

# Optimal Consumption and Insurance: A Continuous-Time Markov Chain Approach

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## Abstract

Personal financial decision making plays an important role in modern finance. Decision problems about consumption and insurance are modelled in a continuous-time multi-state Markovian framework. The optimal solution is derived and studied. The model, the problem, and its solution are exemplified by two special cases: In one model the individual takes optimal positions against the risk of dying; in another model the individual takes optimal positions against the risk of losing income as a consequence of disability or unemployment.

*Key words:* Personal finance, multi-state model, stochastic control, financial decision making, mortality-disability-unemployment risk

*JEL-Classification:* G11, G22, J65

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# 1 Introduction

Optimal personal financial decision making plays an important role in modern financial mathematics and economics. It covers broad disciplines like micro-economic equilibrium theory and portfolio optimization. Merton (1969,1971) initiated the area of continuous-time consumption-investment problems which has become a paradigm for a large set of developments and generalizations. These are often driven by modifications of financial market models or individual preferences. In this article we disregard the investment decision of the individual and concentrate on the consumption decision along with the introduction of insurance decisions of various general types.

Merton (1969,1971) formulated the continuous-time consumption-investment problem. Important general results concern the optimal asset allocation including the so-called mutual fund theorem. Explicit results on asset allocation and consumption are obtained for log-normal prices and so-called HARA utility. Richard (1975) generalized both the general and explicit results to the case where the individual has an uncertain life-time, income while being alive, and, apart from asset allocation and consumption, decides continuously on a life insurance coverage. The uncertain life-time is modelled, in consistency with continuous-time life insurance mathematics, by an age-dependent mortality intensity. Actually, the idea of involving the life insurance decision in the personal decision making of an individual with an uncertain life-time dates further back to Yaari (1965) who studied the issue in a discrete time setting.

The same year in which Merton (1969) first published his ideas, Hoem (1969) demonstrated that the continuous-time finite-state Markov chain is an inevitable tool in the construction of general life insurance products and the modelling of general life insurance risk (that year queuing up for taking small steps that proved to be giant leaps.) The finite-state Markov chain has been studied in the context of life insurance and vice versa since then by Hoem (1988), Norberg (1991) and many others. It provides a model for various kinds of risk connected to an individual's life. One important example to which we return in the core text, is the risk of loss of income in connection with disability or unemployment. Other examples are the risk of increasing mortality intensity in connection with stochastic deterioration of health and the risk connected with ones dependants following from e.g. marriage and parenthood.

Financial decision making in connection with life and pension insurance has in the literature primarily been a matter for the life insurance company. Both asset allocation and adjustment of non-defined payments have been studied as decision processes subject to optimization. These decision problems have often been studied in the context of a quadratic loss function where deviations of wealth and payments from certain targets are punished, see Cairns (2000) for a state of the art exposition of the results. In that area, the life insurance risk was until recently approximated by normal distributions with reference to the fact that the decisions concerned large portfolios of insurance contracts. However, the life insurance market shows a trend towards a larger extent of individualization of these decisions, e.g. in unit-linked insurance contracts where decisions on the asset allocation and composition of payments are partly or fully individualized. Steffensen (2003,2004) models the life insurance risk by a continuous-time Markov chain and solves the decision problem of the insurance company in the cases of quadratic loss and power utility preferences, respectively.

Merton (1969,1971) solved the consumption-investment problem by stochastic control theory. Also the decision problems of the insurance company presented by Cairns (2000) are approached by stochastic control theory. However, in all these problems all risks are modelled as normal. Also in many textbooks on stochastic control theory, applications to primarily normal risk are studied, see e.g. Fleming and Rishel (1975). This is in contrast to Richard (1975) who works in

a survival model through introduction of a mortality intensity. It is also in contrast to Steffensen (2003,2004) who works in a finite-state Markov chain model. Davis' (1993) textbook demonstrates the application of stochastic control theory to non-normal risks.

As mentioned at the end of the very first paragraph we disregard the investment decision in this article. The reason is that it does not add much insight to this work given the results of Merton (1969,1971) and Richard (1975). Our focus is on the consumption and insurance decisions. Richard studies the consumption and life insurance decisions in a survival model where the saving takes place on a private account. We generalize this in two directions. Firstly we model the life insurance risk in a multi-state framework such that e.g. insurance decisions with respect to disability and unemployment can be studied and an optimal position can be taken. This reflects the variety and complexity of real life financial decisions and insurance markets. This is the primary contribution of this article. Secondly we allow for saving in the insurance company. Richard (1975) concludes his article by noting that 'rich, old' people optimally should be sellers of insurance while consuming their wealth. In his article this life insurance is sold although the policy holder has not saved anything in the insurance company. In practice this is not possible since the maximum life insurance sum the policy holder can sell is exactly the savings in the company. Taking this sum to be equal to the savings in the company is exactly what happens when the policy holder holds a life annuity. Therefore, in order to find the optimal position one has to introduce the possibility of saving in a life insurance company. We model this by letting wealth consist of both the balance of a private account and the balance of an account in the insurance company. This is the secondary contribution of this article.

In Section 2 we present the finite-state Markov model, the insurance account and its relation to classical life insurance mathematics. In Section 3 we present the rest of the model, the income process, and the consumption process and how all payment processes affect the private account. In Section 4 we formalize an optimization problem and present its solution. The derivation appears in the Appendix. We study and interpret aspects of the optimal decisions in Section 5. Sections 6 and 7 contain special studies of two important cases. The survival model is studied in Section 6, obtaining as a special case the same results as Richard (1975) obtained. The disability/unemployment model is studied in Section 7 providing further results and insight.

## 2 Life Insurance Mathematics

In this section we present the finite-state Markov model and the insurance payment process. We also introduce an insurance account which turns out to coincide with a traditionally defined reserve in a special case of deterministic payment process coefficients.

We take as given a probability space  $(\Omega, \mathcal{F}, P)$ . On the probability space is defined a process  $Z = (Z(t))_{0 \leq t \leq n}$  taking values in a finite set  $\mathcal{J} = \{0, \dots, J\}$  of possible states and starting, by convention, in state 0 at time 0. We define the  $J + 1$ -dimensional counting process  $N = (N^k)_{k \in \mathcal{J}}$  by

$$N^k(t) = \# \{s \in (0, t], Z(s-) \neq k, Z(s) = k\},$$

counting the number of jumps into state  $k$  until time  $t$ . Assume that there exist deterministic functions  $\mu^{jk}(t)$ ,  $j, k \in \mathcal{J}$ , such that  $N^k$  admits the stochastic intensity process  $(\mu^{Z(t-)k}(t))_{0 \leq t \leq n}$  for  $k \in \mathcal{J}$ , i.e.

$$M^k(t) = N^k(t) - \int_0^t \mu^{Z(s)k}(s) ds$$

constitutes a martingale for  $k \in \mathcal{J}$ . Then  $Z$  is a Markov process. For each state we introduce the

indicator process indicating sojourn,  $I^j(t) = 1[Z(t) = j]$ , and the functions  $\mu^{jk}(t)$ ,  $j, k \in J$  and the intensity process  $\mu^{Z(t)k}(t)$  are connected by the relation  $\mu^{Z(t)k}(t) = \sum_{j:j \neq k} I^j(t) \mu^{jk}(t)$ .

The reader should think of  $Z$  as the state of life of an individual in a certain sense of personal financial decision making which will be described in this section. In order to fix ideas we already now offer the reader some examples of  $Z$  to have in mind. The simplest model is the so-called survival model with only two states, alive and dead. There, the individual jumps from the state of being alive to the absorbing state of being dead with an age-dependent intensity. This model is illustrated in Figure 1. To solve problems in a survival model the setup of the finite state Markov chain is like cracking a nut with a sledge hammer, though. Indeed, we have a much wider set of applications in mind.

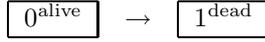


Figure 1: Survival model

Consider the three state model illustrated in Figure 2. The absorbing state 2 is the state of being dead. The individual can jump between two states of being alive, 0 and 1, with certain age-dependent intensities, possibly 0. From each of these states the individual can jump into the state of being dead with an age- and state-dependent intensity. Two examples of states 0 and 1 which are both very apt to think of throughout this paper are the following: If 0 is the state of activity and 1 is the state of disability, we speak of a disability model; if 0 is the state of employment and 1 is the state of unemployment, we speak of an unemployment model.

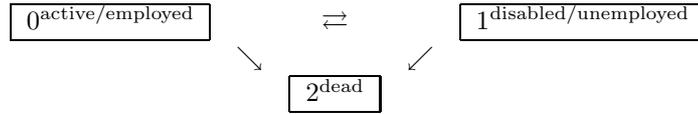


Figure 2: Disability/Unemployment model

We introduce now an insurance payment process  $B = (B(t))_{0 \leq t \leq n}$  representing the accumulated insurance net payments from the insurance company to the policy holder. The insurance payment process is assumed to follow the dynamics

$$dB(t) = \sum_j I^j(t) dB^j(t) + \sum_{k:k \neq Z(t-)} b^{Z(t-)k}(t) dN^k(t),$$

where  $B^j(t)$  is a sufficiently regular adapted process specifying accumulated payments during sojourns in state  $j$  and  $b^{jk}(t)$  is a sufficiently regular predictable process specifying payments due upon transitions from state  $j$  to state  $k$ . We assume that each  $B^j$  decomposes into an absolutely continuous part and a discrete part, i.e.

$$dB^j(t) = b^j(t) dt + \Delta B^j(t),$$

where  $\Delta B^j(t) = B^j(t) - B^j(t-)$ , when different from 0, is a jump representing a lump sum payable at time  $t$  if the policy holder is then in state  $j$ . Positive elements of  $B$  are called benefits whereas negative elements are called premiums.

In the survival model one can think of a life insurance sum paid out upon death before termination. Alternatively, one can think of a so-called deferred temporary life annuity benefit starting upon retirement at time  $m$  and running for  $n - m$  time units until termination time  $n$  or death

whatever occurs first. Such benefits or streams of benefits can e.g. be paid for by a premium rate paid continuously until death or retirement whatever occurs first.

In the disability/unemployment model there are also several possible constructions of insurance payment processes: Insurance sums may be paid out upon occurrence of disability/unemployment or rates of benefits may be paid out as long as the individual is disabled/unemployed, i.e. so-called disability/unemployment annuities. These insurances against disability/unemployment are typically paid for by a premium rate paid continuously as long as the individual is active/employed. All these insurances can now be combined with the different insurances against death and/or survival mentioned in the previous paragraph.

Assume that there exists a constant interest rate  $r$ . In general the reserve is defined as the conditional expected present value of future payments,

$$Y(t) = E^* \left[ \int_t^n e^{-r(s-t)} dB(s) \middle| \mathcal{F}(t) \right]. \quad (1)$$

Here and in the rest of the article we define  $\int_t^n = \int_{(t,n]}$ . The coefficients in the payment process are settled in accordance with the so-called equivalence principle stating that  $Y(0-) = 0$  or, equivalently,  $Y(0) = \Delta B^0(0)$ , see e.g. Norberg (1991). The asterisk decoration of  $E^*$  means that the expectation is taken with respect to a valuation measure which we denote by  $P^*$  and which may be different from the objective measure. We assume that  $Z$  is Markov also under this measure such that we can parametrize this measure by the transition intensities  $\mu^{jk^*}(t)$ ,  $j \neq k$ ,  $j, k \in \mathcal{J}$ .

We introduce a process  $\tilde{Y}$  which is described by the following forward stochastic differential equation,

$$d\tilde{Y}(t) = r\tilde{Y}(t)dt + \sum_{k:k \neq Z(t-)} y^{Z(t-)k}(t) dN^k(t), \quad (2)$$

$$- \sum_j I^j(t) \left( dB^j(t) + \sum_{k:k \neq j} \mu^{jk^*}(t) (b^{jk}(t) + y^{jk}(t)) dt \right) \\ \tilde{Y}(0) = \Delta B^0(0). \quad (3)$$

The process  $y^{jk}$  is here a sufficiently regular predictable process.

We will now show prove a relation stating that if the terminal lump sum benefit is simply the value of  $\tilde{Y}$  prior to termination, independently of the state, then  $Y$  and  $\tilde{Y}$  are equal.

**Proposition 1** *If  $\Delta B^j(n) = \tilde{Y}(n-)$  then*

$$Y = \tilde{Y}.$$

*We then have that*

$$y^{Z(t-)k}(t) = E^* \left[ \int_t^n e^{-r(s-t)} dB(s) \middle| \mathcal{F}(t) \cap \{Z(t) = k\} \right] - Y(t-). \quad (4)$$

**Proof.** *First realize that, according to (2), we have that*

$$e^{-r(n-t)} \tilde{Y}(n-) = \tilde{Y}(t) + \int_t^{n-} d \left( e^{-r(s-t)} \tilde{Y}(s) \right) \\ = \tilde{Y}(t) + \int_t^{n-} \left( -re^{-r(s-t)} \tilde{Y}(s) ds + e^{-r(s-t)} d\tilde{Y}(s) \right).$$

Plugging this relation into (1), using that  $\Delta B^j(n) = \tilde{Y}(n-)$ , and applying (2) now gives the result that the reserve equals  $\tilde{Y}(t)$ ,

$$\begin{aligned} Y(t) &= E^* \left[ \int_t^{n-} e^{-r(s-t)} dB(s) + e^{-r(n-t)} \tilde{Y}(n-) \middle| \mathcal{F}(t) \right] \\ &= \tilde{Y}(t) + E^* \left[ \int_t^n e^{-r(s-t)} \sum_{k:k \neq Z(s-)} \left( b^{Z(s-)k}(s) + y^{Z(s-)k}(s) \right) dM^{k*}(s) \middle| \mathcal{F}(t) \right]. \end{aligned}$$

Here,  $M^{k*}$  is a martingale under  $P^*$  such that the last term vanishes.

We know that  $\tilde{Y}$  upon transition of  $Z$  to  $k$  at time  $t$  equals  $\tilde{Y}(t-) + y^{Z(t-)k}(t)$ . We also know from the definition of  $Y$  that  $Y$  upon transition of  $Z$  to  $k$  at time  $t$  can be written as  $E^* \left[ \int_t^n e^{-r(s-t)} dB(s) \middle| \mathcal{F}(t) \cap \{Z(t) = k\} \right]$ . But if  $Y = \tilde{Y}$  these observations together give (4). ■

This proposition has the consequence that we can skip the decoration of  $\tilde{Y}$  such that  $Y$  follows the SDE given in (2). Thus, although  $\tilde{Y}$  is a retrospectively calculated account, it coincides with the reserve (1). We emphasize that this relation relies heavily on the fact that  $\tilde{Y}(n-)$  is paid out upon termination such that  $\tilde{Y}(n) = 0$ . Henceforth, we use only the letter  $Y$  without the decoration. Since this represents the savings in the insurance company we call  $Y$  for *institutional wealth*.

We emphasize that the results above hold even for non-deterministic  $dB^j(t)$ ,  $b^{jk}(t)$  and  $y^{jk}(t)$ . It is in this assumption that they differ from similar classical results in life insurance mathematics where the processes  $B^j$  and  $b^{jk}$  are assumed to be deterministic, see e.g. Norberg (1991) for a derivation of the following classical results. If  $B^j$  and  $b^{jk}$  are deterministic, then, by the Markov property, the reserve is fully specified by the so-called statewise reserves,

$$Y^j(t) = E^* \left[ \int_t^n e^{-r(s-t)} dB(s) \middle| Z(t) = j \right], \quad (5)$$

since

$$Y(t) = \sum_j I^j(t) Y^j(t).$$

In this case the reserve jump  $y^{jk}$  is, in accordance with (4),

$$y^{jk}(t) = Y^k(t) - Y^j(t).$$

The dynamics of  $Y$  are, in accordance with (2), given by

$$\begin{aligned} dY(t) &= \sum_j I^j(t) dY^j(t) + \sum_j \left( Y^k(t) - Y^{Z(t-)k}(t) \right) dN^k(t) \\ dY^j(t) &= rY^j(t) dt - dB^j(t) - \sum_{k \neq j} \mu^{jk*}(t) \left( b^{jk}(t) + Y^k(t) - Y^j(t) \right) dt, \\ Y^j(n) &= 0. \end{aligned}$$

In this paper we consider a decision problem where, at time  $t$ , the policy holder decides on  $dB^{Z(t)}(t)$ ,  $b^{Z(t)k}(t)$  and  $y^{Z(t)k}(t)$  for all  $k \neq Z(t)$ . This is really an unconventional construction and to a reader with a life insurance background, this may look like a very awkward decision problem. Deciding on  $dB^{Z(t)}(t)$  and  $b^{Z(t)k}(t)$  may seem reasonable but what does it mean that the policy holder decides on the reserve jump  $y^{Z(t)k}(t)$ ?

In practice the policy holder decides on a set of future payments, i.e.  $dB^j(s)$ ,  $b^{jk}(s)$ ,  $s > t$ ,  $j \neq k$ , and on the basis of these, the insurance company calculates the reserve jumps  $y^{Z(t)k}(t)$ ,

$k \neq Z(t)$ . These reserve jumps are the only consequence of the future payments that the policy holder realizes on his account at time  $t$ . But this means that the policy holder in practice indirectly decides on the reserve jumps through specification of the future payments.

In the decision problem studied here the indirect decision on the reserve jump is made direct by reversing the procedure: We assume that the policy holder decides directly at time  $t$  on the reserve jumps  $y^{Z(t)k}(t)$ ,  $k \neq Z(t)$ . In principle the policy holder could then just inform the insurance company about the reserve jump he wants insured. But this is then what he does indirectly by demanding a set of future payments, such that the reserve jump is obtained upon a possible transition. I.e. for a given decision  $y^{Z(t)k}(t)$ ,  $k \neq Z(t)$ , he (or someone else with the required expertise like e.g. the insurance company) calculates a set of future payments  $dB^j(s)$ ,  $b^{jk}(s)$ ,  $s > t$ ,  $j \neq k$  such that (4) is fulfilled.

There is typically a continuum of future payment processes leading to the same reserve jump so the future payments are not uniquely determined by the above procedure. However, he can take any one of these as long as it leads to the right reserve jumps. When the future turns into the present, as time goes by, he decides then continuously on the actual payments to be realized,  $dB^{Z(t)}(t)$  and  $b^{Z(t)k}(t)$ ,  $k \neq Z(t)$ .

In the example section we demonstrate how this machinery works.

### 3 The Model and the Decision Processes

In this paragraph, we introduce an income process  $A = (A(t))_{0 \leq t \leq n}$  representing the accumulated income of the individual. The income process is assumed to follow the dynamics

$$dA(t) = a^{Z(t)}(t) dt + \sum_{k:k \neq Z(t-)} a^{Z(t-)k}(t) dN^k(t),$$

where  $a^j(t)$  and  $a^{jk}(t)$  are assumed to be deterministic functions. Here,  $a^j(t)$  is the rate of income given that the individual is in state  $j$  at time  $t$  and  $a^{jk}(t)$  is the lump sum income at time  $t$  given that the individual jumps from state  $j$  to state  $k$  at time  $t$ . By income we think primarily of labor income which makes sense to the state-dependent income rate  $a^j$  in the disability and unemployment models. The lump sum income is taken into account for the sake of generality, and more creative models could actually defend the possibility of having a lump sum income upon transition: Consider the three state model where state 0 is the state where the rich uncle to whom the individual is the only inheritor, is still alive and state 1 is the state where the rich uncle has passed away. Note that in this case  $\mu^{10}$  should be set to zero just as  $\mu^{20}$  and  $\mu^{21}$  are default set to zero.

In this paragraph, we introduce a consumption process  $C = (C(t))_{0 \leq t \leq n}$  representing the accumulated consumption of the individual. The consumption process is assumed to follow the dynamics

$$dC(t) = c^{Z(t)}(t) dt + \sum_{k:k \neq Z(t-)} c^{Z(t-)k}(t) dN^k(t).$$

Here,  $c^j(t)$  is the rate of consumption given that the individual is in state  $j$  at time  $t$  and  $c^{jk}(t)$  is the lump sum consumption at time  $t$  given that the individual jumps from state  $j$  to state  $k$  at time  $t$ . The processes  $c^j(t)$  and  $c^{jk}(t)$  are decision processes chosen at the discretion of the individual. As for the income process the rate of consumption has an obvious interpretation while a lump sum consumption must be motivated by a more creative pattern of thinking.

The *personal wealth* is accounted for on a bank account of the individual. This bank account is assumed to earn interest at rate  $r$ , the same rate that is earned on the insurance account, and,

hereafter, it accounts for the three payment processes  $A$ ,  $B$ , and  $C$ . Thus the bank account has the following dynamics,

$$\begin{aligned} dX(t) &= rX(t)dt + dA(t) + dB(t) - dC(t), \\ X(0) &= x_0. \end{aligned} \tag{6}$$

Thus, apart from earning interests, this bank account simply accounts for labor income and insurance benefits as income and consumption as outgo.

Finally, we can add up the institutional and personal wealths to derive the dynamics of the total wealth,

$$\begin{aligned} d(X(t) + Y(t)) &= r(X(t) + Y(t))dt + dA(t) - dC(t) \\ &\quad + \sum_{k:k \neq Z(t-)} \left( b^{Z(t-)k}(t) + y^{Z(t-)k}(t) \right) dM^{Z(t)k*}(t). \end{aligned}$$

These dynamics have the following interpretation. Firstly, the total wealth earns interest at rate  $r$ . Secondly, the income process and the consumption process affect the total wealth directly. Thirdly, upon a transition from  $j$  to  $k$  the total wealth increases by  $b^{jk}(t) + y^{jk}(t)$ . From this amount,  $b^{jk}(t)$  is paid from the insurance institution to the individual and added to the bank account. The amount,  $y^{jk}(t)$  is also paid from the insurance company to the individual but kept by the insurance company by adding it to the insurance account. For this total wealth increment of  $b^{jk}(t) + y^{jk}(t)$ , the individual pays a natural premium at rate  $\mu^{jk*}(b^{jk}(t) + y^{jk}(t))$ .

From the dynamics of  $X + Y$  there are three important points to make.

- Assume that  $X$  and  $Y$  appear in the objective function of the decision problem through their sum only. Then it seems possible to replace the two state variables  $X$  and  $Y$  by their sum  $S = X + Y$ . Otherwise we still need the two state variables. One situation where  $X$  and  $Y$  will not appear through their sum only, is if there are specific constraints on e.g.  $Y$ . To keep insurance business separated from banking (loaning) business, one could have the constraint that  $Y(t) \geq 0$  for all  $t$  while there could be no such constraint  $X$ . That is, loaning takes place in the bank, not in the insurance company. We solve the unconstrained problem below but we are still able to carry out special studies with certain constraints.
- If  $b^{Z(t-)k}(t)$  and  $y^{Z(t-)k}(t)$  appear in the objective function through their sum only, they will not be determined uniquely. Below,  $b^{Z(t-)k}$  and  $y^{Z(t-)k}$  will not appear in the objective function at all, so therefore we get a non-unique solution. One situation where  $b^{Z(t-)k}$  and  $y^{Z(t-)k}$  will not appear through their sum only, is if there are specific constraints on e.g.  $b^{Z(t-)k}$ . To prevent individuals from selling life insurance on their own lives, one could have the constraint that  $b^{jk}(t) \geq 0$ . We solve the unconstrained problem below but we are still able to carry out special studies with certain constraints.
- If the continuous insurance payment rate  $b^{Z(t)}(t)$  does not appear in the objective function, then that payment rate cannot be determined by solving the optimization problem since the payment rate has vanished from the dynamics of  $X + Y$ . The payment rate can appear in different forms in the objective function, e.g. through constraints. We solve the unconstrained problem below, though, and  $b^{Z(t)}(t)$  will therefore not be determined.

Above we have mentioned possible constraints like  $Y(t) \geq 0$  and  $b^{jk}(t) \geq 0$  which both can be motivated by not mixing insurance and the creditworthiness of the individual. If such constraints are fulfilled, the insurance company does not need to worry about whether the individual can afford

the insurance contract he enters into. Creditworthiness is completely left up to the bank to decide. This motivation is closely linked to the way banking and life insurance regulation is carried out in practice. We emphasize that below we solve the unconstrained problem in general. But due to the non-uniqueness of  $b^j$ ,  $b^{jk}$ , and  $y^{jk}$ , we can still solve certain relevant constrained problems as it is also seen in the examples.

## 4 The Control Problem and Its Solution

In this section we present the control problem and its solution. Introduce a utility process with dynamics given by

$$dU(t) = u^{Z(t)}(t, c^{Z(t)}(t)) dt + \sum_{k:k \neq Z(t-)} u^{Z(t-)k}(t, c^{Z(t-)k}(t)) dN^k(t) + \Delta U^{Z(t-)}(t, X(t-), Y(t-)) d\varepsilon(t, n).$$

Here,  $w^j(t, c)$  is a deterministic utility function which measures utility of the consumption rate  $c$  given that the individual is in state  $j$  at time  $t$  and  $w^{jk}(t, c)$  is a deterministic utility function which measures utility of the lump sum consumption  $c$  given that the individual jumps from state  $j$  to state  $k$  at time  $t$ . Finally,  $\Delta U^j(n, x, y)$  is a deterministic function which measures utility of the terminal lump sum payout from the two accounts  $x$  and  $y$  given that the individual is in state  $j$  at time  $n$ . We assume that the individual chooses a consumption-insurance process to maximize utility in the sense of

$$\sup E \left[ \int_0^n dU(t) \right].$$

where the supremum is taken over  $b^j, b^{jk}, y^{jk}, c^j, c^{jk}, j \neq k$ .

We specify further the utility functions appearing in the utility process. We are interested in solving the problem for an individual with preferences represented by the power utility function in the sense of

$$\begin{aligned} w^j(t, c) &= \frac{1}{\gamma} w^j(t)^{1-\gamma} c^\gamma, \\ w^{jk}(t, c) &= \frac{1}{\gamma} w^{jk}(t)^{1-\gamma} c^\gamma, \\ \Delta U^j(t, x, y) &= \frac{1}{\gamma} \Delta W^j(t) (x + y)^\gamma. \end{aligned}$$

Here,  $w^j(t)$  is a weight process which gives weight to power utility of the consumption rate  $c$  given that the individual is in state  $j$  at time  $t$ ,  $w^{jk}(t)$  is a weight process which gives weight to power utility of the lump sum consumption  $c$  given that the individual jumps from state  $j$  to state  $k$  at time  $t$ . Finally,  $\Delta W^j(t)$  is a weight function which gives weight to power utility of lump sum consumption given that the individual is in state  $j$  at time  $t$ . The weight functions appear simply as factors in the utility functions. However, it is convenient to think of these weight functions as stemming from a weight process with dynamics given by

$$dW(t) = w^{Z(t)}(t) dt + \sum_{k:k \neq Z(t-)} w^{Z(t-)k}(t) dN^k(t) + \Delta W^{Z(t)}(t) d\varepsilon(t, n).$$

This artificial process - artificial in the sense that it is not a payment process anyone pays - exposes the symmetry in structure across all appearing processes.

For presentation of the results we introduce the abbreviating function

$$h^{jk}(t) = \left( \frac{\mu^{jk}(t)}{\mu^{jk*}(t)} \right)^{1/(1-\gamma)}. \quad (7)$$

The quotient of intensities in (7),  $\mu^{jk}(t)/\mu^{jk*}(t)$ , is actually the reciprocal of one plus the so-called Girsanov kernel that characterizes the measure transformation from  $P$  to  $P^*$  in probabilistic terms. In financial applications, minus the Girsanov kernel is called the market price of risk.

Calculations in Appendix A show that the optimal consumption and insurance strategies are given by the following feed-back functions for  $c^j(t)$ ,  $c^{jk}(t)$ ,  $b^{jk}(t)$ , and  $y^{jk}(t)$ ,

$$c^j(t, x, y) = \frac{w^j(t)}{f^j(t)} (x + y + g^j(t)), \quad (8a)$$

$$c^{jk}(t, x, y) = \frac{w^{jk}(t)}{f^j(t)} h^{jk}(t) (x + y + g^j(t)), \quad (8b)$$

$$\begin{aligned} b^{jk}(t, x, y) + y^{jk}(t, x, y) &= \frac{f^k(t) + w^{jk}(t)}{f^j(t)} h^{jk}(t) (x + y + g^j(t)) \\ &\quad - (a^{jk}(t) + x + y + g^k(t)) \end{aligned} \quad (8c)$$

with  $f$  and  $g$  given below. In the next section we study these optimal control functions in details. Here we just specify the functions  $g$  and  $f$  such that the optimal controls are fully specified by (8). They satisfy the systems of ordinary differential equations given by

$$g_t^j(t) = rg^j(t) - a^j(t) - \sum_{k:k \neq j} \mu^{jk*}(t) (a^{jk}(t) + g^k(t) - g^j(t)), \quad (9)$$

$$g^j(n) = 0,$$

$$f_t^j(t) = r^{*j}(t) f^j(t) - w^j(t) - \sum_{k:k \neq j} \tilde{\mu}^{jk}(t) (w^{jk}(t) + f^k(t) - f^j(t)) \quad (10)$$

$$f^j(n) = \Delta W^j(n),$$

with

$$\tilde{\mu}^{jk}(t) = \mu^{jk}(t) h^{jk}(t)^\gamma = \mu^{jk*}(t) h^{jk}(t),$$

$$\delta = \frac{\gamma}{1-\gamma},$$

$$r^{*j}(t) = -\delta r - \delta (\mu^{j*}(t) - \mu^{j\cdot}(t)) + \mu^{j\cdot}(t) - \tilde{\mu}^{j\cdot}(t).$$

The solution to the system of ordinary differential equations for  $g$  has the Feynman-Kac representation

$$\begin{aligned} g^j(t) &= E_{t,j}^* \left[ \int_t^n e^{-r(s-t)} dA(s) \right] \\ &= \int_t^n e^{-r(s-t)} \sum_k p_{jk}^*(t, s) \left( a^k(s) + \sum_{l:l \neq k} \mu^{kl*}(s) a^{kl}(s) \right) ds. \end{aligned} \quad (11)$$

Thus,  $g^j(t)$  is the conditional expected present value of the future income process where the expectation is taken under  $P^*$ . This is, in other words, the financial value of the future income.

The solution to the system of ordinary differential equations for  $f$  has the Feynman-Kac representation

$$f^j(t) = \tilde{E}_{t,j} \left[ \int_t^n e^{-\int_t^s r^{*Z(\tau)}(\tau) d\tau} dW(s) \right]. \quad (12)$$

Thus,  $f^j(t)$  is the conditional expected value of the future weight process where expectation is taken under an artificial measure  $\tilde{P}$  under which  $N^k$  admits the intensity process  $\tilde{\mu}^{Z(t)k}(t)$ . This is, in other words, an artificial financial value of the future weights in the sense that an artificial stochastic interest rate process and an artificial valuation measure are applied.

## 5 Studies of the Optimal Controls

In this section we study in detail the optimal controls derived in the previous section. We give interpretations of the optimal controls in the general forms in (8). In the succeeding two sections we study two important special constructions of the underlying process  $Z$ . There we pay further attention to the optimal controls.

We note two important points on uniqueness. These relate to the remarks at the end of Section 3. Firstly, there is no condition on the continuous payment rate  $b^j(t)$ . This was foreseen in Section 3. Secondly, the decision processes  $b^{jk}(t)$  and  $y^{jk}(t)$  are not uniquely determined since there is only one equation for their sum. Also this was foreseen in Section 3. Special cases are, of course, the cases where the one or the other is default set to zero. If we put  $b^{jk}(t)$  or  $y^{jk}(t)$  equal to zero in (8c), respectively, we get unique optimal controls for  $y^{jk}(t)$  and  $b^{jk}(t)$ , respectively. This illustrates how the lack of uniqueness, makes it possible to study certain constrained control problems after all.

We now take a closer look at the optimal controls. First we give interpretations of them as they appear in (8). In all three formulas appear the sum  $x + y + g^j(t)$ . This can be interpreted as the total wealth of the individual given that he is in state  $j$  at time  $t$ . This total wealth consists of personal wealth  $x$ , institutional wealth  $y$ , and human wealth  $g^j$ . Recall that  $g^j$  is the financial value of future income given that the individual is in state  $j$ . Furthermore, in (8c) appears the sum  $a^{jk}(t) + x + y + g^k(t)$ . This can be interpreted as the total wealth of the individual upon transition from state  $j$  to state  $k$  at time  $t$  before the effect of insurance. This wealth consists of the lump sum income upon transition  $a^{jk}(t)$  and then again of personal wealth  $x$ , institutional wealth  $y$ , and human wealth  $g^k(t)$ . Here the human wealth is measured given that the individual is in state  $k$  at time  $t$ . We emphasize that this is the wealth before a possible insurance sum is paid out or a reserve jump has been added to the institutional wealth. With these interpretations of total wealth in mind we can now interpret the three control functions:

- The optimal continuous consumption rate in (8a) is a fraction of total wealth. The fraction  $w^j(t)/f^j(t)$  measures the utility of present consumption against utility of consumption in the future. Recall that  $f^j(t)$  is an artificial value of the future weights. The optimal consumption rate is in related problems typically formed by a similar fraction of total wealth.
- The optimal lump sum consumption upon transition in (8b) is also a fraction of wealth. The fraction  $h^{jk}(t)w^{jk}(t)/f^j(t)$  consists of two elements. The fraction  $w^{jk}(t)/f^j(t)$  measures the utility of consumption upon transition against utility of future consumption. However, future consumption is calculated given that the individual is in state  $j$  at time  $t$  and not given that the individual is in state  $k$  at time  $t$ . This is explained by the fact that the risk connected to the jump is partly 'insured away'. The price of this insurance is, together with the individual attitude towards risk, hidden in  $h^{jk}$ . This explains the first element of the factor  $h^{jk}(t)w^{jk}(t)/f^j(t)$ .
- The optimal insurance sum plus reserve jump upon transition in (8c) can be interpreted as a protection of wealth. In the optimal decision one should not distinguish between an

insurance sum that is a sum added to personal wealth, and a reserve jump that is a sum added to institutional wealth. The allocation of the total jump should be determined by other considerations. However, how does this optimal total insurance sum protect wealth? The optimal sum measures the difference between a fraction  $h^{jk}(t)(f^k(t) + w^{jk}(t)) / f^j(t)$  of present wealth  $x + y + g^j(t)$  and wealth upon transition  $a^{jk}(t) + x + y + g^k(t)$ . If the fraction  $h^{jk}(t)(f^k(t) + w^{jk}(t)) / f^j(t)$  is 1 then this difference reduces to  $-(a^{jk}(t) + g^k(t) - g^j(t))$  which is minus the human wealth sum at risk. Thus, this is really the wealth that is potentially lost upon transition and which should be protected by an opposite insurance position. However, in the calculation of the optimal protection two further considerations should be taken into account: 1) The utility of future wealth in case of no transition is measured against the utility of future wealth in case of transition in the ratio  $(f^k(t) + w^{jk}(t)) / f^j(t)$ . If utility of future wealth given a transition is lower than without transition, i.e.  $(f^k(t) + w^{jk}(t)) / f^j(t) < 1$ , then one should underinsure ones wealth under risk,  $-(a^{jk}(t) + g^k(t) - g^j(t))$ , and vice versa. 2) If the protection is 'expensive', i.e.  $h^{jk} < 1$ , then one should also underinsure ones wealth under risk in order to 'pick up' some of this market price of risk. These underinsurances are implemented by weighting the present human wealth with the ratio  $(f^k(t) + w^{jk}(t)) / f^j(t)$  and  $h^{jk}(t)$ .

Now, take a closer look at the controls  $c^j$  and  $c^{jk}$ . For fixed  $Z(t) = j$ , we can study the optimally controlled processes  $X^j$  and  $Y^j$  that solve the following ordinary differential equations

$$\begin{aligned}\frac{d}{dt}X^j(t) &= rX^j(t) + a^j(t) + b^j(t) - c^j(t), \\ \frac{d}{dt}Y^j(t) &= rY^j(t) - b^j(t) - \sum_{k:k \neq j} \mu^{jk*}(t)(b^{jk}(t) + y^{jk}(t)).\end{aligned}$$

Since  $X^j$  and  $Y^j$  evolve deterministically, we can study the state-wise controls  $c^j(t, X^j(t), Y^j(t))$  and  $c^{jk}(t, X^j(t), Y^j(t))$  as functions of time. With a slight abuse of notation we denote these deterministic functions by  $c^j(t)$  and  $c^{jk}(t)$ . Furthermore, we consider the optimal wealth upon transition before consumption which is given by

$$\begin{aligned}q^{jk}(t, x, y) &= b^{jk}(t, x, y) + y^{jk}(t, x, y) + a^{jk}(t) + x + y + g^k(t) \\ &= \frac{f^k(t) + w^{jk}(t)}{f^j(t)} h^{jk}(t)(x + y + g^j(t)).\end{aligned}$$

Also  $q^{jk}(t, X(t), Y(t))$  can be studied as a function of time, accordingly denoted by  $q^{jk}(t)$ . From (8) we can, by using

$$\begin{aligned}c_t^j(t) &= \frac{\partial}{\partial t} c^j(t, X^j(t), Y^j(t)) \\ &\quad + \frac{\partial}{\partial x} c^j(t, X^j(t), Y^j(t)) \frac{dX^j(t)}{dt} \\ &\quad + \frac{\partial}{\partial y} c^j(t, X^j(t), Y^j(t)) \frac{dY^j(t)}{dt}\end{aligned}$$

and similar formulas for  $c^{jk}$  and  $q^{jk}$ , derive the following simple exponential differential equations

for  $c^j(t)$ ,  $c^{jk}(t)$ , and  $q^{jk}(t)$ ,

$$\begin{aligned} c_t^j(t) &= c^j(t) \left( \frac{1}{1-\gamma} (r + \mu^{j^*}(t) - \mu^{j^{\cdot}}(t)) + \frac{w_t^j(t)}{w^j(t)} \right), \\ c_t^{jk}(t) &= c^{jk}(t) \left( \frac{1}{1-\gamma} (r + \mu^{j^*}(t) - \mu^{j^{\cdot}}(t)) + \frac{w_t^{jk}(t)}{w^{jk}(t)} + \frac{h_t^{jk}(t)}{h^{jk}(t)} \right), \\ q_t^{jk}(t) &= q^{jk}(t) \left( \frac{1}{1-\gamma} (r + \mu^{j^*}(t) - \mu^{j^{\cdot}}(t)) + \frac{f_t^k(t) + w_t^{jk}(t)}{f^k(t) + w^{jk}(t)} + \frac{h_t^{jk}(t)}{h^{jk}(t)} \right). \end{aligned}$$

By the definition of  $h$  in (7) and introducing  $\mu^{jk^*}(t) = (1 + \Gamma^{jk}(t)) \mu^{jk}(t)$ , we can calculate that  $h_t^{jk}(t)/h^{jk}(t) = -\Gamma_t^{jk}(t)/((1-\gamma)(1+\Gamma^{jk}(t)))$ . If we define the weights according to the usual impatience factor, i.e.  $w^j(t)^{1-\gamma} = \exp(-\iota t)$  we can furthermore calculate that  $w_t^j(t)/w^j(t) = -\iota/(1-\gamma)$ . Plugging in these relations and introducing  $(\Gamma\mu)^{j^{\cdot}}(t) = \sum_{k:k \neq j} \Gamma^{jk}(t) \mu^{jk}(t)$ , we get the following simple differential equations for the optimal controls  $c^j(t)$  and  $c^{jk}(t)$ ,

$$\begin{aligned} c_t^j(t) &= c^j(t) \frac{1}{1-\gamma} \left( r - \iota + (\Gamma\mu)^{j^{\cdot}}(t) \right), \\ c_t^{jk}(t) &= c^{jk}(t) \frac{1}{1-\gamma} \left( r - \iota + (\Gamma\mu)^{j^{\cdot}}(t) - \frac{\Gamma_t^{jk}(t)}{1 + \Gamma^{jk}(t)} \right). \end{aligned}$$

## 6 The Survival Model

In this section we specialize the results in Section 4 into the case of a survival model. The idea is to study optimal consumption and insurance decisions of an individual who has utility of consumption while being alive including utility of lump sum consumption upon termination. Furthermore he (or rather his inheritors) has utility of consumption upon death before termination. In Figure 3, we have specified a set of income process coefficients and a set of utility weight coefficients.

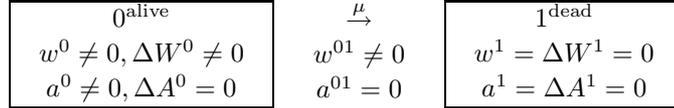


Figure 3: Survival model with income and utility weights

All statewise coefficients are zero in the state 'dead'. This means that there is no income and no utility of consumption in that state. Weights on utility of consumption in the state 'alive' are specified by the coefficients  $w^0$  and  $\Delta W^0$ , and weight on utility of a lump sum payment upon death is specified by  $w^{01}$ . Income is specified by the rate  $a^0$  and other income coefficients are set to zero such that there is no lump sum income upon death or upon survival until termination. We start out by specifying the functions  $f$  and  $g$  for this special case.

According to (10)  $f^1 = 0$  and  $f^0$  is characterized by

$$\begin{aligned} f_t^0(t) &= -w^0(t) + f^0(t) r^*(t) - \tilde{\mu}(t) (w^{01}(t) - f^0(t)), \\ f^0(n) &= \Delta W^0(n), \\ r^* &= -\delta r - \delta (\mu^*(t) - \mu(t)) + \mu(t) - \tilde{\mu}(t) \end{aligned}$$

This differential equation has the solution and Feynman-Kac representation, respectively,

$$\begin{aligned} f^0(t) &= \int_t^n e^{-\int_t^s r^* + \tilde{\mu}} (w^0(s) + \tilde{\mu}(s) w^{01}(s)) ds + e^{-\int_t^n r^* + \tilde{\mu}} \Delta W^0(n), \\ &= \tilde{E}_{t,0} \left[ \int_t^n e^{-\int_t^s r^*(\tau) d\tau} dW(s) \right]. \end{aligned} \quad (13)$$

According to (9)  $g^1 = 0$  and  $g^0$  is characterized by

$$\begin{aligned} g_t^0(t) &= r g^0(t) - a^0(t) + \mu^*(t) g^0(t), \\ g^0(n) &= 0. \end{aligned} \quad (14)$$

This differential equation has the solution and Feynman-Kac representation, respectively,

$$\begin{aligned} g^0(t) &= \int_t^n e^{-\int_t^s r + \mu^*} a^0(s) ds \\ &= E_{t,0}^* \left[ \int_t^n e^{-r(s-t)} dA(s) \right]. \end{aligned} \quad (15)$$

We can now specify the optimal controls in terms of  $f^0$  and  $g^0$ . We get the controls

$$\begin{aligned} c^0(t, x, y) &= \frac{w^0(t)}{f^0(t)} (x + y + g^0(t)), \\ c^{01}(t, x, y) &= \frac{w^{01}(t)}{f^0(t)} h^{01}(t) (x + y + g^0(t)), \\ b^{01}(t, x, y) + y^{01}(t, x, y) &= \frac{w^{01}(t)}{f^0(t)} h^{01}(t) (x + y + g^0(t)) - x - y \\ &= c^{01}(t, x, y) - x - y. \end{aligned} \quad (16)$$

What happens upon death in that case is that the benefit  $b^{01}(t, x, y)$  is paid out and  $c^{01}(t, x, y)$  is consumed. If  $x$  and  $y$  are the accounts just prior to death, these accounts upon death will then be  $x + b^{01}(t, x, y) - c^{01}(t, x, y)$  and  $y + y^{01}(t, x, y)$ , respectively. But according to (16) these accounts are the same with opposite sign. These accounts are now rolled forward earning interest but experiencing no cash flow to time  $n$  where  $Y(n-)$  is paid out (positive or negative). But this benefit exactly covers the deficit at the bank account so both accounts close at zero.

We can now specify a set of payments after time  $t$  which gives the right reserve jump  $y^{01}(t, x, y)$  in accordance with (4). There are several solutions from which we take the payment stream specifying that the sum  $\Delta B^1(n)$  is paid out at time  $n$  if the policy holder is dead then, i.e.

$$dB(s) = \Delta B^1(s) I^1(s) d\varepsilon^n(s), s > t.$$

For this payment process for we get the following calculation for isolation of  $\Delta B(n)$  in (4),

$$\begin{aligned} y^{01}(t, x, y) &= E^* \left[ \int_t^n e^{-r(s-t)} dB(s) \Big| Z(t) = 1 \right] - y \\ &= e^{-r(n-t)} \Delta B^1(n) - y \\ &\Rightarrow \\ \Delta B^1(n) &= e^{r(n-t)} (y^{01}(t, x, y) + y). \end{aligned} \quad (17)$$

We now have two equations (16) with three unknowns ( $b^{01}(t, x, y)$ ,  $y^{01}(t, x, y)$ ,  $\Delta B^1(n)$ ) and all solutions are equally optimal. The natural one to take is the one where  $\Delta B^1(n) = 0$ ,  $y^{01}(t, x, y) = -y$ , and  $b^{01}(t, x, y) = c^{01}(t, x, y) - x$ , such that all accounts are set to zero upon death.

We also specify the simple exponential differential equation characterizing the statewise consumptions,

$$c_t^0(t) = c^0(t) \frac{1}{1-\gamma} (r - \iota + \Gamma(t) \mu(t)), \quad (18)$$

$$c_t^{01}(t) = c^{01}(t) \frac{1}{1-\gamma} \left( r - \iota + \Gamma(t) \mu(t) - \frac{\Gamma_t(t)}{1+\Gamma(t)} \right). \quad (19)$$

We now consider two examples, where the control in this special survival case can be specified further. These examples correspond to the cases where there is no bank account and no insurance account, respectively.

**Example 2 No insurance account**

We can put the insurance account equal to zero without losing expected utility by specifying that the natural premium for the optimal death sum is the only payment to the insurance account, i.e.

$$-b(t) = \mu^*(t) b^1(t).$$

Realize from (2) and  $y^{01}(t, x, y) = -y$  that then  $Y(t) = 0$  for all  $t$ . Then we can put  $y = 0$  in all controls and skip the dependence on  $y$ , i.e.

$$\begin{aligned} c^0(t, x) &= \frac{w^0(t)}{f^0(t)} (x + g^0(t)), \\ c^{01}(t, x) &= \frac{w^{01}(t)}{f^0(t)} h^{01}(t) (x + g^0(t)) \\ b^{01}(t, x) &= c^{01}(t) - x. \end{aligned}$$

The formulas are identical to those by Richard (1975,(42,43)). In comparison we mention that Richard (1975) uses the following notation (Richard notation  $\equiv$  notation here):  $a \equiv f^{1-\gamma}$ ,  $b \equiv g$ ,  $h \equiv w^{1-\gamma}$ ,  $m \equiv (w^1)^{1-\gamma}$ ,  $\lambda \equiv \mu$ ,  $\mu \equiv \mu h^{\gamma-1}$ .

In Richard (1975) the mortality is modelled such that the probability of survival until termination  $n$ ,  $\exp(-\int_t^n \mu)$ , is zero for all  $t$ . This is obtained by  $\mu \rightarrow \infty$  for  $t \rightarrow n$ . Furthermore, it is assumed that  $\mu^*(t)/\mu(t) \rightarrow 1$  for  $t \rightarrow n$ . But then the last term of (13) is zero and  $\Delta W(n)$  is superfluous: If we know that we will not survive time  $n$ , the utility of consumption at time  $n$  plays no role for our decision. With  $\Delta W(n) = 0$ , (13) and (15) are identical to those by Richard (1975,(41,25)).

One problem, from a practical point of view, with this construction is that the optimal insurance sum may become negative. If the individual (and his inheritors) has relatively large utility from consuming while being alive compared to consuming upon death, he should optimally risk losing parts of his wealth as he grows old. When there is no institutional wealth he does so by selling life insurance. But the way individuals sell life insurance in practice is instead by holding life annuities based on institutional wealth. Therefore, a much more realistic special case is given now in an example with no bank account.

**Example 3 No bank account**

We can put the bank account equal to zero without losing expected utility by specifying that income minus consumption goes directly into the insurance account, i.e.

$$B = C - A.$$

In this concrete case this corresponds to letting  $-b^0(t) = a^0(t) - c^0(t)$ , i.e. the excess of income over consumption is paid as premium on the insurance contract, and  $b^{01}(t) = c^{01}(t)$ , i.e. upon

death the insurance benefit is consumed (by the inheritors). Realize from (6) and (16) that then  $X(t) = 0$  for all  $t$  and we can put  $x = 0$  and skip the dependence on  $x$  in all controls, i.e.

$$\begin{aligned} c^0(t, y) &= \frac{w^0(t)}{f^0(t)}(y + g(t)), \\ c^{01}(t, y) &= \frac{w^{01}(t)}{f^0(t)}h(t)(y + g(t)) \\ &= b^{01}(t). \end{aligned} \tag{20}$$

Note that since now  $b^{01} = c^{01}$ , the differential equation (19) holds also for the optimal death sum.

**Remark 4** The optimal consumption rate (20) solves the problem of optimal design of a life annuity. If from time  $t$  there are no more incomes, i.e.  $g(t) = 0$ , the optimal life annuity rate is given by the fraction  $w^0(t)/f^0(t)$  of the reserve. If  $w^0(t)$  is constant and there is no utility from benefits upon death or termination, i.e.  $w^{01}(t) = \Delta W^0(n) = 0$ , we get the optimal annuity rate equals the reserve divided by

$$\int_t^n e^{-\int_t^s r^* + \tilde{\mu}} ds.$$

This is just the present value of the life annuity with interest  $r^*$  and mortality rate  $\tilde{\mu}$ . For the logarithmic investor ( $\gamma = 0$ ) we get the simpler  $\int_t^n e^{-\int_t^s \mu} ds$ . It is interesting to see how the life annuity rate evolves over time, but this question is answered by (18),

$$c_t^0(t) = c^0(t) \frac{1}{1 - \gamma} (r - \iota + \Gamma(t) \mu(t)).$$

If the insurance is priced fair, this simplifies to

$$c_t^0(t) = c^0(t) \frac{r - \iota}{1 - \gamma}.$$

Whether this annuity is decreasing or increasing depends on whether the impatience factor  $\iota$  is larger or smaller than the interest rate  $r$ . Typically, one would take the impatience factor to be larger than the interest rate and then the optimal annuity rate decreases exponentially.

## 7 The Disability/Unemployment Model

In this section we specialize the results in Section 4 into the special case of a survival model. The idea is to study the optimal consumption and insurance decisions of an individual who has utility of consumption as long as he is alive. The utility may change, however, as he jumps into a state where he loses his income. This state may be interpreted as a disability state or unemployment state, depending on the study. In Figure 3, we have specified a set of income process coefficients and a set of utility weight coefficients.

All statewise coefficients are zero in the state 'dead'. This means that there is no income and no utility of consumption in that state. Weight on utility of consumption in the state 'active/employed' are specified by the coefficient  $w^0$ . Lump sum consumption upon termination or transition between states is given no weight, i.e.  $\Delta W^0 = \Delta W^1 = w^{01} = 0$ . Weight on utility of consumption in the state 'disabled/unemployed' is specified by the coefficient  $w^1$ . The income in that state is set to zero,  $a^0 = 0$ . Letting  $\rho = 0$  may be less realistic in an unemployment interpretation than in a disability interpretation but is nevertheless assumed to obtain explicit results.

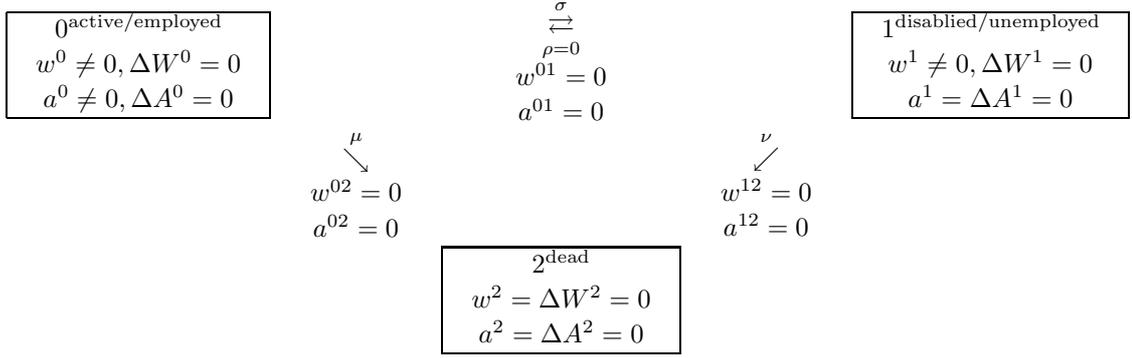


Figure 4: Disability/Unemployment model

According to (10)  $f^2 = 0$  and  $f^1$  and  $f^0$  are characterized by

$$\begin{aligned}
f_t^1(t) &= (r^{*1}(t) + \tilde{\nu}(t)) f^1(t) - w^1(t), f^1(n) = 0, \\
r^{*1}(t) &= -\delta r - \delta(\nu^* - \nu) + \nu - \tilde{\nu}, \\
f_t^0(t) &= (r^{*0}(t) + \tilde{\mu}(t) + \tilde{\sigma}(t)) f^0(t) - w^0(t) - \tilde{\sigma}(t) f^1(t), f^0(n) = 0 \\
r^{*0}(t) &= -\delta r - \delta(\mu^*(t) + \sigma^*(t) - \mu(t) - \sigma(t)) + \mu(t) + \sigma(t) - \tilde{\mu}(t) - \tilde{\sigma}(t).
\end{aligned}$$

This differential equation has the solution and Feynman-Kac representation, respectively,

$$\begin{aligned}
f^1(t) &= \int_t^n e^{-\int_t^s r^{*1} + \tilde{\nu}} w^1(s) ds \\
&= \tilde{E}_{t,1} \left[ \int_t^n e^{-\int_t^s r^{*1}} dW(s) \right], \\
f^0(t) &= \int_t^n e^{-\int_t^s r^{*0} + \tilde{\mu} + \tilde{\sigma}} (w^0(s) + \tilde{\sigma}(s) f^1(s)) ds \\
&= \tilde{E}_{t,0} \left[ \int_t^n e^{-\int_t^s r^{*Z(\tau)}(\tau) d\tau} dW(s) \right].
\end{aligned}$$

We specify further this solution in the special case where  $\sigma^* = \sigma$ ,  $\nu^* = \mu^*$  and  $\nu = \mu$ . This means that disability/unemployment risk is priced by the objective measure and that the mortality risk is not changed when jumping into state 1. In that case we have that  $r^{1*} = r^{0*} \equiv r^*$  and  $\tilde{\sigma} = \sigma$ . If we furthermore have that the utility is the same for the states 0 and 1, i.e.  $w^0 = w^1 \equiv w$ , we get that  $f^0 = f^1$  is given by

$$\begin{aligned}
f^0(t) &= \int_t^n e^{-\int_t^s r^* + \tilde{\mu}} w(s) ds \\
&= \tilde{E}_{t,0} \left[ \int_t^n e^{-\int_t^s r^*(\tau) d\tau} dW(s) \right].
\end{aligned}$$

According to (9)  $g^2 = g^1 = 0$  and  $g^0$  is characterized in the same way as in (14) and (15) with  $\mu^*$  replaced by  $\mu^* + \sigma^*$ .

Assume now, as in the previous section, that the insurance account is set to zero upon death, i.e.

$$y^{02}(t, x, y) = y^{12}(t, x, y) = -y. \quad (21)$$

We can now specify the optimal controls in terms of  $f$  and  $g$ . The consumption upon transition is zero since we have no utility of consumption upon transition. Furthermore, the optimal life insurance sum becomes minus the personal wealth, i.e.

$$\begin{aligned} c^{01} &= c^{02} = c^{12} = 0, \\ b^{02} &= b^{12} = -x. \end{aligned} \tag{22}$$

The more interesting controls are the consumption rates as active and disabled and the optimal protection against disability risk,

$$\begin{aligned} c^0(t, x, y) &= \frac{w^0(t)}{f^0(t)} (x + y + g^0(t)), \\ c^1(t, x, y) &= \frac{w^1(t)}{f^1(t)} (x + y), \\ b^{01}(t, x, y) + y^{01}(t, x, y) &= \frac{f^1(t)}{f^0(t)} h^{01}(t) (x + y + g^0(t)) - (x + y). \end{aligned} \tag{23}$$

Concerning the transitions into state 2 we have from (21) and (22) the relations

$$\begin{aligned} b^{02}(t, x, y) + y^{02}(t, x, y) &= -(x + y), \\ b^{12}(t, x, y) + y^{12}(t, x, y) &= -(x + y), \end{aligned} \tag{24}$$

which have the same interpretations as in Section 6: Upon death the bank account balance  $x + b^{02}(t, x, y)$  (or  $x + b^{12}(t, x, y)$  if the policy holder dies as disabled) equals minus the insurance account balance  $y + y^{02}(t, x, y)$  (or  $y + y^{12}(t, x, y)$  if the policy holder dies as disabled). This just means that these accounts, one being negative the other, earn interest until time  $n$  where the terminal benefit  $\tilde{Y}(n)$  is paid to the bank such that both accounts close at zero. As in Section 6 we can now find payments upon death and after death such that (24) holds. But then we can just as well take the insurance account to zero already upon death by requiring that  $y^{02}(t, x, y) = y^{12}(t, x, y) = -y$ .

Before looking closer at the insurance decision we specify the simple exponential differential equation characterizing the statewise consumptions,

$$\begin{aligned} c_t^0(t) &= c^0(t) \frac{1}{1-\gamma} (r - \iota + \Gamma^{02}(t) \mu(t) + \Gamma^{01} \sigma(t)), \\ c_t^1(t) &= c^1(t) \frac{1}{1-\gamma} (r - \iota + \Gamma^{12}(t) \nu(t)). \end{aligned}$$

The optimal protection against loss of income is given in (23). There we see that for the special case with the same utility in states 0 and 1, i.e.  $f \equiv f^0 = f^1$ , the optimal protection reduces to  $h^{01}(t) (x + y + g^0(t)) - (x + y)$ . If furthermore, the price of this protection is calculated by the  $P$ -intensity, i.e.  $h^{01} = 1$ , then the protection reduces to  $g^0(t)$ . Thus, under these circumstances the individual should fully protect the financial value of future income. If utility of consumption as disabled is lower than utility of consumption as active and/or if the protection is expensive in the sense of  $h^{01} > 1$ , then one should underinsure the potential loss.

We now consider two examples, where the control in this special disability/unemployment case can be specified further. These examples correspond to the cases where there is no bank account and no insurance account, respectively.

**Example 5 No insurance account.**

We can put the insurance account equal to zero without loss of value by specifying that the natural premium for the optimal death sum is the only payment to the insurance account, i.e.

$$\begin{aligned} -b^0(t) &= \sigma^*(t)b^{01}(t) + \mu^*(t)b^{02}(t), \\ -b^1(t) &= \mu^*(t)b^{12}(t), \\ y^{12}(t) &= 0. \end{aligned}$$

Realize from (2) and  $y^{02}(t, x, y) = y^{12}(t, x, y) = -y$  that then  $Y(t) = 0$  for all  $t$  and we can put  $y = 0$  and skip the dependence on  $y$  in all controls, i.e.

$$\begin{aligned} c^0(t, x) &= \frac{w^0(t)}{f^0(t)}(x + g^0(t)), \\ c^1(t, x) &= \frac{w^1(t)}{f^1(t)}x, \\ b^{01}(t, x) &= \frac{f^1(t)}{f^0(t)}h^{01}(t)(x + g^0(t)) - x. \end{aligned}$$

The remark at the end of Example 2 applies again here: The insurance sum may become negative, and in practice negative insurance sums are not obtained by an individual's selling of life insurance but by putting the wealth saved in the institution at risk through some annuity contract. Therefore, a much more realistic special case is given now in an example with no bank account.

**Example 6 No bank account.**

We can put the bank account equal to zero without losing expected utility by specifying that income minus consumption goes directly into the insurance account, i.e.

$$B = C - A.$$

In this concrete case this corresponds to letting  $-b^0(t) = a^0(t) - c^0(t)$ , i.e. as active the excess of income over consumption is paid as premiums on the insurance contract,  $b^1(t) = c^1(t)$ , i.e. as disabled the annuity benefit is fully consumed, and  $b^{01}(t) = c^{01}(t) - a^{01}(t) = 0$ , i.e. there is no lump sum death benefit paid out. Realize from (6) that then  $X(t) = 0$  for all  $t$ . We can then put  $x = 0$  in all controls and skip the dependence on  $x$ , i.e.

$$\begin{aligned} c^0(t, y) &= \frac{w^0(t)}{f^0(t)}(y + g^0(t)), \\ c^1(t, y) &= \frac{w^1(t)}{f^1(t)}y, \\ y^{01}(t, y) &= \frac{f^1(t)}{f^0(t)}h^{01}(t)(y + g^0(t)) - y. \end{aligned} \tag{25}$$

The question is now, what should the policy holder actually do in order to demand the optimal reserve jump  $y^{01}(t, y)$ . Let us consider the case where the policy holder demands to optimal reserve jump by purchasing an optimal disability annuity. In general, we have that the disability annuity rate solves the equivalence principle upon transition

$$y + y^{01}(t, y) = \int_t^n e^{-\int_t^s r + \nu^*} b^1(s) ds,$$

which by the optimization relation (25) leads to

$$\frac{f^1(t)}{f^0(t)}h^{01}(t)(y + g^0(t)) = \int_t^n e^{-\int_t^s r + \nu^*} b^1(s) ds.$$

If the disability annuity demanded is constant this leads to the optimal annuity rate

$$b^1 = \frac{f^1(t)}{f^0(t)} h^{01}(t) \frac{y + g^0(t)}{\int_t^n e^{-\int_t^s r + \nu^* ds}}.$$

This rate becomes particularly simple in the special case where preferences in the states 0 and 1 are equal and where insurance is priced fair, i.e.  $h^{01}(t) f^1(t) / f^0(t) = 1$ . However in that case one could also come up with a very simple non-constant solution. The differential equation for  $Y^0(t) + g^0(t)$ ,

$$\begin{aligned} \frac{d}{dt} (Y^0(t) + g^0(t)) &= (r + \mu(t)) Y^0(t) - c^0(t) + a^0(t) - \sigma(t) y^{01}(t, Y^0(t)) \\ &\quad + (r + \mu(t) + \sigma(t)) g^0(t) - a^0(t) \\ &= (r + \mu(t)) (Y^0(t) + g^0(t)) - c^0(t) \end{aligned}$$

should be equal to the differential equation for the value of the future annuity benefits,

$$\frac{d}{dt} \left( \int_t^n e^{-\int_t^s r + \mu b^1(s)} ds \right) = (r + \mu(t)) \int_t^n e^{-\int_t^s r + \mu b^1(s)} ds - b^1(t).$$

But these are the same exactly if

$$b^1(t) = c^0(t).$$

Thus the policy holder obtains the optimal reserve jump by demanding a disability annuity with a time dependent payment rate corresponding to his optimal consumption rate given that he is still in state 0. It is very intuitive that he then gets full protection if the disability rate equals his optimal consumption in state 0 since this gives him the opportunity, in case of disability, to continue consuming 'as if nothing had happened'. If instead the policy holder is underinsured, i.e.  $h^{01}(t) f^1(t) / f^0(t) < 1$ , because he has lower utility from consumption as disabled than as active and/or because the protection is expensive, then he would have to demand a correspondingly lower disability annuity.

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## Appendix A

For solution of the control problem we introduce a value function

$$V(t, x, y) = \sup E_{t,x,y}^j \left[ \int_t^n dU(s) \right],$$

where  $E_{t,x,y}^j$  denotes conditional expectation given that  $X(t) = x$ ,  $Y(t) = y$ , and  $Z(t) = j$ . The HJB equation for this value function is as follows,

$$\begin{aligned} V_t^j(t, x, y) = & \inf \left[ -\frac{1}{\gamma} w^j(t)^{1-\gamma} c^j(t)^\gamma \right. \\ & - V_x^j(t, x, y) (rx + a^j(t) + b^j(t) - c^j(t)) \\ & - V_y^j(t, x, y) \left( ry - b^j(t) - \sum_{k:k \neq j} \mu^{jk*}(t) (b^{jk}(t) + y^{jk}(t)) \right) \\ & \left. - \sum_{k:k \neq j} \mu^{jk}(t) \left( \frac{1}{\gamma} w^{jk}(t)^{1-\gamma} c^{jk}(t)^\gamma + V^k(t, x^{jk}(t), y + y^{jk}(t)) - V^j(t, x, y) \right) \right] \end{aligned}$$

with

$$x^{jk}(t) = x + a^{jk}(t) + b^{jk}(t) - c^{jk}(t).$$

We now guess that the HJB equation is solved by the following function with according derivatives,

$$\begin{aligned} V^j(t, x, y) &= \frac{1}{\gamma} f^j(t)^{1-\gamma} (x + y + g^j(t))^\gamma, \\ V_t^j(t, x, y) &= \frac{1-\gamma}{\gamma} f^j(t)^{-\gamma} f_t^j(t) (x + y + g^j(t))^\gamma \\ &\quad + f^j(t)^{1-\gamma} (x + y + g^j(t))^{\gamma-1} g_t^j(t), \\ V_x^j(t, x, y) &= f^j(t)^{1-\gamma} (x + y + g^j(t))^{\gamma-1}, \\ V_y^j(t, x, y) &= f^j(t)^{1-\gamma} (x + y + g^j(t))^{\gamma-1}. \end{aligned}$$

Firstly we consider the first order conditions for the elements of the consumption process. The conditions for  $c$  and  $c^k$  becomes

$$\begin{aligned} c^j(t) &= \frac{w^j(t)}{f^j(t)} (x + y + g^j(t)), \\ c^{jk}(t) &= \frac{w^{jk}(t)}{f^k(t) + w^{jk}(t)} (a^{jk}(t) + b^{jk}(t) + x + y + y^{jk}(t) + g^k(t)). \end{aligned} \quad (26)$$

Here, one should note that  $b^{jk}$  and  $y^{jk}$  appear in the first place in the relation for  $c^{jk}$ .

Secondly we consider the first order conditions for the elements of the insurance contract. These conditions are given by the following relation

$$b^{jk} + y^{jk} = \frac{h^{jk}(t) f^k(t)}{f^j(t)} (x + y + g^j(t)) - (x + a^{jk}(t) - c^{jk} + y + g^k(t)). \quad (27)$$

Here, there are several points to make. Firstly, there is no condition on  $b^j$ . Secondly,  $b^{jk}$  and  $y^{jk}$  are not uniquely determined since the first order condition only puts a condition on their the sum. Thirdly,  $c^{jk}$  appears on the right hand side. Thus, we have to calculate the solution of two equations (26) and (27) with two unknowns  $c^{jk}(t)$  and  $b^{jk}(t) + y^{jk}(t)$ . The solution is

$$\begin{aligned} c^{jk}(t) &= \frac{w^{jk}(t)}{f^j(t)} h^{jk}(t) (x + y + g^j(t)), \\ b^{jk} + y^{jk} &= \frac{f^k(t) + w^{jk}(t)}{f^j(t)} h^{jk}(t) (x + y + g^j(t)) \\ &\quad - (a^{jk}(t) + x + y + g^k(t)). \end{aligned}$$

Plugging these control candidates and the value function and its derivatives into the HJB equation leads to the ordinary differential equations for  $f$  and  $g$  presented in (9) and (10). That these ordinary differential equations actually have solutions, see (11) and (12), verifies that the suggested value function is the right one and that the control candidates above are indeed the optimal ones.