlying state variables and a completely affine model results. The generalization relative to Duffee (2002) comes from the fact that we also take the reciprocal of the diagonal elements of V_{t-1} that might approach zero for some values of x_{t-1}. Although this means the variance of the pricing kernel goes to infinity in those cases, bond prices are not affected as this variance does not appear in the bond pricing dynamics (see Equation 3), whereas in the covariance terms these elements cancel.

The coefficients A_τ and B_τ can be found recursively by substituting Equations 4, 5, and 7 in Equation 3:

$$A_\tau + B_\tau x_t = A_1 + B_1 x_t + A_{\tau-1} + B_{\tau-1}((I_n - \Phi)\theta + \Phi x_t) - \frac{1}{2} B_{\tau-1}\Sigma V_t \Sigma' B'_{\tau-1} + B_{\tau-1}\Sigma\Lambda \begin{bmatrix} 1 \\ x_t \end{bmatrix} \quad (8)$$

with A_0 and B_0 equal to zero and A_1 and B_1 being determined by the assumed process for the short rate. As this equation has to be valid for every value of the state variables the following recursive equations result:

$$A_\tau = A_1 + A_{\tau-1} + B_{\tau-1}\left((I_n - \Phi)\theta + \Sigma\Lambda_0\right) - \frac{1}{2}(B_{\tau-1}\Sigma)^2\Gamma_0 \quad (9)$$

$$B_\tau = B_1 + B_{\tau-1}\left(\Phi + \Sigma\Lambda_x\right) - \frac{1}{2}(B_{\tau-1}\Sigma)^2\Gamma_x \quad (10)$$

where Λ_0 and Γ_0 represent the first column of Λ and Γ respectively, and Λ_x and Γ_x denote the remaining columns.

2.2 Short rate dynamics

The first two factors of our term structure model are the excess short interest rate (denoted i_t^e) and excess expected short term inflation.[6] Both variables are taken in excess of their

[6] As the interaction between the two factors goes both ways in our model, and as the their volatilities are proportional, very similar results are obtained if the real short rate is modeled instead of the nominal one.

assumed equilibrium values. Let π_{t+1} denote the inflation rate from t to $t+1$, and let z_t be its ex-ante expectation at date t (minus its unconditional mean μ_π). In the first step the following system is estimated by means of the extended Kalman filter (Harvey 1989):[7]

$$i_t = \mu_i + i_t^e \tag{11a}$$

$$\pi_{t+1} = \mu_\pi + z_t + s_{t+1} + \omega_\pi v_{t-1}^{1/2} \xi_{\pi,t+1} \tag{11b}$$

$$i_t^e = \phi_{ii} i_{t-1}^e + \phi_{iz} z_{t-1} + \sigma_{ii} v_{t-1}^{1/2} \varepsilon_{i,t} \tag{11c}$$

$$z_t = \phi_{zi} i_{t-1}^e + \phi_{zz} z_{t-1} + v_{t-1}^{1/2} \left(\sigma_{zi}\varepsilon_{i,t} + \sigma_{zz}\varepsilon_{z,t}\right) \tag{11d}$$

$$s_{t+1} = -s_t - s_{t-1} - s_{t-2} + \omega_s v_{t-1}^{1/2} \xi_{s,t+1} \tag{11e}$$

$$v_{t-1} = Max[1 + \gamma_i i_{t-1}^e + \gamma_z z_{t-1}, 0] \tag{11f}$$

where s_t denotes the seasonal contribution to inflation at time t, and $\xi_{\pi,t}$ and $\xi_{s,t}$ are standard normally distributed error terms, independent from ε_t. As the excess short rate and expected inflation both have expectation zero, the normalization of the intercept in the variance equation is unrestrictive. In state space formulation, the model has two measurement equations (11a and 11b) and five state equations (11c, 11d, 11e and two definition equations for lagged seasonals). The short term interest rate equation is measured without error as it is perfectly observable. Consequently, the state variable i_t^e is perfectly observable as well.

The dynamic structure for the first two factors of our term structure model does not conform to the specification framework of Dai and Singleton (2000) as it is not in canonical form. The reason for choosing a different form is that our factors are (nearly) observable, and

[7]The extension is due to the variance equation, that includes state variables. Consequently, the true variance process is not known exactly, but has to be estimated as well. The resulting inconsistency does not seem to be very important, though in short samples the mean reversion parameters are often biased upwards, see Lund (1997), Duan and Simonato (1999), De Jong (2000), Bolder (2001), Chen and Scott (2003), Duffee and Stanton (2004), and De Rossi (2006).

their economic interpretation precludes the canonical form. On the one hand, the volatility of at least one of the factors should include a level effect as short term interest rates and expected inflation are clearly asymmetric processes. On the other hand, the causality between interest rates and inflation goes both ways. Ex-ante inflation is expected to be negatively affected by the short-term interest rate (ϕ_{zi} should be negative), since lower interest rates stimulate output, which in turn induces inflation. Exactly for this reason, high ex-ante inflation rate induces the central bank to raise interest rates, implying a positive value of ϕ_{iz}. In the canonical form, this combination of requirements is not allowed.

Given that our model does not fit into the standard classification of term structure models, the validity of our model is not obvious. The restrictions imposed in the canonical model guarantee the model satisfies the multivariate Feller condition. This condition implies that the deterministic process is such that given any arbitrary starting point at the border of the area with variance zero, the system moves directly into the direction with positive variances. In the continuous-time model this condition is important for basically two reasons. First, the condition is necessary to prove pathwise uniqueness for the stochastic differential equations that underly the bond pricing formulae. Second, the condition guarantees that volatilities always stay positive. In a discrete-time model, the first reason is not important as we are dealing with difference equations (Equations 9 and 10) instead of differential equations, for which uniqueness is not an issue. The second argument, though relevant, is also less important in our setting, as under the assumption of normality the probability of negative variances remains positive, irrespective the Feller condition. That is why we have to restrict v_t, which implies our bond pricing formulae are not exact, but only approximations.[8] The accuracy of

[8]Dai, Le, and Singleton (2005) derive exact bond pricing expressions in a discrete-time model by assuming a Poisson mixture of standard gamma distributions for the volatility factors. Their model assumes only latent factors, which can be arranged in canonical form. Analytical expressions are obtained by assuming some yields

this approximation naturally depends on the probability and size of negative outcomes.

One way to mitigate the approximation errors is to impose the Feller condition also in our discrete-time model. A method that works exactly in continuous time is likely to work reasonable in discrete time as well. In System 11 this would imply the condition: $\phi_{iz}\gamma_i^2 - \phi_{zi}\gamma_z^2 + (\phi_{zz} - \phi_{ii})\gamma_i\gamma_z = 0$.[9] We do not impose this restriction however as it seriously hampers the proper interaction between short rates and inflation. Moreover, the condition seems overly restrictive as all combinations of short rates and expected inflation that result in a zero variance are considered, irrespective the likelihood of these combinations given the interaction between the two variables.

In System 11, another mechanism is used to minimize the approximation error. By assigning the same volatility process to all state variables that also jointly determine this volatility (Equations 11c, 11d and 11f), it is precluded that volatilities that are already zero are pushed into negative direction due to stochastic shocks. With respect to the deterministic part of the volatility process, the maximum likelihood procedure will automatically preclude parameters resulting in frequent variance estimates of zero as the likelihood approaches minus infinity in this case unless both disturbance terms are exactly zero.[10] The fact that unexpected inflation shocks and shocks to the seasonal pattern share the same volatility process helps in this respect further, though this assumption is not necessary. Spreij, Veerman, and Vlaar (2007) have simulated bond prices for a two-factor completely affine term structure model based on System 11. They show that ignoring the zero variance restriction in the bond pricing formula hardly affects theoretical yields (the error is less than one basis point), even

are observed without measurement error.

[9]More generally, for every row i of Γ_x the inproduct of $\Gamma_{x(i)}$ and $\Phi(\Gamma_{x(i)})'_{\perp}$ should be zero.

[10]In order to preclude numerical problems, the minimum value for v_t in the estimation procedure is set at 10^{-9}.

though variances are restricted every now and then. However, these restrictions are hardly ever caused by a violation of the Feller condition, but are due to the use of the normal distribution.[11] Any term structure model estimated with the extended Kalman filter (the method recommended by Duffee and Stanton (2004)) uses the same approximation.

2.3 The prices of risk

In order to complete the term structure model, System 11 has to be augmented by a specification of the price of interest rate risk. We suppose this price is partly stochastic, reflecting time-variation in risk appetite/aversion. This stochastic risk aversion element in the price of risk (denoted a_t) is the third factor in our term structure model.[12] The shocks to this (unobservable) state variable are assumed to be homoscedastic. Therefore, no restrictions are required on the influence of expected inflation or the short rate on this risk factor. The reverse feedback is not allowed as this could lead to prolonged periods with zero variances.

In terms of the general model of Subsection 2.1 our model comprises the following characteristics:

$$\text{State variables:} \quad x_t = \begin{bmatrix} i_t^e & z_t & a_t \end{bmatrix}'$$

$$\text{with means:} \quad \theta = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}'$$

$$\text{Lagged dependence:} \quad \Phi = \begin{bmatrix} \phi_{ii} & \phi_{iz} & 0 \\ \phi_{zi} & \phi_{zz} & 0 \\ \phi_{ai} & \phi_{az} & \phi_{aa} \end{bmatrix}$$

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[11] In the model, the deterministic process pushes the volatility in the negative area (starting from zero) if expected inflation is less then 0.4% whereas the short rate is above 3.1%. Given the high macroeconomic costs of deflation and the central bank's close control of the short term rate, this combination is highly unlikely from an economic point of view. Indeed, the simulations confirmed the improbability of this outcome.

[12] As there is no feedback from this risk aversion factor to short rates or inflation, our specification implies a 2-factor model under the risk neutral measure Q, but a 3-factor model under the physical measure P. Previous studies including a factor that does not influence the short term dynamics are Brennan and Schwartz (1979), who directly include the long term yield, and Chacko (1997), who uses an independent latent factor. In the context of stock market valuation, Lettau and Wachter (2007) model an independent risk preference shock to explain the value premium.

$$\text{Variance factors:} \quad \Gamma = \begin{bmatrix} 1 & \gamma_i & \gamma_z & 0 \\ 1 & \gamma_i & \gamma_z & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Prices of risk:} \quad \Lambda = \text{unrestricted } 3 \times 4 \text{ matrix}$$

$$\text{State error loadings:} \quad \Sigma = \begin{bmatrix} \sigma_{ii} & 0 & 0 \\ \sigma_{zi} & \sigma_{zz} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The model contains maximal flexibility in the sense that no unnecessary restrictions are imposed. The ones in Γ and Σ are simply normalization and the zeros in Σ are either necessary to avoid negative variances or not restrictive.[13] The flexible specification might compensate for the fact that a two-step procedure is used, and the fact that only one of our state variables is truly unobservable (and thereby fully flexible).

The measurement errors in bond prices are assumed to be uncorrelated and with identical variance for all maturities. It is assumed to be heteroscedastic with the same volatility specification as for the short term and inflation dynamics. Higher likelihood values are obtained if correlation between the errors is taken into account or if variances are allowed to differ among maturities. These higher likelihood values are not accompanied by lower average measurement errors however. Especially for the longer maturities the fit is worse. As less observations are available, the contribution of these maturities on the likelihood value is smaller. In a free estimate this will result in more emphasis on a good fit for relatively short maturities at the cost of the higher ones. A higher variance estimate would decrease the penalty for measurement errors on these high maturities.

The second step in the estimation procedure involves sixteen parameters: the twelve prices of risk (Λ), three AR parameters (Φ), and the measurement error loading (denoted ω_l).

[13]Instead of restricting σ_{ai} and σ_{az} to zero, it is also possible to restrict ϕ_{ai} and ϕ_{az}. Although some of the parameter values will change, the value of the likelihood and the model predictions remain unaffected.

3 Estimation results

Table 1 shows the estimation results for the short time dynamics (see the appendix for the data sources). The model is estimated over the period 1959:IV - 2006:II. As inflation and interest rate data contain near unit roots, it is important to take as much data into account as possible, in order to reduce the well-known uncertainty in estimates of the speed of mean reversion of highly persistent processes. Moreover, in order to model the interaction between interest rates and inflation, one should preferably incorporate several inflationary periods. As the parameters of the model are used in the second step to fit the term structure over a shorter sample, the means μ_i and μ_π are not estimated but imposed to be equal to the mean value over the latter sample.[14] The average inflation over this period was just over 3% which is substantially higher than the current inflation target (below, but close to 2%) of the European System of Central Banks (ESCB). The average short rate was about 5.7%.

Table 1: Estimation results short term dynamics

μ_i	5.687	(-)
μ_π	3.005	(-)
ϕ_{ii}	0.927	(28.2)
ϕ_{iz}	0.112	(1.5)
ϕ_{zi}	-0.005	(0.1)
ϕ_{zz}	0.941	(15.7)
γ_i	0.176	(6.1)
γ_z	0.213	(5.0)
σ_{ii}	0.791	(13.2)
σ_{zi}	0.166	(1.5)
σ_{zz}	0.513	(2.8)
ω_π	1.819	(11.4)
ω_s	0.192	(4.2)

Estimation sample 1959:IV - 2006:II
Absolute heteroscedasticity-consistent t-values in parenthesis.

[14] Including extra factors representing the time-varying long-run means of the inflation and/or the short rate process turned out to be unsuccessful. Dewachter, Lyrio, and Maes (2004) found a time-varying long-run expectation for inflation to be very important in modeling the German term structure. However, as their model does not include stochastic risk aversion, this 'long-run tendency' variable merely captures the time-varying price of interest rate risk. The time-series pattern for this variable seems unrealistic, as, for instance, expected long-run inflation was negative between 1998 and 2000 according to their results. For our German/euro area data the estimated variance for this factor was zero when included.

With respect to the dynamics of the system, the results are encouraging. The (complex) eigenvalues of the Φ-matrix are equal to 0.934, which implies a halftime of shocks of just over $2\frac{1}{2}$ years. The dynamics are as expected: higher inflation induces higher short rates, whereas higher short rates depress inflation somewhat. The latter effect is far from significant though. This might have to do with the short time lag in our model (one quarter), whereas the impact of monetary policy on inflation probably takes longer. The fact that ϕ_{zi} can be restricted to zero does not mean that the Feller condition can be imposed without much cost, as the variance is not only very significantly affected by inflation, but also by the short term interest rate.[15] These pronounced level effects produce the well known asymmetry in inflation and interest rates, with extremely high rates being much more likely than very low ones.

With respect to the error loadings, the variance of the nominal short rate is more than twice as large as the one on expected inflation. Although the correlation between the two is positive (0.31), these results indicate that real short rates are slightly more volatile than nominal ones. Whether this also holds for long term rates depends on the term premia. Recent experience with French index-linked and nominal bonds suggests the volatility of real bonds is somewhat (about 10%) lower than the one of nominal bonds. Using only swap market data, which are available since the end of 1987, a negative relationship between the term premium and expected inflation was found however. This would imply that 'expected inflation adjusted' nominal yields are more volatile than pure nominal ones. Although this result is not necessarily at odds with the French experience, due to possible differences in risk

[15]The Feller condition is indeed rejected as the system will remain in the zero-variance region for a while if expected inflation is below 0.4% but short rates are still above 3.1%. Such a constellation is hard to conceive from a monetary policy perspective however. Although imposing the Feller condition is acceptable in terms of log-likelihood value (it decreases by just 0.7 points), the interaction between interest rates and inflation is seriously affected, as ϕ_{zi} becomes slightly positive (0.015), and ϕ_{iz} reduces to only 0.054. Both the positive impact of interest rates on inflation and the reduced impact of inflation on interest rates are unappealing from an economic point of view. Consequently, these parameters are likely to lead to less plausible simulation properties.

premia for real and nominal bonds, lower risk premia in a high inflation environment seem very unlikely. The reason for this empirical finding might be that the main high inflation period during the last twenty years was caused by the German unification. This episode was atypical in the sense that only the German economy was booming causing relatively high inflation, whereas most of the rest of the world was in recession. As long term bonds are more affected by foreign developments than short term ones, this asymmetry caused German long term yields to decline even though inflation was still rising. In order to better capture the relationship between inflation and term premia, we decided to include data from the seventies as well. Unfortunately, swap market data are not available, so bond yields are used for the period before 1987:IV. Table 2 shows the results. Even though the parameters for inflation and short rates are taken from the first step, these series are still included when estimating the second step as the optimal predictor for expected inflation will be different when the long rates are taken into account.

Lagged expected inflation has a negative impact on the risk aversion, though not significantly so, whereas the short interest rate has a significant positive impact. The negative

Table 2: Estimation results prices of risk

ϕ_{ai}	0.158	(2.9)		
ϕ_{az}	-0.209	(1.2)		
ϕ_{aa}	0.951	(41.7)		
Λ	-2.54	-6.18	15.89	0.61
	(1.0)	(3.2)	(5.0)	(1.9)
	5.10	3.02	-8.34	0.24
	(2.1)	(1.6)	(2.3)	(1.2)
	0.79	-1.67	1.18	0.75
	(0.9)	(1.2)	(0.4)	(1.6)
ω_l	0.152	(31.1)		

Data included: Inflation over next quarter 1959:IV - 2006:II
Three-month money market rate 1959:IV - 2006:II
1, 2, 4, 7, 10 year zero-coupon rates 1972:III - 2006:II
15-year zero-coupon rate 1986:II - 2006:II
30-year zero-coupon rate 1996:I - 2006:II
60-year zero-coupon rate 2001:III - 2006:II

Absolute two-step-consistent t-values in parenthesis.

value for ϕ_{az} does not imply however that inflation and bond risk premia are inversely related as higher inflation strongly increases the impact of short term interest rate risk on the term premium (coefficient $\Lambda_{1,3}$). The dynamic properties of the model seem to be good, as the third eigenvalue of the system is only 0.951.[16] Consequently, the model is likely to have desirable simulation properties. If all parameter would have been estimated simultaneously, the first two eigenvalues would increase to 0.970, thereby more than doubling the halftime of inflation and short term interest rate shocks. Therefore, the one-step results were discarded as the simulation properties would be seriously hampered.[17]

Considering the price of risk parameters in Λ, only five out of twelve coefficients are significantly different from zero at the 10% level, which seems to be rather common for term structure models.[18] This is probably due to the fact that the risk parameters always appear in conjunction with the autoregressive parameters (see Equations 9 and 10). If the parameter uncertainty of the first step parameters is ignored in the second step, six elements of Λ are even significant at the 1% level. Especially shocks to expected inflation and short interest rates influence bond prices significantly. The significant negative coefficients related to short rate uncertainty (first row of Λ) and inflation risk (second row), strongly suggest that a completely affine specification, in which the prices of risk are proportional to volatility, has

[16]In this two-step procedure, the other two eigenvalues are not affected as there is no feedback from the risk aversion factor to short rates or inflation.

[17]Cassola and Barros Luís (2003) found a highest eigenvalue of even 0.991 in their Gaussian two-factor model estimated on monthly German data over the period 1972–1998. Over the smaller sample 1986–1998 (so excluding the high inflation period) a much lower eigenvalue was found (0.968). Fendel (2005) also finds an eigenvalue (related to inflation) of 0.99 for German monthly data from 1979 to 1998, in his three-factor Gaussian model with explicit links to inflation and the output gap. In a complementary AR(1) regression on inflation he finds a coefficient of only 0.97 however, implying a half-time of inflation shocks of even less than $2\frac{1}{2}$ years.

[18]The two-step covariance matrix is based on a first order Taylor approximation of the gradient functions around the true parameters, see Ang, Piazzesi, and Wei (2006). As the hessian of the second step parameters was not positive definite due to numerical problems, it is replaced by the crossproduct of gradients. Consequently, the t-values for the second step are not robust.

to be rejected. As these factors involve a level effect, this proportionality restriction is also imposed for these rows by the essentially affine specification of Duffee (2002). Consequently, that specification is also strongly rejected by the data. The loadings for the bond price measurement errors on the heteroscedasticity factor (which has expected value of one) is just 15 basis points. Given the wide range of maturities considered, and the long sample period, this results seems remarkably good.

In order to find out more on the empirical fit of the model, Table 3 shows the root mean squared pricing errors for three different subperiods and 17 maturities. The errors for time t are hereby calculated as realizations minus predictions given the expected state variables (using the Kalman filter on the inflation data, short rates and long rates of the eights maturities considered) conditional on time t information. The results confirm relatively low pricing errors. For most maturities, the root mean squared error is well below ten basis points. The assumption of equal measurement error variances for all maturities seems reasonable. Only for the one-year rate (in the early years) and 45 and 60-year rates the errors are somewhat larger. These good results are all the more surprising as only one of our state variables (the stochastic risk aversion element) is fully flexible. The short rate is completely

Table 3: Root mean squared pricing errors long term yields

Maturity (years)	1972:III-2006:II	1996:I-2006:II	2001:III-2006:II
1	20.8	7.3	6.1
2	10.1	4.8	3.9
3	11.6	3.2	2.9
4	11.3	4.8	3.4
5	9.9	6.4	4.2
6	8.3	7.0	4.9
7	7.2	7.0	5.5
8	6.8	6.7	6.0
9	7.4	6.4	6.5
10	9.1	6.2	7.1
12		6.2	7.9
15		7.2	8.5
20		7.7	7.5
25		6.7	4.6
30		8.7	6.5
45			18.0
60			13.8

Results in basis points given predicted state variables.

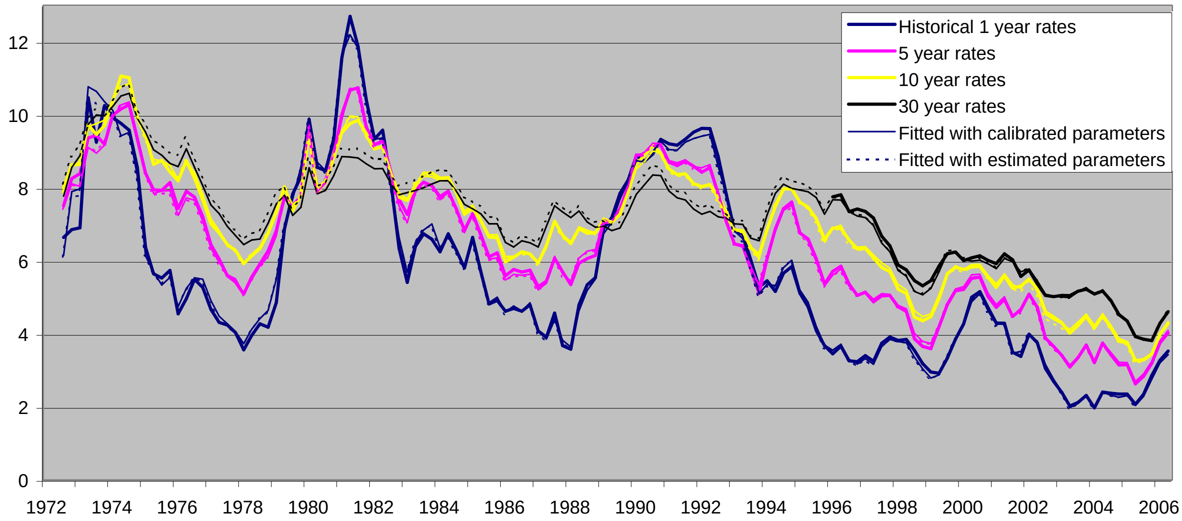
observable and expected inflation is closely linked to actual inflation.

Figure 1 gives some further insight in the flexibility of the model. It shows the time series pattern of yields with maturities of one, five, ten and thirty years for the period 1972:III - 2006:II (data on the 30 year yield starts in 1996). The time series data (thick solid lines) are accompanied by two predictions of our model: one with the estimated parameter values (dotted lines) and one with partly calibrated parameters (thin solid lines) to better fit the current situation.[19] As price stability is the overarching goal of the ESCB, an expected inflation rate well above the target seems unrealistic. Therefore, we reduce μ_π to 2%. Lower expected inflation is likely to be accompanied by on average lower short term interest rates, which we calibrate at 4%. As on average lower inflation and short rates should lead to lower volatility as well, the intercept in the variance equation is lowered such that a zero variance is reached at the same short rate and expected inflation combinations. The change in average volatility is moreover assumed to lead to an equivalent reduction in the average price of risk, which is incorporated by a proportional decrease in the first two elements of Γ_0. Finally, the simulations showed that inflation and short rates were still too volatile according to current standards. Therefore, the first two rows of Σ are multiplied by 0.7 and 0.95 respectively, such as to bring the average volatility of the factors in accordance with the empirical values over the last decade. These values are calibrated such that the equilibrium yield curve fulfils the maximum return assumption of the Dutch pension supervisor (maximum expected return of 4.5% on fixed income assets).[20] Both specifications mimic the yield dynamics remarkably well. For instance, both the inverted yield curve at the beginning of the eighties and the

[19]For the calibrated model, new optimal state variables given the new parameters are calculated.

[20]For maturities between 8.25 and 66 years, the equilibrium yield is slightly higher than 4.5%. As the duration of fixed income assets of the average Dutch pension fund is only six years and as the excess return on high maturities is only marginal, the relevance of exceeding the maximum is limited.

Figure 1: Fit of term structure model given predicted state variables

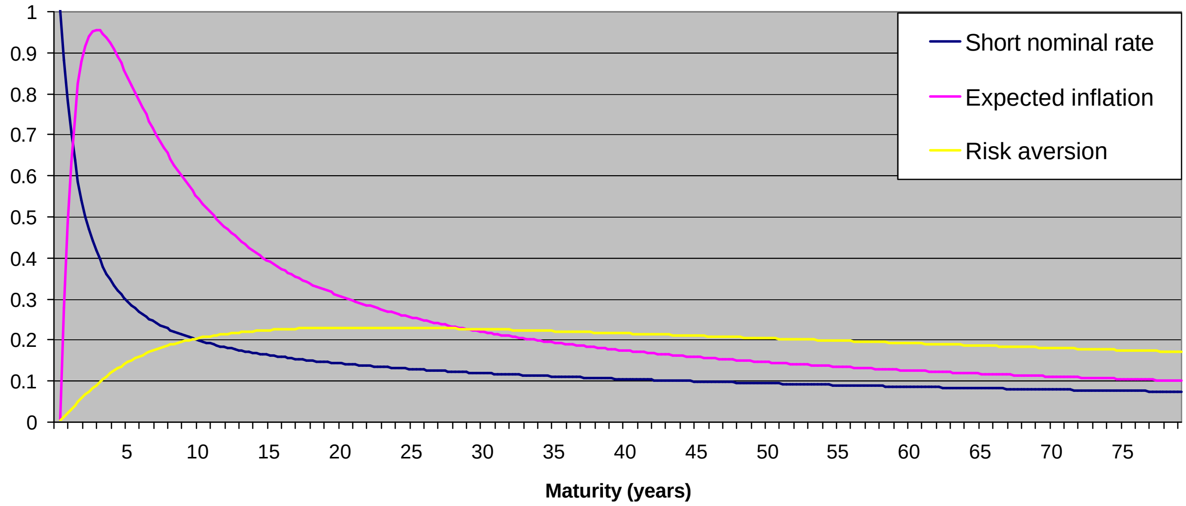


nineties and the sudden rise in long term yields in 1994 are captured without much problems. The relatively poor fit for the one-year rate is primarily located in the first two years of the sample. The difference between the predictions of the estimated and the partly calibrated model is fairly small. The lower equilibrium values for short rates and inflation is compensated by higher predictions for the risk aversion factor.

In order to get an idea what is driving the long term yields, Figure 2 shows the implied factor loadings of the calibrated model. All three factors positively affect long term yields. For the very short rates, the current monetary policy stance, reflected in the nominal short rate, is most important. Above one year maturity expected inflation becomes dominant, whereas for maturities of ten years and more the risk aversion factor is most relevant.[21] These results are intuitively appealing. As inflation is a stationary process, one should not expect a large impact of this factor on very long term rates. Long term bonds are more prone to interest rate risk and therefore are more affected by changes in risk appetite. The declining impact

[21]The volatility of the risk aversion factor is about three times higher than that of the other two factors.

Figure 2: Factor loadings for nominal term structure given calibrated parameters



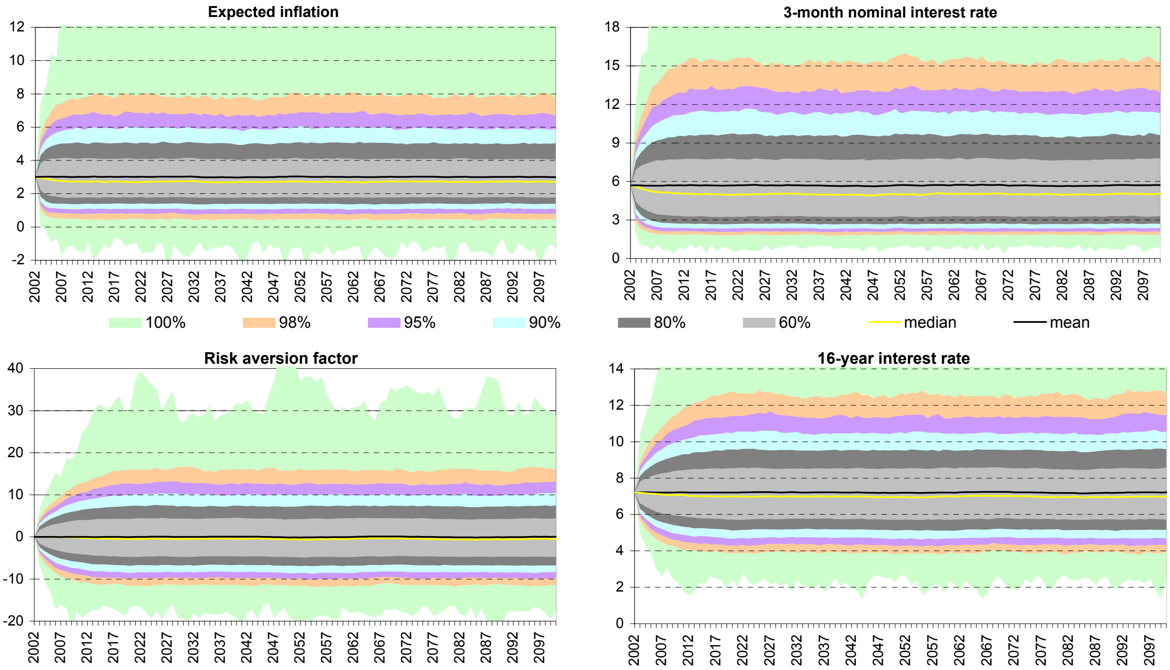
of risk aversion on very long term bonds might be explained by the demand for long term bonds by pension funds and life insurance companies. As their obligations stretch far into the future, increasing the duration of their assets decreases their risk instead of increasing it.

4 Simulation results

Although the dynamic properties of the model seem promising, one can only be sure by trying. In order to find out the unconditional properties of the model, we simulated 10000 paths for a period of 98 years (392 quarters) with the model. Figure 3 shows the results regarding the three factors and the nominal 16-year interest rate for the estimated model. We assumed all state variables were zero in the starting year (2002). The 16-year rate is chosen as the average duration of pension liabilities in the Netherlands is about 16 years. Consequently, the evolution of this yield is crucial for pension funds who choose to use a duration approach to value their liabilities.

The short term dynamics looks quite reasonable. Both expected inflation and short term interest rates are clearly asymmetric. Short rates are never negative but can become as high

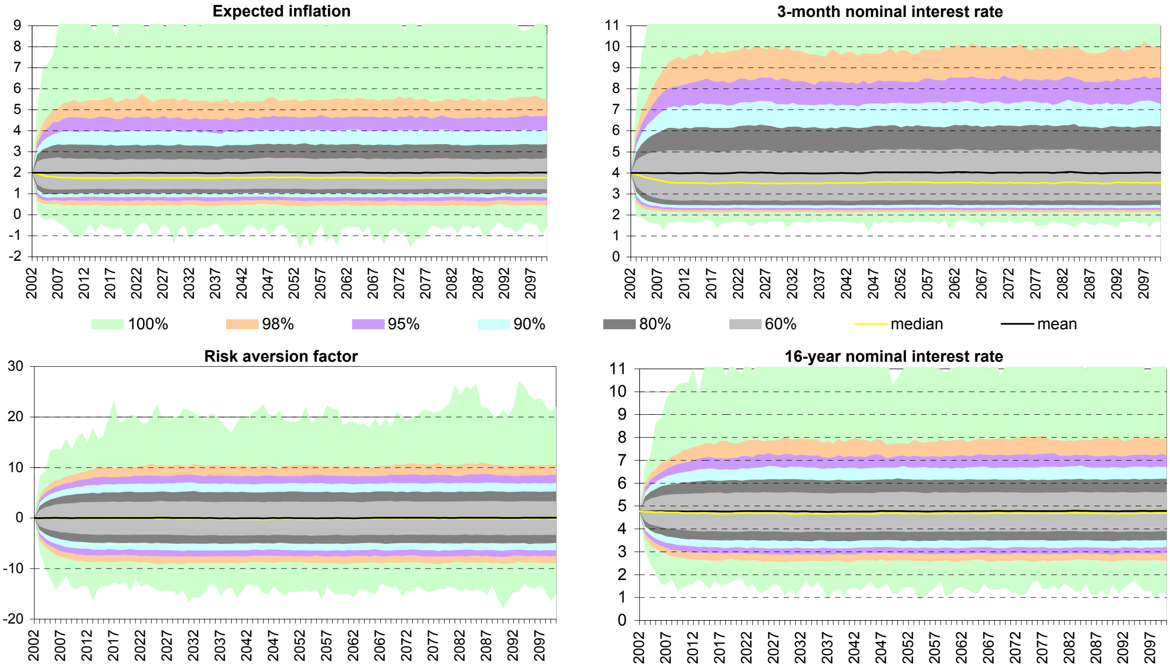
Figure 3: Simulated state variables and 16 year rate given estimated parameters



as 25% in extreme scenarios, though they are higher than 15% in about 1% of all scenarios. The risk aversion factor inherits this asymmetry to some extent. All three factors show stationary behavior, the pattern after 10 years looks very similar to the one after 98 years. The 16-year rate also shows asymmetric behavior, with a mean value of just over 7% and 1 and 99 percentiles of about 4% and 12.5% respectively. As the 16-year rate in 2005 was even slightly lower than 4%, these historical results are clearly not representative any more. Figure 4 shows the results for the calibrated model with adjusted means and volatilities. Both inflation and interest rates seem to fluctuate within a reasonable range. The probability of a 16-year rate below 4% is higher than 20% and even values below 3% occur with an almost 3% probability.

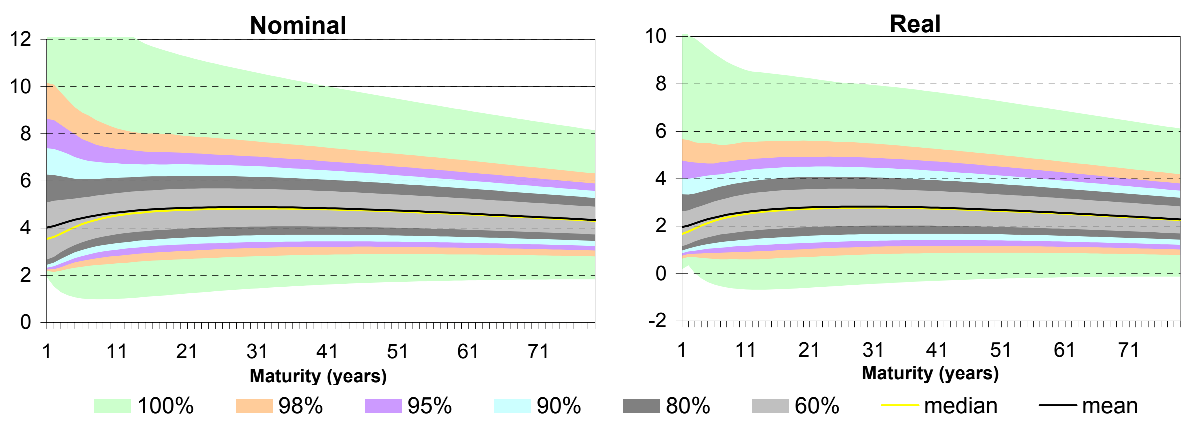
Figure 5 shows some percentiles of the distribution of the entire nominal and real term structure with maturities from 1 to 79 years (the longest maturity used in the pension asset and liability model Palmnet (Van Rooij, Siegmann, and Vlaar 2004)) in the last quarter of our

Figure 4: Simulated state variables and 16 year rate given calibrated parameters



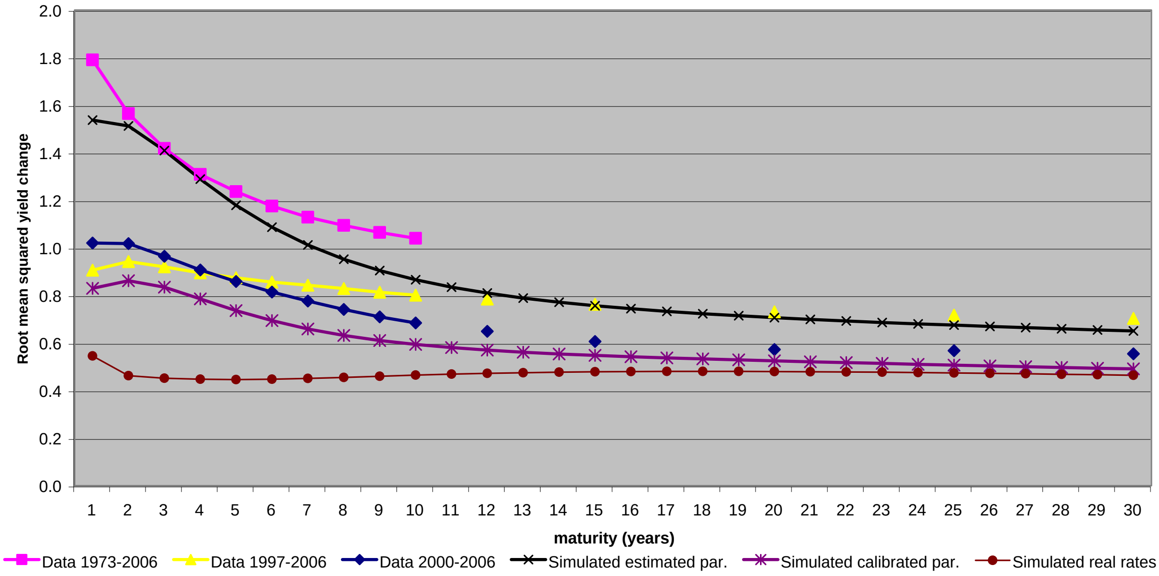
simulation. The real term structure is hereby calculated as the nominal one minus expected inflation according to the same model, thereby abstracting from inflation risk premia. As to be expected, real interest rates show less dispersion than nominal ones. The unconditional term structure is upward sloping for maturities up to 20 to 30 years after which it is downward sloping. This pattern is confirmed by the (scarce) historical data. All in all, the range of possible interest rates seems to be reasonable, even out of sample. This is however not the

Figure 5: Nominal and real term structure given calibrated parameters



only important characteristic. Volatility is important as well. Figure 6 gives a good indication of this feature. It shows the root mean squared annual change in predicted as well as realized interest rates. For the predictions this value was calculated by taking the interest rate change over the last year of the simulations (last quarter of 2100 minus last quarter of 2099).

Figure 6: Term structure of annual yield volatility



The volatility patterns of our predictions are similar to the historical ones. As long maturities are not available for a long time, it is difficult to judge the results, but in general our calibrated model captures historical volatilities quite well. Whether the pattern over the last six and a half years, over the last ten years or the one over the last thirty four years will turn out to be the most representative for the future remains to be seen. In any case, our model is flexible enough to accommodate all desired assumptions in this respect.

Apart from the volatility of nominal rates, the picture also shows the volatility of real interest rates according to our calibrated model. For relatively short maturities, the volatility of real rates is substantially lower than for nominal ones. As inflation is a stationary process however, the difference becomes smaller for very long maturities. Since the duration of real pension obligations (including indexation) of pension funds is probably more than twenty years, these result indicate that real pension obligations are only slightly less volatile than nominal ones.

5 Conclusions

In this paper, a new affine three factor term structure model is introduced. The model is more flexible than allowed according to the Dai and Singleton (2000) classification, as it permits interaction between short rates and inflation in both first and second moments. Moreover, the price of risk specification generalizes the essentially affine formulation by Duffee (2002) as proportionality between volatility and the price of risk is also not imposed for factors that affect volatility. By allowing for un unbalanced panel, long-term maturities are included, without relying on a short sample for all variables. The determining state variables in the model are expected inflation, nominal short rates and a stochastic risk aversion factor. The model is estimated on German/eurozone data for maturities from one to sixty years. Inflation and short term interest rate data start in the last quarter of 1959, most long term yields in the third quarter of 1972, whereas data on the sixty year maturities only start in the third quarter of 2001. In order to preserve good simulation properties, the model is estimated in a two-step procedure. In the first step, the short-term dynamics for inflation and short-term interest rates is estimated. In the second step, the prices of risk are determined, conditional on the parameters of the first step.

The results are very good. Within sample, the root mean squared pricing error is less than ten basis points for most maturities. Moreover, the state variables are clearly stationary, which makes it possible to use the model for long run simulations. Indeed, the possible range of simulated outcomes looks reasonable both for nominal and real rates, and the model reproduces first and second moments of (changes in) historical term structures correctly.

All in all, our model seems to encompass all desired properties for a successful asset and liability analysis for pension funds or life insurers. The model is sufficiently flexible to fit all forms of the yield curve that were observed in the past. It produces reasonable simulation results for maturities from one to 79 years, that comply with the return assumptions of the Dutch pension supervisor, without approaching the zero lower bound, and with realistic volatilities of yield changes for all maturities. Moreover, the modeled interaction between interest rates and inflation helps in analyzing the cost of indexation policies.

Appendix: Data sources

Inflation is based on the consumer price index (CPI) over the last month of the quarter as published in the International Financial Statistics of the IMF. Over the period 1959:IV – 1990:IV data for Western-Germany are used, over the period 1991:I – 1998:IV data for unified Germany, and for the period 1999:I – 2006:III the Harmonized Index of Consumer Prices (HICP) for the euro area.

Our short-term interest rate is the three-month money market rate. The data are taken from the Bundesbank website (www.bundesbank.de). For the period 1959:IV - 1969:IV no daily data were available, so monthly averages (last month of the quarter) are used. For the period 1970:I – 1990:II, end of quarter money market rates as reported by Frankfurt banks are taken, whereas for the period 1990:III – 2006:II 3-month Frankfurt Inter Bank Offered Rates (FIBOR) are included.

For long term yields, we used end of month zero-coupon rates based on swap market data for maturities from one to sixty years as published by De Nederlandsche Bank (www.dnb.nl). These data are available from the third quarter of 2001 on. For the 2 to 10, 12, 15, 20, 25 and 30 year maturities end-of-period zero-coupon swap market data from J.P.Morgan were used as well. For the 2 to 10 year maturities all available data were used (starting 1987:IV), whereas our starting date for the 12 and 15 year rates is 1993:III, and for the 20, 25 and 30 year rates 1996:I. For earlier dates, the data for these longer maturities seemed unreliable (very flat and sometimes erratic), probably because of lack of liquidity in the market. For the one-year rates we used money market rates as well. For the period 1981:II – 1990:II end of quarter money market rates as reported by Frankfurt banks are taken, whereas for 1990:III – 2001:II, the twelve month FIBOR is used (both from the Bundesbank website). For the period 1972:III-1987:III zero coupon yields with maturities of one to fifteen years (from the Bundesbank website) based on Government bonds were used as well (15 year rates start in June 1986). No adjustments were made to correct for possible differences in credit risk of swaps on the one hand and German bonds on the other. The biggest difference in yield between the two term structures (for the 2-year yield) in 1987:IV was only 12 basis points. For the period 1987:IV-1993:II, bond yield data are also used for the 15 year maturity. For this sample, the 15 year rate was computed as the 10 year swap rate plus the difference between the 15 year bond rate and the 10 year bond rate.

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