

RISK ASSESSMENT OF LIFE INSURANCE CONTRACTS: A COMPARATIVE STUDY IN A LÉVY FRAMEWORK

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ABSTRACT

Common features in life insurance contracts are an interest rate guarantee and policyholder participation in the returns of the insurer's reference portfolio, which can be of substantial value. The aim of this paper is to analyze the model risk involved in pricing and risk assessment that arises from the process specification of the reference portfolio. This is, in general, the most important source of model risk and is analyzed by comparing results from the standard Black-Scholes setting with a Lévy-type model based on a Normal Inverse Gaussian process. We focus on the dependence of the insurer's insolvency risk associated with fair contracts on the specification of the underlying asset process using lower partial moments. We show that a misspecification of the underlying stochastic asset model may not only result in serious mispricing, but also lead to an inadequate assessment of insurers' shortfall risk. Model risk can thus imply substantial solvency risk for insurance companies.

KEYWORDS: Lévy processes, model risk, insolvency risk, risk assessment, participating life insurance contracts.

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1. INTRODUCTION

In recent years, implicit options in life insurance contracts have attracted substantial attention, especially after British Equitable Life had to close to new business in 2000 due to an improper hedging of provided options. Participating life insurance contracts feature embedded options in the form of an interest rate guarantee and participation in the return generated by the insurer's reference portfolio. Requirements concerning proper valuation of these embedded options have been implemented in the context of international accounting standards (e.g., fair value principle in IAS 39) and play a substantial role within regulatory frameworks (e.g., Solvency II in the European Union). Since insurance companies normally cannot or do not follow perfect hedging strategies, they remain at risk. Failure to quantify and handle this risk adequately can endanger an insurer's ability to meet financial obligations and, ultimately, even lead to bankruptcy. In this respect, consideration of model risk is of central importance in practice, e.g., for determination of solvency capital. Generally, the most relevant source of model risk is the asset specification of the reference portfolio, which will have a significant impact on the price and risk of participating contracts and, in particular, the options embedded in them and can thus imply substantial solvency risk for insurance companies. The aim of this study is to analyze this model risk in more detail by comparing the risk of fair contracts in two settings: the standard Black-Scholes framework and a Lévy-type model.

Fair pricing of various types of participating life insurance contracts has been studied by several authors, including Briys and de Varenne (1994), Grosen and Jørgensen (2000, 2002), Hansen and Miltersen (2002), and Ballotta, Haberman, and Wang (2006). All this cited literature builds on the assumption of a geometric Brownian motion model for the dynamics of the asset base.

However, in recent years, the dynamics of financial markets have changed dramatically and the investment strategies of many insurers have become riskier, with the consequence that insurance companies' portfolios exhibit stronger fluctuations. Recently, models based on Lévy processes as the driving noise of the underlying reference portfolio have been applied in the literature to evaluate participating life insurance contracts. These models allow for jumps in asset price paths and skewness (asymmetry) and excess kurtosis (fat tails) in asset price re-

turns, all of which are commonly observed features across almost all financial asset classes.

Ballotta (2005) employs a Merton-style jump-diffusion model to evaluate a participating contract common in the United Kingdom. Kassberger, Kiesel, and Liebmann (2007) generalize Ballotta (2005) by comparing the impact of different Lévy process specifications on the pricing and fair contract parameters for various popular designs of participating life insurance policies. Their analysis revealed significant model risk, as reflected in contract prices. Consideration of risk is an important aspect that is usually omitted in the pricing process, as demonstrated by the above-mentioned articles, even though all types of fair contracts do not necessarily bear the same risk. Gatzert and Kling (2007), for instance, address this problem and compare the risk of fair contracts for different types of participating life insurance policies within a Black-Scholes framework, thus identifying key risk drivers. Consideration of risk will be of particular interest and relevance when the dynamics of the insurer's reference portfolio are based on a Lévy process that allows to even better capture the properties of real-world return distributions.

The aim of this paper is to extend previous literature by analyzing the model risk that arises from the insurer's asset dynamics for the participating life insurance contract design detailed in Ballotta, Haberman, and Wang (2006). This is done by comparing results from the Black-Scholes framework, based on a geometric Brownian motion, with outcomes from an asset returns process using a Lévy process, specifically, a Normal Inverse Gaussian process. The main focus of the paper is the assessment of risk aspects associated with fair contracts under both model specifications. To capture the downside risk, we use lower partial moments as the relevant risk measures, comparing shortfall probability and expected shortfall.

We proceed as follows. First, we assume that the insurance company invests in a reference portfolio that evolves according to a geometric Brownian motion. In this setting, we identify contract parameters that lead to the same market value and measure the corresponding risk under the objective real-world measure. In a second step, the risk of the contracts found in the first step under the Brownian motion setting is recalculated, this time assuming that the reference portfolio follows a Lévy process instead of a geometric Brownian motion, which results in

the contracts no longer being fair. Hence, in a third step we recalibrate the contract parameters to obtain fair contracts under the Lévy model and then, again, calculate the associated risk. By proceeding this way, we can successively and independently identify the effects of the asset model on pricing and risk assessment. Further, we investigate to what extent the results are subject to parameter risk.

The remainder of the paper is organized as follows. In Section 2, the basic model framework for the participating life insurance contract is described and the two asset models to be considered are introduced. Section 3 compares fair contracts under Brownian motion with those found employing Lévy process models. The risk corresponding to fair contracts is analyzed in Section 4. Section 5 contains a sensitivity analysis to assess the parameter risk in both models. We conclude in Section 6.

2. BASIC MODEL FRAMEWORK

The liability structure of the insurance company is implied by participating life insurance contracts and based on a model suggested by Ballotta, Haberman, and Wang (2006).

2.1 The Life Insurance Contract

To initiate the contract, policyholders pay a single premium P_0 . The company's initial equity capital is denoted by E_0 . The sum of the initial contributions $A_0 = E_0 + P_0$ is invested in the reference portfolio. Hence, for $0 < k \leq 1$, it holds that $P_0 = k \cdot A_0$ and $E_0 = (1 - k) \cdot A_0$, where k represents the leverage of the company. The dynamics of the reference portfolio will be discussed in detail in the next section.

Let P denote the policyholder's account, which is the book value of the policy reserves. The policy reserve P is a year-to-year, or cliquet-style, guarantee, i.e., it annually earns the greater of the guaranteed interest rate or a fraction α of the annual surplus generated by the insurer's investment portfolio. For $t = 1, 2, \dots, T$, the development of the policy reserve is given by

$$P(t) = P(t-1) \cdot \left(1 + \max \left[g, \alpha \left(\frac{A(t)}{A(t-1)} - 1 \right) \right] \right),$$

with $P(0) = P_0$. At maturity, a fraction $\delta \in [0,1]$ (δ denotes the terminal surplus participation coefficient) of the terminal surplus $B(T)$ is distributed to the policyholders according to the parameter k .¹ If the company is insolvent at time of maturity, it will not be able to pay the policyholders' claims $P(T)$ in full. In this case, policyholders receive the total value of the reference portfolio at maturity. The value of liabilities $L(T)$ can be summarized as²

$$\begin{aligned} L(T) &= P(T) + \delta [kA(T) - P(T)]^+ - [P(T) - A(T)]^+ \\ &= P(T) + \delta B(T) - D(T), \end{aligned} \tag{1}$$

where $D(T)$ denotes the default put option. The residual claim of the equityholders $E(T)$ is determined as the difference between market value of the reference portfolio $A(T)$ and the policyholders' claim $L(T)$, i.e.,

$$E(T) = A(T) - L(T) = \max(A(T) - P(T), 0) - \delta B(T) \geq 0. \tag{2}$$

2.2 Modeling Asset Prices with the Black-Scholes and Lévy Models

In this section, the traditional Brownian motion based Black-Scholes setting is juxtaposed with a popular representative of the class of Lévy models, namely the Normal Inverse Gaussian (NIG) model.

2.2.1 The Black-Scholes model for asset prices

Let $(W^{\mathbb{P}}(t))$, $0 \leq t \leq T$, be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and (\mathcal{F}_t) , $0 \leq t \leq T$, be the filtration generated by the Brownian motion. In the standard Black-Scholes framework, the total market value of assets A evolves according to a geometric Brownian motion (under the objective measure \mathbb{P})

¹ See also Grosen and Jørgensen (2002).

$$dA(t) = mA(t)dt + \sigma A(t)dW^{\mathbb{P}}(t), \quad (3)$$

with constant asset drift m , volatility σ , and a \mathbb{P} -Brownian motion $W^{\mathbb{P}}$, assuming a complete, perfect, and frictionless market. The solution of the stochastic differential equation in (3) is given by³

$$\begin{aligned} A(t) &= A(0) \cdot \exp\left((m - \sigma^2/2)t + \sigma \cdot W^{\mathbb{P}}(t)\right) \\ &= A(t-1) \cdot \exp\left(m - \sigma^2/2 + \sigma \cdot (W^{\mathbb{P}}(t) - W^{\mathbb{P}}(t-1))\right). \end{aligned} \quad (4)$$

Under the unique equivalent martingale measure \mathbb{Q} , the drift changes to the risk-free interest rate r , and the solution of the stochastic differential equation under \mathbb{Q} , is then obtained analogously to Equation (4):

$$A(t) = A(t-1) \cdot \exp\left(r - \sigma^2/2 + \sigma \cdot (W^{\mathbb{Q}}(t) - W^{\mathbb{Q}}(t-1))\right),$$

where $W^{\mathbb{Q}}$ is a \mathbb{Q} -Brownian motion.

2.2.2 A Lévy model for asset prices

In recent years, exponential Lévy models have become a viable alternative to the geometric Brownian motion for modeling price processes of financial assets. The class of Lévy models encompasses the Brownian model as a special case but, in contrast to the Brownian model, allows for jumps in the price paths, and skewness and excess kurtosis in asset return distributions, and thus takes into account features often observed in real-world asset prices.

A Lévy process is a process with independent and stationary increments that is continuous in probability. Due to these properties, a Lévy process is uniquely defined by one of its marginal distributions. In other words, to define a Lévy process, it is sufficient to state only one marginal distribution. Within the class of Lévy processes, it is convenient to work with a parametric subclass, one of the most popular of which is the Normal Inverse Gaussian class. The Normal Inverse

² Early surrenders or deaths are ignored in this setting. See also Ballotta, Haberman, and Wang (2006).

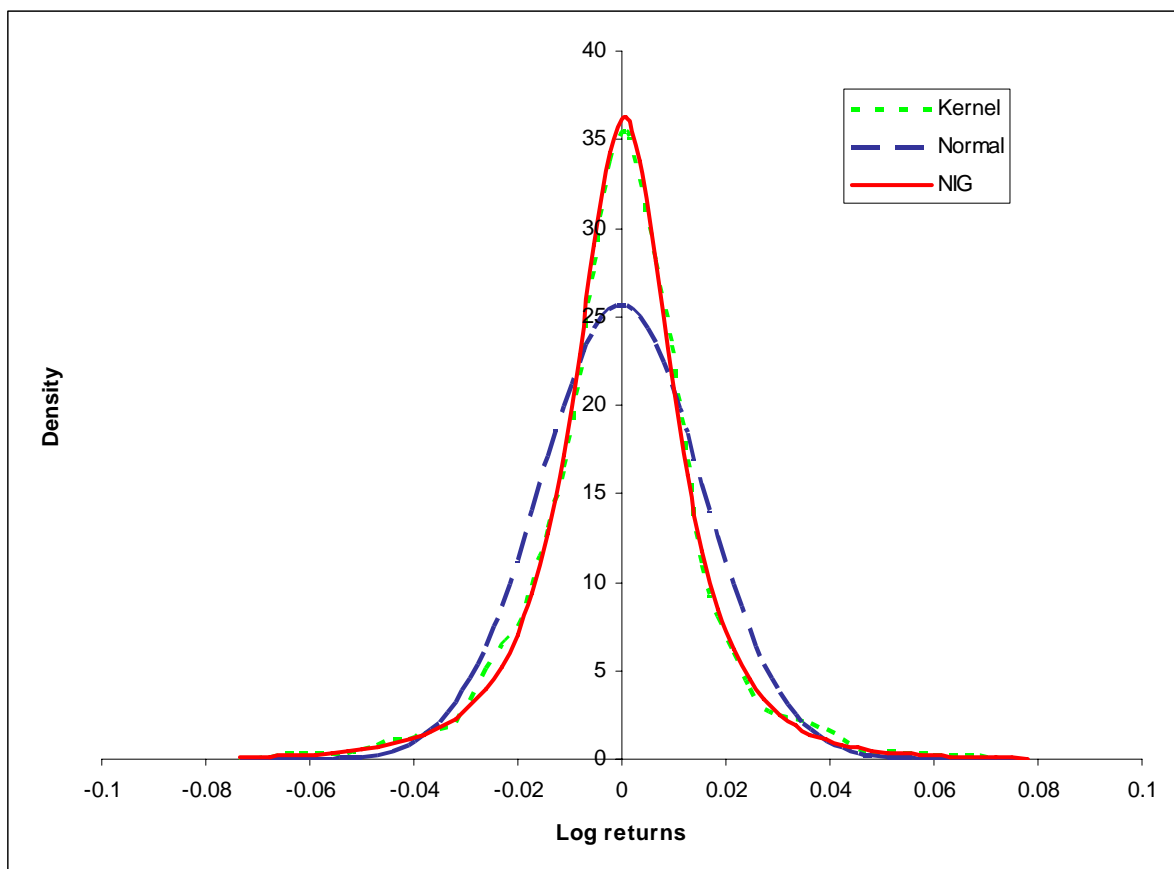
³ For details, see, e.g., Bjørk (2004).

Gaussian distribution $NIG(\alpha, \beta, \delta, \mu)$ was first proposed in a financial context by Barndorff-Nielsen (1997) for modeling log returns of stock prices and has since been used by many other authors. In the insurance context, it was employed in Kassberger, Kiesel, and Liebmann (2007).

To explain and justify the use of the NIG distribution, Figure 1 shows a kernel density estimate of daily EuroStoxx 50 log returns observed between January 2000 and January 2006, and maximum likelihood fits of the NIG and normal distributions. The figure demonstrates that the NIG distribution captures empirical features much better than the normal distribution. The normal distribution has too little mass for returns close to zero, too much mass for medium-sized returns, and too little mass in the tails. Its thin-tailed-ness, i.e., the fact that the normal distribution attributes too little probability to extreme returns, is particularly dangerous, since it can lead to severe underestimation of financial risk.

FIGURE 1

Empirical log returns of EuroStoxx 50 (green curve) and maximum-likelihood fits of NIG (red) and normal (blue) distributions.



The density of a $NIG(\alpha, \beta, \delta, \mu)$ random variable is

$$d_{NIG}(x) = \frac{\alpha\delta}{\pi} \exp\left(\delta\sqrt{\alpha^2 - \beta^2} + \beta(x - \mu)\right) \frac{K_1\left(\alpha\sqrt{\delta^2 + (x - \mu)^2}\right)}{\sqrt{\delta^2 + (x - \mu)^2}}$$

with $\alpha > 0$, $-\alpha < \beta < \alpha$ and $\delta > 0$. Here,

$$K_1(z) = \frac{1}{2} \cdot \int_0^{\infty} \exp\left(-\frac{1}{2} \cdot z \cdot \left(x + \frac{1}{x}\right)\right) dx, \quad z > 0,$$

is the modified Bessel function of the third kind with index 1. The characteristic function of a $NIG(\alpha, \beta, \delta, \mu)$ random variable has the form

$$\phi_{NIG}(u) = \exp\left(\delta\sqrt{\alpha^2 - \beta^2} - \delta\sqrt{\alpha^2 - (\beta + iu)^2} + iu\mu\right). \quad (5)$$

The characteristic function can be used for efficient pricing of standard European options using the Fast Fourier Transform algorithm (see, e.g., Carr and Madan, 1999). Expected value, variance, skewness, and kurtosis of a $NIG(\alpha, \beta, \delta, \mu)$ distributed random variable X are given by

$$\begin{aligned} E(X) &= \mu + \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}} \\ Var(X) &= \frac{\alpha^2\delta}{(\alpha^2 - \beta^2)^{3/2}} \\ Skew(X) &= \frac{E\left[(X - E(X))^3\right]}{Var(X)^{3/2}} = \frac{3\beta}{\alpha\sqrt{\delta\sqrt{\alpha^2 - \beta^2}}} \\ Kurt(X) &= \frac{E\left[(X - E(X))^4\right]}{Var(X)^2} = 3 + 3 \frac{\alpha^2 + 4\beta^2}{\delta\alpha^2\sqrt{\alpha^2 - \beta^2}}. \end{aligned} \quad (6)$$

The parameter α relates to steepness, β to asymmetry, δ to scale, and μ to the location of the density. For $\beta = 0$, $Skew(X)$ becomes zero, and thus the NIG distribution is symmetric.

We will use the corresponding NIG process in our Lévy model and assume that annual log returns $X(t) - X(t-1)$ under the empirical measure \mathbb{P} are $NIG(\alpha, \beta, \delta, \mu)$ distributed. Consequently, the asset price follows an exponential Lévy process with

$$A(t) = A(t-1) \cdot \exp(X(t) - X(t-1)),$$

where $(X(t))$ is a NIG process with $X(t) \sim NIG(\alpha, \beta, \delta t, \mu t)$. In order for the exponential moment $E(\exp(X(t)))$ to exist (which becomes necessary when pricing options involving $A(t)$), we need the additional condition $\alpha \geq \beta + 1$.

Kassberger, Kiesel, and Liebmann (2007) conduct a study on the impact of the type of Lévy process on fair valuation by comparing fair contract values for the NIG process with those derived from the Meixner-process. They find that the actual type of Lévy process used is not of central importance for pricing, as long as it is able to accurately replicate higher moments of empirical log returns (in particular, skewness and kurtosis). For this reason, we limit our considerations to the NIG process, and assume that similar conclusions obtain when using other types of Lévy processes such as variance gamma (see, e.g., Madan and Seneta, 1990) or Meixner processes (see, e.g., Schoutens, 2003).

2.3 Option Pricing in General Lévy Models

Normally, Lévy models lead to incomplete markets with an infinite number of martingale measures \mathbb{Q} . In principle, there are two ways to determine the pricing measure. One can start from the statistical measure \mathbb{P} and determine the equivalent market measure (also called risk-neutral measure) \mathbb{Q} by transforming \mathbb{P} in a suitable fashion. If a statistical measure is to be used, the measure \mathbb{Q} and thus also option prices will depend on the type of transform used. Alternatively, one can directly model under \mathbb{Q} without any explicit reference to the statistical measure \mathbb{P} . This can be done by calibrating the parameters of the driving process

to the prices of options in the reference portfolio, if such prices are observable. Which approach is more suitable depends on the problem and data at hand.

Replacing the Black-Scholes model, and the complete market paradigm underlying it, with a general Lévy model, which in all but trivial cases leads to incomplete markets, might be seen as a serious drawback. However, this is not the case at all. The Black-Scholes model relies on perfect hedging arguments, but insurance companies typically cannot and do not following perfect hedging strategies for participating contracts. In incomplete market situations, there typically is not just one arbitrage-free price, but a whole range of arbitrage-free prices. Thus, as prices become more indicative in nature, rather than firmly known, supplementary information, such as risk figures, becomes all the more important. It is these risk figures that are particularly prone to model error.

2.3.1 The Esscher transform

Since the risk involved in an insurance contract must be calculated under the real-world measure \mathbb{P} , whereas pricing needs to be carried out under a risk-neutral measure \mathbb{Q} , we have to be able to relate these measures. One way to achieve this is the so-called Esscher transform (see Gerber and Shiu, 1994, 1996). A change of a measure \mathbb{P} to an equivalent measure \mathbb{Q} is called an Esscher transform if the measures are related by

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = Z_t^\theta = \frac{\exp(\theta X_t)}{E_{\mathbb{P}}(\exp(\theta X_t))}$$

for some real number θ . \mathbb{Q} is called an Esscher measure. Now θ can be chosen such that the discounted asset price process is a martingale under \mathbb{Q} . Alternatively, one can start with \mathbb{Q} and obtain \mathbb{P} through the formula

$$\left. \frac{d\mathbb{P}}{d\mathbb{Q}} \right|_{\mathcal{F}_t} = \tilde{Z}_t^\theta = \frac{\exp(\theta X_t)}{E_{\mathbb{Q}}(\exp(\theta X_t))}.$$

If, under the real-world measure \mathbb{P} , the annual log returns $X(t) - X(t-1)$ follow a $NIG(\alpha, \beta, \delta, \mu)$ distribution with $\alpha \geq \frac{1}{2}$ and $(r - \mu)^2 \leq \delta^2(2\alpha - 1)$, an

Esscher transform can be performed with parameter (see Kassberger, Kiesel, and Liebmann, 2007)

$$\theta^* = -b - \frac{1}{2} + \frac{r - \mu}{2} \sqrt{\frac{4\alpha^2}{(r - \mu)^2 + \delta^2} - \frac{1}{\delta^2}}.$$

Then, under the corresponding Esscher measure \mathbb{Q} , the discounted asset price process $(A(0) \cdot \exp(-rt + X(t)))$ will be a \mathbb{Q} martingale. The NIG process $(X(t))$ under \mathbb{P} with $X(t) \sim \text{NIG}(\alpha, \beta, \delta t, \mu t)$ will also be a NIG process under \mathbb{Q} with $X(t) \sim \text{NIG}(\alpha, \theta^* + \beta, \delta t, \mu t)$. Thus, the NIG class is closed under the Esscher transform.

2.3.2 Calibration to option prices

Even if the data needed to estimate the real-world distribution of the annual return from a time series are available, these data reflect the past and may not accurately capture the market's opinion on the future. Therefore, it may be more appropriate to try to extract the market's opinion on the pricing measure from option prices, which reflect market participants' expectations of future prices of the underlying. To do so, one can a priori fix a martingale measure \mathbb{Q} , choose a family of distributions such as NIG, and calibrate the parameter set $\alpha, \beta, \delta, \mu$ in order to match market prices of traded options. Details of such a procedure can be found in Schoutens (2003), where it is shown that NIG provides the best fit to S&P500 option prices among those Lévy models with three free parameters. The parameters that give the best fit in the NIG and Brownian frameworks are reported in Table 1; these will be used for our analysis as well (see also Kassberger, Kiesel, and Liebmann, 2007). To obtain a martingale measure, the fourth parameter μ was chosen such that $E_{\mathbb{Q}}[\exp(Y)] = \exp(r)$, assuming the riskless interest rate r to be 3.5% and Y to follow a NIG distribution with parameters as in Table 1. This was done by solving $\phi_{\text{NIG}}(-i) = \exp(r)$ for μ (see Equation (5)).

TABLE 1

Risk-neutral parameters according to Schoutens (2003) and corresponding risk-neutral moments for S&P 500 index.

	NIG	BM
$\alpha_{\text{S\&P}}$	6.1882	-

$\beta_{S\&P}$	-3.8941	-
$\delta_{S\&P}$	0.1622	-
$\mu_{S\&P}$	0.1421	-
std deviation	0.2363	0.1812
skewness	-2.1374	0
kurtosis	12.9374	3

The parameters in Table 1 are the option implied risk-neutral parameters for the S&P 500 index in a NIG model. However, as we want to model a more realistic reference portfolio, that is, one for an insurer that has investments other than and in addition to those in its S&P 500 index portfolio, the reference portfolio is set to consist of 25% S&P 500 index and 75% bonds. For the sake of simplicity, we consider the bonds riskless and not subject to price fluctuations. The standard deviation of this new portfolio is one-fourth of the original one, while skewness and kurtosis remain unchanged, as they are invariant under scaling. The corresponding parameters $(\alpha, \beta, \delta, \mu)$ of the reference portfolio are obtained by applying the following scaling result for NIG distributions: If $X \sim NIG(\bar{\alpha}, \bar{\beta}, \bar{\delta}, \bar{\mu})$, then $cX \sim NIG(\bar{\alpha}/c, \bar{\beta}/c, \bar{\delta}c, \bar{\mu}c)$. In our case, c equals 0.25. The parameters (α, β, δ) for the portfolio follow directly from the above scaling formula and are exhibited in the left column of Table 2. The parameter μ is a weighted average of index and bond returns.

In the case of a portfolio consisting of 100% stocks, as considered in Ballotta, Haberman, and Wang (2006), the standard deviation of the reference portfolio is substantially higher, while skewness and kurtosis remain unchanged. In this setting, the numerical results all remain valid.

TABLE 2

Risk-neutral and real-world parameters and moments for a portfolio with 25% stocks and 75% bonds.

	Q		P	
	NIG	BM	NIG	BM
α	24.7496	-	24.7496	-
β	-15.5734	-	-9.6571	-
δ	0.04055	-	0.04055	-
μ	0.06615	-	0.06615	-

drift	-	0.035	-	0.05
std deviation	0.0591	0.0453	0.0458	0.0453
skewness	-2.1374	0	-1.2177	0
kurtosis	12.9374	3	8.2238	3

Table 2 demonstrates the impact the change of measure has on the moments. Unlike for the Brownian motion, all reported moments of the real-world NIG distribution are lower (in terms of their absolute values) than their risk-neutral counterparts.

For risk measurement purposes, the asset portfolio dynamics under the real-world measure \mathbb{P} are needed. To relate \mathbb{P} and \mathbb{Q} , we assume a real-world drift of $m = 5\%$ and calculate the real-world parameters via an Esscher transformation.⁴ Thus, $E_{\mathbb{P}}[\exp(Y)] = \exp(m)$ needs to be solved for θ^* , where $Y \sim NIG(\alpha, \beta + \theta^*, \delta, \mu)$. β then changes to $\beta^* = \beta + \theta^* = -9.6571$, and the moments also change (see right column in Table 2).

3. FAIR VALUATION OF LIFE INSURANCE LIABILITIES

The aim of this section is to lay the foundation for the risk assessment of fair contracts by investigating the model risk involved in pricing the contracts that occurs due to asset process specifications.

3.1 Fairness condition

Valuation of the claims given in Equation (1) can be conducted using risk-neutral valuation. Hence, the market value (which is calculated under the risk-neutral measure \mathbb{Q}) is

$$\begin{aligned}
 \Pi^* &= E^{\mathbb{Q}}(e^{-r}L(T)) \\
 &= E^{\mathbb{Q}}(e^{-r}P(T)) + E^{\mathbb{Q}}(e^{-r}\delta B(T)) - E^{\mathbb{Q}}(e^{-r}D(T)) \\
 &= \Pi^P + \Pi^B - \Pi^{DPO}.
 \end{aligned} \tag{7}$$

⁴ The exact value of m is of little relevance as long as it remains in a sensible range. The important aspect for the following considerations is that negative skewness and excess kurtosis carry over from \mathbb{P} to \mathbb{Q} .

Equation (7) shows that the value of the policyholder payoff is composed of three parts. The first two terms sum up to the value of the insurance liabilities, i.e., the payoff promised to the policyholder in terms of the policy reserves Π^P and the terminal bonus participation Π^B . The third part is the value of the default put option Π^{DPO} , which reduces the insurance liabilities ($\Pi^P + \Pi^B$) to Π^* . This leads to the implication that companies with higher default risk should charge a lower competitive premium, whereas insurers with lower default risk can charge higher rates. Hence, in the valuation of liabilities, possible insolvency of the company is explicitly taken into account by the default put option.

Several parameters have an impact on the value of the default put option. For instance, with increasing leverage coefficient k less initial equity is available, which raises the risk of default (see Equation (1)). In contrast, distribution of terminal surplus participation is optional and thus does not cause additional risk for the insurer. Insolvency risk is induced only by the guaranteed interest rate and the annual surplus participation.

In a no-arbitrage setting, an up-front premium is regarded as fair if it equals the market value of the liabilities at time $t = 0$, which can be expressed by the following equilibrium condition:

$$\Pi^* = P_0. \tag{8}$$

This requirement is equivalent to the condition that the value of the equityholders' payoff be equal to their initial contribution as all other combinations would imply arbitrage opportunities. We will use Equation (8) to identify parameters of fair contracts by calibrating the guaranteed interest rate g , the annual surplus participation rate α , and the terminal surplus participation rate δ . Unless otherwise stated, the following parameter set is the basis for all analyses:

$$m = 5\%, r = 3.5\%, T = 10, E_0 = 10, P_0 = 100.$$

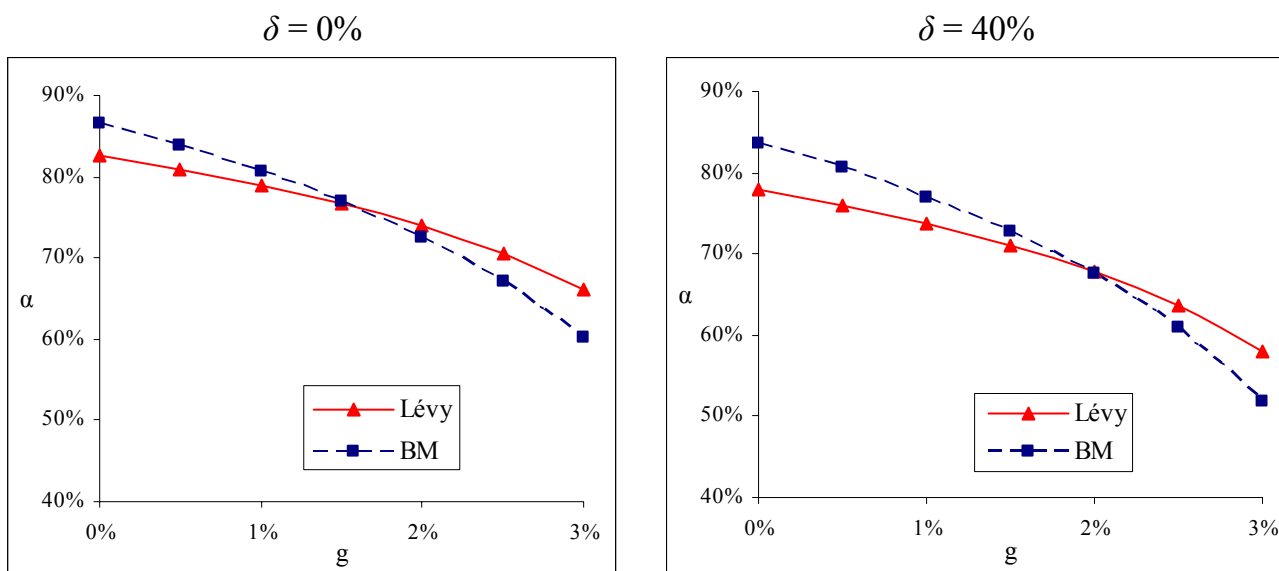
Asset specifications are fixed as shown in Table 2, i.e., parameters under \mathbb{P} are derived from assumptions on the real-world drift m . The numerical results are obtained using Monte Carlo simulation with 200,000 runs.⁵

⁵ See, e.g., Glasserman (2004).

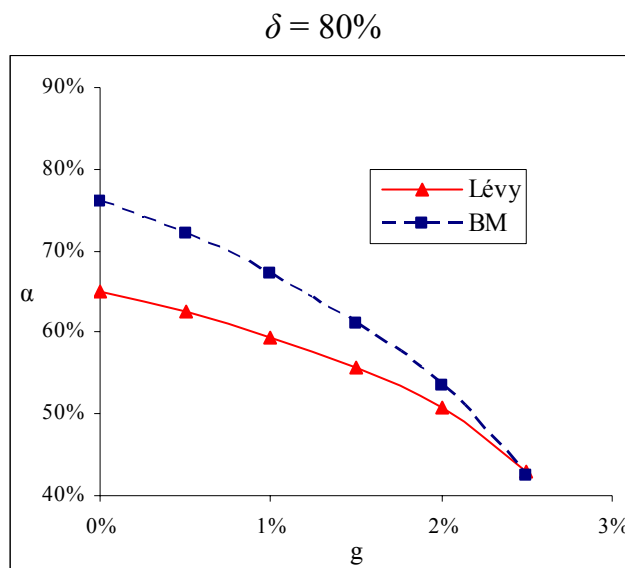
In a first step, we compare fair contracts derived from the Black-Scholes and Lévy models. The guaranteed interest rate g and annual surplus participation rate α are calibrated to satisfy the equilibrium set out in Equation (8) using the Newton method for numerical root finding. Fair parameter combinations (g, α) are displayed in Figure 2 for different choices of terminal surplus participation rate ($\delta = 0\%$, 40% , and 80%) under a Brownian motion (“BM,” dashed blue line in Figure 2) and a NIG process (“Lévy,” solid red line).⁶

FIGURE 2

Fair contracts under Brownian motion and NIG specifications.



⁶ A detailed analysis of feasible sets of individual contract parameters when employing fair valuation within a Black-Scholes framework can be found in Ballotta, Haberman, and Wang (2006).



Notes: g = guaranteed interest rate, α = annual surplus participation, δ = terminal surplus participation, Lévy = under NIG, BM = under Brownian motion.

To ensure the fairness condition, in both models, the annual surplus participation rate decreases as the guaranteed interest rate increases. When no terminal surplus participation is offered ($\delta = 0\%$), the isoquant obtained under the NIG model is above the BM isoquant if $g \geq 1.5\%$, i.e., the insurance company can offer higher annual surplus participation in this case. The isoquants calculated under the two models run near each other and intersect in the range considered. With increasing terminal surplus participation δ , the intersection point is shifted to the right. For example, when $\delta = 80\%$ and $g < 2.5\%$, pricing the contract under the Black-Scholes model leads to higher fair annual participation rates than under the NIG model.

3.2 Decomposition of policyholder claims into building blocks

To investigate the sources of the differences observed in Figure 2, we decompose the market value of the policyholder claims into its building blocks. Table 3 shows the total value of the contract Π^* and its components as given in Equation (7): the value of the policy reserves Π^P , the value of the terminal bonus Π^B , and the default put option value Π^{DPO} . Furthermore, the default-value-to-liability ratio d^* is given, which is a common measure of insurance company solvency.⁷ The default-value-to-liability ratio can be directly derived from the value of the

⁷ Cf., e.g., Butsic (1994) and Barth (2000).

claims in Equation (7); it is defined as the default put option value divided by the expected value of the liabilities and thus permits a comparison of insurance companies of different sizes:

$$d^* = \frac{\Pi^{DPO}}{\Pi^P + \Pi^B}.$$

To analyze how asset model specifications affect the contract components, we proceed in three steps. First, we calculate the component values for fair contracts under the Black-Scholes model as displayed in Figure 2. In the second step, the values of the contracts found in the first step and characterized by parameter combinations that are fair under Black-Scholes are calculated, now assuming that the reference portfolio follows an exponential NIG process instead of a geometric Brownian motion, resulting in the contracts no longer being fair. Hence, in the third step, the annual surplus participation rate α is recalibrated to obtain a fair contract under the Lévy model, as shown in Figure 2, and then, again, the corresponding component values are calculated. Table 3 shows selected results for a guaranteed interest rate of 0.50%, 1.50%, and 2.50% in the case where $\delta = 40\%$.⁸

TABLE 3

Decomposition of fair contract value in building blocks under Brownian motion and NIG specifications for $\delta = 40\%$.

	g	α	Π^*	Π^P	Π^B	Π^{DPO}	d^*
BM	0.50%	80.58%	100.00	99.03	1.25	0.28	0.28%
Lévy not fair		80.58%	101.18	102.47	1.33	2.62	2.52%
Lévy		76.04%	100.00	100.42	1.81	2.23	2.18%
BM	1.50%	72.67%	100.00	99.05	1.52	0.57	0.57%
Lévy not fair		72.67%	100.40	101.42	1.79	2.81	2.73%
Lévy		71.06%	100.00	100.71	1.96	2.67	2.60%
BM	2.50%	60.93%	100.00	99.26	1.85	1.11	1.09%
Lévy not fair		60.93%	99.40	100.05	2.38	3.03	2.95%
Lévy		63.69%	100.00	101.13	2.13	3.26	3.14%

⁸ Similar results are obtained for $\delta = 0\%$ and 80% .

Notes: g = guaranteed interest rate, α = annual surplus participation, δ = terminal surplus participation, Π^ = value of policyholder claims in \$, Π^P = value of policy reserves in \$, Π^B = value of terminal bonus payment in \$, Π^{DPO} = value of default put option in \$, $d^* = \text{default-value-to-liability ratio} = \Pi^{DPO} / (\Pi^P + \Pi^B)$, BM = under Brownian motion, Lévy = under NIG, Lévy not fair = NIG for fair parameter combinations under Brownian motion.*

We look first at Black-Scholes model where fair contracts are calibrated under the Brownian motion setting, but the market exhibits Lévy characteristics (“BM” and “Lévy not fair”). Since the contracts are no longer fair in this case, the market value of the policyholder’s claims Π^* differs from the policyholder’s initial payment of \$100. For $g = 0.50\%$, the market value increases to \$101.18. As the guaranteed interest rate increases, the market value decreases, e.g., for $g = 2.50\%$, Π^* decreases from \$100 to \$99.40, which is less than the initial up-front premium. The difference in the market value of the claims under the Lévy specification may appear minor, but this is an inaccurate observation as the individual contract components are significantly shifted when asset specifications are changed. It is the counterbalancing effects of Π^P , Π^B , and Π^{DPO} that make the change in the total market value appear negligible. This effect was also observed in Ballotta (2005) in a similar context for a different type of contract. In particular, the overall decrease under the Lévy model for $g = 2.50\%$ is mainly caused by a higher default put option value, which even dominates the increase in the terminal bonus component. In the Black-Scholes setting, the values of all contract components are clearly underestimated compared to the Lévy model.

The increase in the default put option also affects the default-value-to-liability ratio. Even though the market value of the claims Π^* is lower for $g = 2.50\%$ when using Lévy processes, the insurer’s insolvency risk increases significantly ($d^* = 1.09\%$ vs. $d^* = 2.95\%$), illustrating that the Black-Scholes model underestimates the insurer’s default risk. Note that the value of the default put option corresponds to the lower partial moment of degree one (termed unconditional expected shortfall).⁹ The use of the risk-neutral measure \mathbb{Q} is appropriate for risk pricing, i.e., for premium calculation; however, measuring the real-world risk of future insolvency in scenario analyses should be conducted under the objective real-world measure \mathbb{P} , which will be done in the next section.

⁹ See Wirch and Hardy (1999).

We now turn to the third situation in Table 3, where fair pricing is completely conducted under Lévy specifications (“Lévy”). For fixed g , the annual surplus participation rate α is adjusted so as to obtain fair contracts. Now, the values of all components Π^P , Π^B , and Π^{DPO} are higher compared to the Black-Scholes scenario, but still result in the same contract value Π^* . The share of the default put option in the value of the policyholder’s payoff is significantly higher (e.g., for $g = 2.50\%$: $d^* = 1.09\%$ vs. $d^* = 3.14\%$).

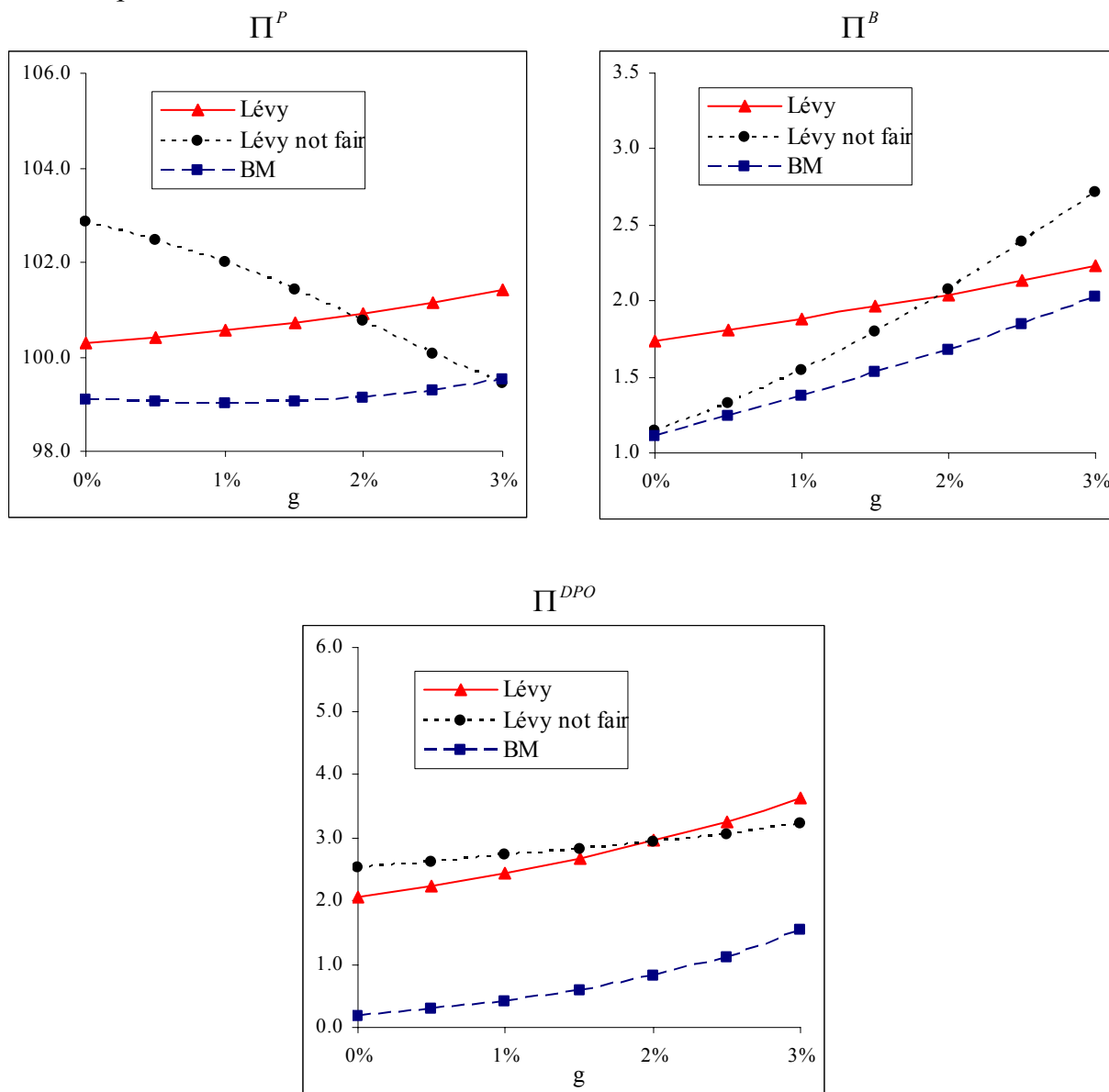
To provide further insight and a broader view, Π^P , Π^B , and Π^{DPO} are graphically displayed in Figure 3 as functions of the guaranteed interest rate for the three situations described in Table 3 (“BM”: dashed blue line; “Lévy not fair”: dashed black line; “Lévy”: solid red line) for $\delta = 40\%$. For every g , different annual surplus participation is provided as given in Figure 2. In particular, the participation rate is reduced as the guaranteed interest rate increases.

It can be seen that even though the isoquants in Figure 2 run near each other and even intersect, the values of the individual components shown in Figure 3 can differ tremendously. In particular, the Black-Scholes model leads to an underpricing of all components Π^P , Π^B , and Π^{DPO} of fair insurance contracts given in Figure 2 compared to the fair Lévy model.

Further, Figure 3 shows that the two curves “BM” and “Lévy not fair” seem to behave oppositely for Π^P and Π^B . In the first graph Π^P , the curves converge as g increases, whereas in the second graph Π^B , they diverge. Moreover, the trend of the curves “BM” and “Lévy” is noticeable. The distance between the curves of the bonus option value Π^B for fair parameters under both Lévy and Black-Scholes is lessened when the guaranteed interest rate increases. In contrast, the distance between the three default put option curves remains fairly stable across changes in contract design.

FIGURE 3

Decomposition of fair contract value in building blocks under Brownian motion and NIG specifications for $\delta = 40\%$.



Notes: g = guaranteed interest rate, δ = terminal surplus participation, Π^P = value of policy reserves in \$, Π^B = value of terminal bonus payment in \$, Π^{DPO} = value of default put option in \$, BM = under Brownian motion, Lévy = under NIG, Lévy not fair = NIG for fair parameter combinations under BM.

In summary, decomposition into individual contract components revealed that considerable model risk is involved in pricing the contract components, even though fair contracts have the same market value under both models and even

though the isoquants of fair contracts appear to run near to each other. The value of the default put option is generally underestimated in the Black-Scholes model, which implies that the insurer's insolvency risk might not be adequately reflected when that model is employed, a topic given more detailed treatment in the next section, where the empirical risk of fair contracts is measured and compared for Brownian motion and Lévy settings.

4. RISK OF FAIR CONTRACTS UNDER BLACK-SCHOLES AND LÉVY PROCESS SPECIFICATIONS

In the following, we consider only European-style contracts and interpret risk solely as possible shortfall at maturity (see, e.g., Grosen and Jørgensen, 2000). A *shortfall* is defined as occurring when the value of assets at maturity does not cover the guaranteed book value of the policy reserves, i.e., $A(T) < P(T)$. In the following analysis, lower partial moments are used to measure insolvency risk.

Lower partial moments belong to the class of downside-risk measures that describe the lower part of the density function; hence only negative deviations are taken into account. Under the objective measure \mathbb{P} , shortfall probability and (unconditional) expected shortfall are, respectively, given by

- shortfall probability: $SP = \mathbb{P}(A(T) < P(T))$,
- expected shortfall: $ES = E^{\mathbb{P}}([P(T) - A(T)]^+)$.

We now calculate the risk that corresponds to the contracts with parameter combinations (g, α, δ) from Figure 2 measured with lower partial moments. Figure 4 depicts the risk of these contracts and shows a comparison of results under Brownian motion and NIG process specifications.

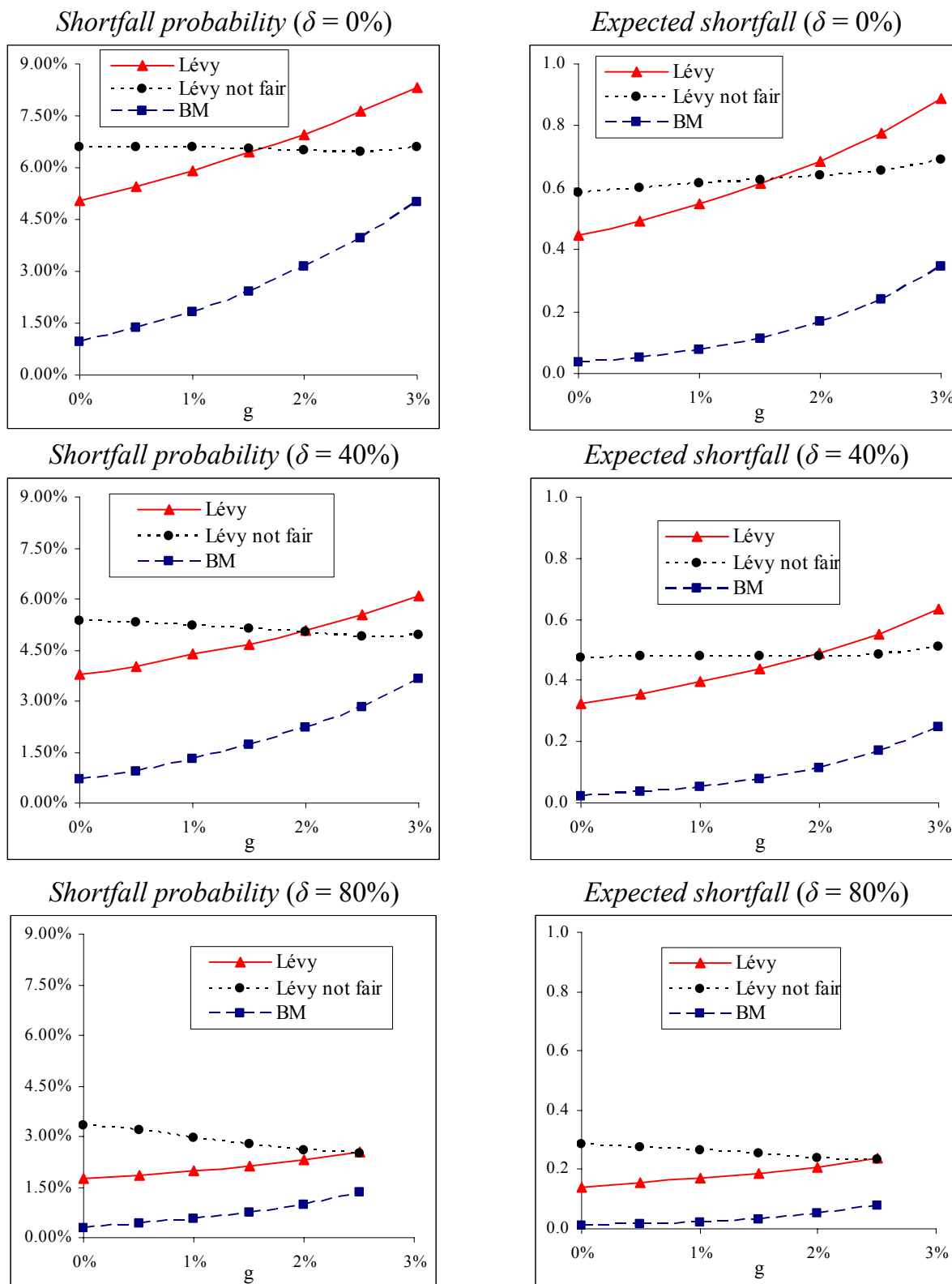
From left to right, the graphs in Figure 4 display shortfall probability and expected shortfall. Shortfall risk is given for the three situations described in Table 3—“BM,” “Lévy not fair,” and “Lévy.” Every point on the curve “BM” depicts the risk of a fair contract as displayed in Figure 2 under the standard Black-Scholes model. Hence, for different levels of g , different annual surplus participation α is provided according to Equation (8). For example, if $\delta = 40\%$, Figure 2 shows that $g = 1.5\%$ implies $\alpha = 72.67\%$ in the Brownian motion case. The

corresponding shortfall probability in the middle graph of Figure 4 (“BM”) is 1.71%. Under NIG specifications, however, these contracts are no longer fair, as was illustrated in Table 3. Hence, every point on the curve “Lévy not fair” corresponds to the risk of the same contract that underlies the “BM” curve. When the asset process actually follows NIG specifications, the risk of a contract with $g = 1.5\%$ and $\alpha = 72.67\%$ is 5.11%. To make the contract fair under the NIG model, the annual surplus participation rate α is lowered to 71.06%, which leads to a corresponding shortfall probability of 4.68% (“Lévy”).

The graphs demonstrate how the risk of fair contracts varies: both shortfall probability and expected shortfall increase in contract with a guaranteed interest rate, despite the simultaneous decrease in the annual surplus participation. This tendency is independent of both the underlying asset process (Brownian motion and NIG process) and the choice of risk measure (shortfall probability and expected shortfall). However, the insurer’s risk level differs tremendously depending on the choice of the underlying asset process, a fact that could lead to serious misestimation in the context of, for example, solvency capital.

Increasing δ , allows the annual surplus participation coefficient to be lowered, which leads to a considerable reduction in risk. This occurs because risk does not depend on δ , only on α . Thus, terminal bonus participation is a key feature in reducing shortfall risk. In the standard Black-Scholes (Lévy) case, for $g = 1.50\%$, shortfall probability is reduced from 1.22% (3.39%) to 0.53% (1.51%) when terminal bonus participation is raised from 40% to 80%.

FIGURE 4: Shortfall probability and expected shortfall of fair contracts in Fig. 2.



Notes: g = guaranteed interest rate, δ = terminal surplus participation, BM = under Brownian motion, Lévy = under NIG, Lévy not fair = NIG for fair parameter combinations under BM.

In general, the distributional characteristics of the underlying asset process will have a stronger impact on the value of the policyholder claims for higher participation rates. For high participation rates, the guaranteed policy reserves at maturity mainly consist of the annual surplus participation, which thus induces a larger influence of the asset process on pricing and risk assessment. With increasing guarantees and decreasing surplus participation, this effect is weakened.

Figure 4 illustrates that the model risk involved in the asset process may be misleading with respect to contract prices and the insurer's actual risk level. Both curves that represent the risk of contracts under the Lévy model ("Lévy" and "Lévy not fair") run above the Brownian-motion-based risk curve, which indicates that insurers may need to account for substantially higher risk.

5. SENSITIVITY ANALYSIS AND PARAMETER RISK

In this section we analyze the robustness of our previous results with respect to changes in input parameters. Starting with the parameters given in the previous sections, we vary the volatility of the insurer's portfolio and initial equity and investigate the effect on fair values and risk figures. In each case, fair contracts and shortfall probabilities are provided under Brownian motion and a NIG process for $\delta = 40\%$.

5.1 Volatility

To see the effect of increasing the asset volatility, the portfolio weights are changed to 50% (previously 25%) in stocks and 50% (previously 75%) in bonds. Following the procedure described in Section 2.3.2 with an adjusted real-world drift $m = 6.5\%$, the risk-neutral and real-world parameters for the new portfolio are obtained (see Table 4).

TABLE 4

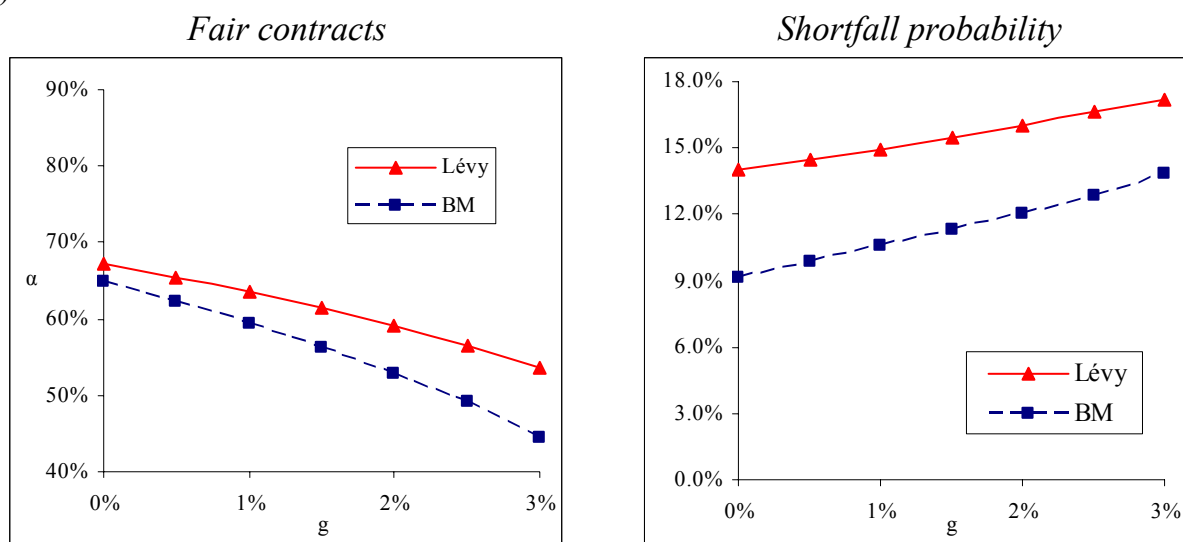
Risk-neutral and real-world parameters and moments for a portfolio with 50% stocks and 50% bonds.

	\mathbb{Q}		\mathbb{P}	
	NIG	BM	NIG	BM
α	12.3763	-	12.3763	-
β	-7.7881	-	-5.8551	-
δ	0.0811	-	0.0811	-
μ	0.0942	-	0.0942	-
drift	-	0.035	-	0.065
std deviation	0.1182	0.0906	0.0979	0.0906
skewness	-2.1375	0	-1.5092	0
kurtosis	12.9374	3	9.4296	3

Figure 5 displays parameter combinations (g, α) of fair contracts and the corresponding shortfall probabilities for the 50/50 portfolio. Compared to the original portfolio in Figure 2, the annual surplus participation rate α is substantially lower for a fixed guaranteed interest rate under both the Lévy and the Brownian model.

FIGURE 5

Fair contracts and corresponding shortfall probability for 50/50 portfolio (Table 4) and $\delta = 40\%$.



Notes: g = guaranteed interest rate, α = annual surplus participation, δ = terminal surplus participation, BM = under Brownian motion, Lévy = under NIG.

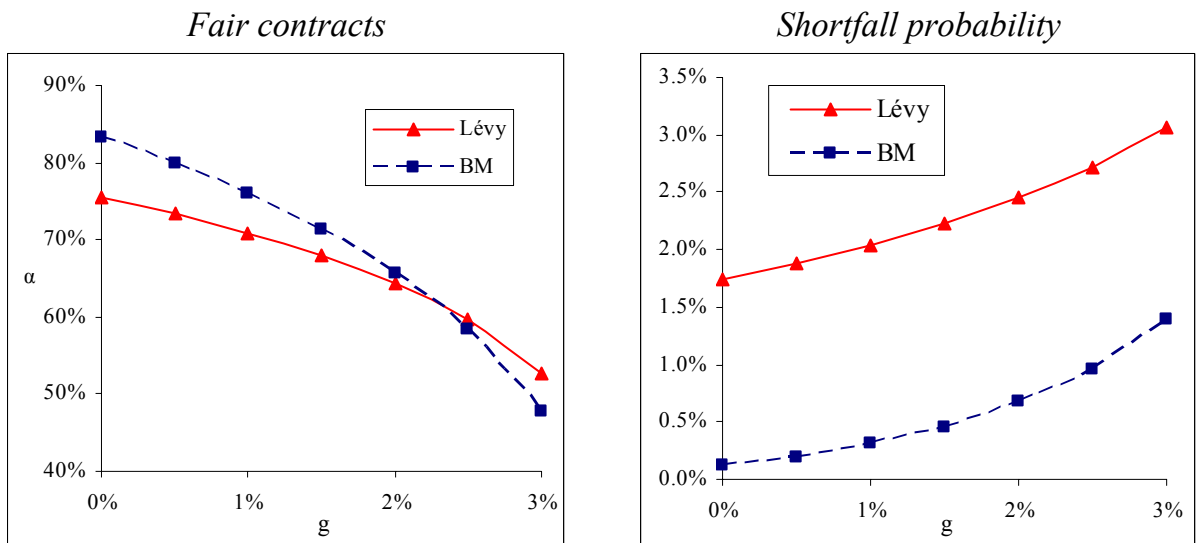
The more volatile the market, the lower the annual surplus participation offered to the policyholder in the case of a fixed guaranteed interest rate. Despite the much lower participation rates, Figure 5 shows that shortfall probability levels change enormously as g increases—by approximately 8 percentage points. The general tendencies are similar to the original case: shortfall probability increases when the guaranteed interest rate increases, despite decreasing participation rates.

5.2 Equity capital

The effect of variations in equity capital is shown in Figure 6. Here, we raise the initial contribution of the equityholders from $E_0 = 10$ to $E_0 = 15$, keeping everything else constant. This implies a decrease in the leverage coefficient from $k = 91\%$ to 87% .

FIGURE 6

Fair contracts and corresponding shortfall probability for $E_0 = 15$ and $\delta = 40\%$.



Notes: g = guaranteed interest rate, α = annual surplus participation, δ = terminal surplus participation, BM = under Brownian motion, Lévy = under NIG.

As illustrated in Figure 6, an increase in equity capital leads to a lower annual surplus participation rate and to a substantial reduction in the shortfall probability. For example, when increasing E_0 to 15 under the Lévy process for $g = 2.5\%$,

α decreases from 63.69% to 59.54%, and the shortfall probability decreases from 5.56% to 2.72%.

6. SUMMARY

Risk-neutral valuation of participating life insurance contracts is often conducted within a Black-Scholes framework, in which the underlying asset is driven by a Brownian motion. We extend the literature by examining the model risk associated with the asset process and compare the effects of the Brownian and Normal Inverse Gaussian settings on the pricing and risk assessment of fair contracts. Our results show that a misspecification in the choice of the asset model can have a substantial impact and that significant model risk is involved in pricing and risk measurement. For instance, shortfall probability and expected shortfall of fair contracts were more than two to three times higher under the Lévy process compared to the Brownian case.

A decomposition of the market value of liabilities into individual building blocks—guaranteed maturity payment including annual bonus, terminal bonus, and default put option value—provided additional insight into the insurer’s actual risk and pricing situation. The analysis revealed that there is already considerable model risk involved in pricing contract components. Even though fair contracts have the same market value under both models and their isoquants under each appear to run very near to each other, the components’ values differ substantially and are generally lower in the Brownian setting. This finding is of particular importance in the context of the default put option, the value of which was several times higher under the NIG process than for Brownian motion.

We further showed that different types of fair contracts can pose substantially different risks to insurers. This result is significant because in the Lévy setting, higher risk needs to be taken into account in pricing fair contracts. This is true for both shortfall probability and expected shortfall and implies that the insurer’s insolvency risk might not be sufficiently reflected when a Brownian motion is used as the driving noise for the asset return process. We found that the guaranteed interest rate is the key risk driver for both models. Additionally, terminal bonus participation is a key feature in reducing risk: promising a higher share in the terminal bonus lowers the guaranteed interest rate and the annual surplus participation, thus reducing risk.

The results are highly dependent on the choice of model input parameters, and further research on estimating input parameters is vital. Overall, the impact of the asset process specifications should not be neglected as considerable model risk is involved both in the pricing and risk assessment of fair participating life insurance contracts. Our findings indicate that insurers should use alternative models instead of, or in addition to, Brownian-motion-based ones for valuation and risk assessment when attempting to determine, e.g., necessary economic capital in order to reduce insolvency risk.

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