Valuation of Bivariate Minimum Guarantees through Option Modelling and Copulas

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Abstract

Pension plans and life insurances offering minimum performance guarantees are very common worldwide. In some markets, for example the brazilian case, besides the minimum guaranteed rate, the costumers of some defined contribution plans have the right to receive, over their savings, the positive difference between the return of a specified investment fund, usually a fixed income fund, and the minimum guaranteed rate, commonly defined as the composition of a fixed interest rate and a floating inflation rate. This instrument can be characterized as an option to exchange one asset, the minimum guaranteed rate, for another, the return of the specified investment fund. In this paper, we provide a closed formula to evaluate this liability that depends on two stochastic rates assuming bivariate normality. We also provide some examples assuming others copulas functions, using Monte Carlo simulation, and compare the effects of the copula and marginals specification in the price of the option. The model makes use of a one-factor Vasicek framework for the term structures of interest rate and inflation rate.

Keywords: Bivariate interest rate options; Copulas; Option to exchange; Minimum guarantees.

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1 Introduction

Pension plans, life insurances and also several financial products offering minimum rates of return guarantees are very common worldwide. In some markets, besides the minimum guaranteed rate, the costumers of some defined contribution plans have the right to receive, over their savings, the positive difference between the return of a specified investment fund, usually a fixed income fund, and the minimum guaranteed rate, commonly defined as the composition of a fixed interest rate and a floating inflation rate. This characteristic may differentiate the products sold in high inflation markets.

The instrument described above can be characterized as an option to exchange one asset, the minimum guaranteed rate, for another, the return of the specified investment fund. Margrabe (1978) was the first author to present closed formulas for the valuation of options to exchange one asset for another. Based on a Black-Scholes framework, the derived formulas depend on the linear correlation between the returns of the underlying assets.

In this paper, we provide a closed formula to evaluate this liability that depends on two stochastic rates. The model makes use of one-factor Vasicek framework for the term structures of interest rate and inflation rate and considers multivariate normality as the joint distribution of these variables.

We also provide some pricing examples assuming others copulas functions, using Monte Carlo simulation, and compare the effects of the copula and marginals specification in the price of the option. For this, we use t-Student copula and t-Student distributions. These distributions are very often applied for the modeling of returns in financial markets. In order to characterize the option's final pay-off, we define for discrete time:

$$C_T = 1.\prod_{t=1}^{T}(1 + IF_t)$$

$$C_T^g = 1.\prod_{t=1}^{T}(1 + Inflation_t).(1 + FR) \tag{1}$$

Where C_T represents the market value at time T of one monetary unit where interest is accrued according to the return in the investment fund (IF). C_T^g represents the market value at time T of one unit of account where interest is accrued according to the inflation rate and a fixed return (FR). Thus, the pay-off of a "call" option to exchange at time T is:

$$O_T^{ex.c} = \beta.(C_T - C_T^g)^+ \tag{2}$$

Where β is the percentage of the excess return the insurer gives to the policyholder ($0 \leq \beta \leq 1$). The pay-off of a "put" option to exchange at time T is:

$$O_T^{ex.p} = (C_T^g - C_T)^+ \tag{3}$$

No matter what happens, the "call" option $O_t^{ex.c}$ can never be worth more than C_t, for every t. Considering $\beta = 1$, if this relationship is not true, in a complete market an arbitrageur could make a riskless profit by buying C_t and selling the option, thus representing an upper bound. For the lower bound, we have to consider two portfolios:

Portfolio 1: one european "call" option to exchange plus C_t^g.
Portfolio 2: C_t.

The value of portfolio 1 in time T is equal to $max(C_T, C_T^g)$. Portfolio 2 is worth C_T. Hence, portfolio 1 is always worth as much as portfolio 2 at maturity. Then: $O_t^{ex.c} + C_t^g \geq C_t$ or $O_t^{ex.c} \geq (C_t - C_t^g)^+$.

The policyholders have the right to withdrawal their savings any time during the life of the contract. This fact characterizes the options as "american"-type ones. In this paper, we derive closed formulas for the "european"-type options. For the "call" option to exchange, the price is the same, as can be derived from the bounds described above, but for the "put" option, the right to make withdrawals at any time represents an additional liability for the insurers.

During the last few years, many authors have applied no-arbitrage pricing theory from financial economics to calculate the value of embedded options in open pension funds and life insurance contracts. Initially, the work was focused on valuing return guarantees embedded in equity-linked insurance policies, see for example, Brennan and Schwartz (1976), Hipp (1996), Boyle and Hardy (1997) and Bacinello and Persson (2002). In equity-linked contracts, the minimum return guarantee can be identified as an equity put-option, and hence the "classical" Black and Scholes (1973) option pricing formula can be used to determine the value of the guarantee.

Among others papers in the literature that deal with the valuation of embedded return guarantees, or participating life contracts, we also refer to Aase and Persson

(1997), Grosen and Jorgensen (1997, 2000, 2002), Pelsser (2003), Vellekoop, Kamp and Post (2006), Coleman, Li and Patron (2006) and Bauer et al. (2006). For the modeling and hedging of guaranteed annuity options (GAO), we refer to Ballotta and Haberman (2003, 2006). Regarding the valuation of minimum guarantees in guaranteed investment contracts (GIC) sold by investment banks, see Walker (1992) and Milevsky and Kim (1997).

Miltersen and Persson (1999) evaluate closed formulas for the pricing of return guarantees in a Heath-Jarrow-Morton framework, but those formulas were based in a one-stochastic-underlying-asset option. Van den Goorbergh et al. (2005) use dynamic copulas for the pricing of bivariate options on two stock indexes. In this paper, we evaluate a closed formula for the pricing of a bivariate option to exchange, with interest and inflation rates as underlying assets. For this, in section 2, we present the models for the term structure of interest and inflation rates. In section 3, we derive a closed formula for this option. In section 4, we provide some numerical examples and sensitivity analysis. In section 5, we present some simulations results using copulas functions for the modeling of the dependence structure and t-Student distributions for the marginals, and finally, in section 6, we conclude the paper.

2 Interest and inflation rates modeling

The uncertainty in the economy is characterized by a filtered probability space (Ω, $(\mathcal{F}_t)_{t>0}$, $\mathcal{F}$, $\mathbb{P}$) satisfying the usual conditions. We also assume the existence of a pricing measure $\mathbf{Q}$ under which discounted bonds prices are martingales. The market is represented by a bi-dimensional Ito process whose dynamics under the risk-neutral equivalent martingale measure $\mathbf{Q}$ are described by the following equations:

$$P(t) = (P_0(t), P_0^g(t)) \quad for \;\; 0 \le t \le T$$

with

$$dP_0(t) = r(t).P_0(t)dt \qquad P_0(0) = 1 \tag{4}$$

$$dP_0^g(t) = g(t).P_0^g(t)dt \qquad P_0^g(0) = 1 \tag{5}$$

On a filtered probability space which supports two standard Brownian motion processes $W_r(t)$ and $W_g(t)$ with correlation coefficient ρ^*. For the short rates, we define the following stochastic differential equations in a Vasicek term structure:

$$dr(t) = \alpha(\eta - r(t))dt + \gamma.dW_r(t) \tag{6}$$

$$dg(t) = \kappa(\theta - g(t))dt + \sigma.dW_g(t) \tag{7}$$

$$dW_r.dW_g = \rho^*.dt$$

For (4)-(7), t denotes the time, W Wiener processes, r the short rate of return, g the short rate of inflation, σ and γ denote the volatilities. From now on, we are going to use the following notation: $x(t) = x_t$.

The equations (6) and (7) are the stochastic processes of the short interest rate and inflation rate, known as Ornstein-Uhlenbeck processes with mean reversion, also used, among others, by Vasicek (1977) and Jamshidian (1989). For the modeling of short interest rates, these processes have a disadvantage because they allow the rates to become negative. This is not a problem for the modeling of inflation rates, since they can assume negative values.

Besides this, according to Duffie (1996), *"Gaussian short-rate models are nevertheless useful, and frequently used, since the probability of negative interest rates within a reasonable short time, with reasonable choices for the coefficient functions, is relatively small. Since any model is only an approximation, there may, therefore, be applications for which it is worth the trouble of having negative interest rates if the tractability that is offered in return is sufficiently great"*. Finally, among the numerous models for inflation proposed in literature, we recall Chan (1998), who models the behavior of the inflation rate directly by means of an Ornstein-Uhlenbeck diffusion process.

If we apply the Ito's lemma for the function $f(t, r_t) = e^{\alpha.t}\alpha.(\eta - r_t)$, we have:

$$df(t, r_t) = d[e^{\alpha.t}\alpha(\eta - r_t)] = -\gamma.\alpha.e^{\alpha.t}dW_r(t) \tag{8}$$

Isolating r_t in equation (8), we get:

$$r_t = r_0.e^{-\alpha.t} + \eta(1 - e^{-\alpha.t}) + \gamma.\int_0^t e^{-\alpha.(t-s)}dW_r(s) \tag{9}$$

Therefore, using Ito isometry, we have:

$$r_t \overset{Q}{\sim} N\Big(r_0.e^{-\alpha.t} + \eta(1 - e^{-\alpha.t}), \frac{\gamma^2}{2\alpha}(1 - e^{-2\alpha.t})\Big) \tag{10}$$

One should note that $P_0(t)$ represents the result of an investment of $\$C_0$ in the short interest rate, during the period of $[0, t]$. In fact, applying Ito's lemma to $ln[P_0(t)]$:

$$P_0(t) = C_t = C_0.exp\Big(\int_0^t r_s ds\Big) \tag{11}$$

The equation (11) represents the accumulation of interest rates over time. In this sense, it can be understood as C_t in continuous time. These calculations are analogous for the inflation rate. Thus, the pay-off of $O_T^{ex.c}$ can be written as:

$$O_T^{ex.c} = \beta.\Bigg(C_t.exp\Big(\int_t^T r_s ds\Big) - C_t^g.exp\Big(\int_t^T g_s ds + (T-t)ln(1+FR)\Big)\Bigg)^+ \tag{12}$$

In order to derive a closed formula for this expression, we must know the distribution of $exp\Big(\int_t^T r_s ds\Big)$ and $exp\Big(\int_t^T g_s ds\Big)$, which is obtained through the **Lemma A** in the Appendix. The variable $exp\Big(\int_t^T r_s ds\Big)$ is normally distributed with mean (n) and variance (k^2) given by:

$$n = n(r_t, (T-t), \alpha, \eta, \gamma) = (T-t)\eta + \frac{1}{\alpha}(r_t - \eta)\big(1 - e^{-\alpha(T-t)}\big)$$

$$k^2 = k^2((T-t), \alpha, \gamma) = \frac{\gamma^2}{2\alpha^3}\big(4e^{-\alpha(T-t)} - e^{-2\alpha(T-t)} + 2\alpha(T-t) - 3\big)$$

Also, according to this lemma, the variable $x(t,T) = \int_t^T g_s ds$ has a Gaussian distribution with mean $m = m(g_t, (T-t), \kappa, \theta, \sigma)$ and variance $j^2 = j^2((T-t), \kappa, \sigma)$ under the probability measure $\mathbf{Q}$. Then:

$$m = m(g_t, (T-t), \kappa, \theta, \sigma) = (T-t)\theta + \frac{1}{\kappa}(g_t - \theta)\big(1 - e^{-\kappa(T-t)}\big)$$

$$j^2 = j^2((T-t), \kappa, \sigma) = \frac{\sigma^2}{2\kappa^3}\big(4e^{-\kappa(T-t)} - e^{-2\kappa(T-t)} + 2\kappa(T-t) - 3\big)$$

Lemma 1 The correlation coefficient between $y(t,T) = \int_t^T r_s ds$ and $x(t,T) = \int_t^T g_s ds$ is equal to ρ, where:

$$\rho = \frac{\frac{\rho^*\sigma.\gamma}{\kappa.\alpha}\Big(\tau + \frac{e^{-\kappa.\tau}-1}{\kappa} + \frac{e^{-\alpha.\tau}-1}{\alpha} - \frac{e^{-(\kappa+\alpha)\tau}-1}{\kappa+\alpha}\Big)}{k.j}$$

Proof. We know that:

$$y(t,T) = \int_t^T r_s ds = \frac{1}{\alpha}(1-e^{-\alpha(T-t)})r_0 + \frac{1}{\alpha}\int_t^T (1-e^{-\alpha(u-t)})\alpha.\eta du + \frac{\gamma}{\alpha}\int_t^T (1-e^{-\alpha(u-t)})dW_r(u)$$

$$x(t,T) = \int_t^T g_s ds = \frac{1}{\kappa}(1-e^{-\kappa(T-t)})g_0 + \frac{1}{\kappa}\int_t^T (1-e^{-\kappa(u-t)})\kappa.\theta du + \frac{\sigma}{\kappa}\int_t^T (1-e^{-\kappa(u-t)})dW_g(u)$$

Then:

$$Cov^Q\big(y(t,T), x(t,T)\big) = E^Q\Bigg(\Big(\frac{\gamma}{\alpha}\int_t^T (1-e^{-\alpha(u-t)})dW_r(u)\Big).\Big(\frac{\sigma}{\kappa}\int_t^T (1-e^{-\kappa(u-t)})dW_g(u)\Big)\Bigg) =$$

$$\frac{\gamma}{\alpha}\frac{\sigma}{\kappa}E^Q\Bigg(\int_t^T (1-e^{-\alpha(u-t)}).(1-e^{-\kappa(u-t)}).\rho^* du\Bigg) = \frac{\rho^*\sigma.\gamma}{\kappa.\alpha}\Big(\tau + \frac{e^{-\kappa.\tau}-1}{\kappa} + \frac{e^{-\alpha.\tau}-1}{\alpha} - \frac{e^{-(\kappa+\alpha)\tau}-1}{\kappa+\alpha}\Big)$$

where $\tau = (T-t)$ $\diamond$

3 Valuation of a closed formula considering bivariate normality

In order to solve the price of the "call" option, we need to find:

$$O_t^{ex.c} = \beta.B(t,T)E^Q\Big(max(C_t e^{y(t,T)} - C_t^g e^{x(t,T)+(T-t)ln(1+FR)}, 0)\Big)$$

according to the standard results of Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983), where $B(t,T)$ is the risk-free discount factor.

$$B(t,T) = e^{-R(t,T).(T-t)}$$

Where $R(t,T)$ is the "market risk free interest rate" for the period $[t,T]$. We know that $y(t,T)$ and $x(t,T)$ are Gaussian. Then:

$$O_t^{ex.c} = \beta.B(t,T)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} max(C_t e^{n+k.z} - C_t^g e^{m+j.w+(T-t)ln(1+FR)}, 0).f^Q(w,z)dwdz$$

Where z and w are standard Gaussian variables. So $f^Q(w,z)$ is the density of a bivariate normal distribution.

$$f^Q(w,z) = \frac{1}{2.\pi.\sqrt{1-\rho^2}}.exp\Big[\frac{1}{2(1-\rho^2)}(z^2 - 2\rho zw + w^2)\Big]$$

Where ρ is the linear correlation coefficient between z and w. Based on these assumptions, we derive the following closed formula for the "call" option.

$$O_t^{ex.c} = \beta.\Big[B(t,T)C_t e^{n+\frac{1}{2}k^2} N(d_1) - B(t,T)C_t^g e^{m+\frac{1}{2}j^2+(T-t)ln(1+FR)} N(d_2)\Big] \quad (13)$$

Where:

$$d_1 = \frac{n - m - ln\big(\frac{C_t^g}{C_t}\big) - (T-t)ln(1+FR) + k^2 - \rho.k.j}{\sqrt{k^2 - 2.\rho.k.j + j^2}}$$

$$d_2 = \frac{n - m - ln\big(\frac{C_t^g}{C_t}\big) - (T-t)ln(1+FR) - j^2 + \rho.k.j}{\sqrt{k^2 - 2.\rho.k.j + j^2}} = d_1 - \sqrt{k^2 - 2.\rho.k.j + j^2}$$

The closed formula for the "put" option is given by:

$$O_t^{ex.p} = B(t,T)C_t^g e^{m+\frac{1}{2}j^2+(T-t)ln(1+FR)} N(d_1^p) - B(t,T)C_t e^{n+\frac{1}{2}k^2} N(d_2^p) \quad (14)$$

Where:

$$d_1^p = \frac{m - n + ln\big(\frac{C_t^g}{C_t}\big) + (T-t)ln(1+FR) + j^2 - \rho.k.j}{\sqrt{k^2 - 2.\rho.k.j + j^2}} = -d_2$$

$$d_2^p = \frac{m - n + ln\big(\frac{C_t^g}{C_t}\big) + (T-t)ln(1+FR) - k^2 + \rho.k.j}{\sqrt{k^2 - 2.\rho.k.j + j^2}} = -d_1$$

From (13) and (14), we derive the following parity between the two options:

$$O_t^{ex.p} + B(t,T)C_t e^{n+\frac{1}{2}k^2} = \frac{1}{\beta}.O_t^{ex.c} + B(t,T)C_t^g e^{m+\frac{1}{2}j^2+(T-t)ln(1+FR)}$$

The parameters n and m can be specified as functions of k and j, respectively, as we show below:

$$B(t,T) = E^Q\Big(e^{-\int_t^T r_s ds}\Big) = e^{-n+\frac{1}{2}k^2} \quad then \quad n = \frac{k^2}{2} - ln\big(B(t,T)\big)$$

Analogously, we can derive a formula for m. Consider $B^G(t,T) = e^{-G(t,T).(T-t)}$, where $G(t,T)$ is the "market inflation rate" for the period $[t,T]$. Thus:

$$B^G(t,T) = E^Q\Big(e^{-\int_t^T g_s ds}\Big) = e^{-m+\frac{1}{2}j^2} \quad then \quad m = \frac{j^2}{2} - ln\big(B^G(t,T)\big)$$

4 Closed formula: results and sensitivity analysis

The results obtained in the previous sections have been used to study the behavior of the options under different scenarios. Throughout the following analysis, unless otherwise stated, the basic set of parameters is:

$$\beta = 1 \quad C_t^g = C_t = 1 \quad T - t = 5 \quad \alpha = 0.30 \quad \gamma = 0.01 \quad \kappa = 0.01 \quad \sigma = 0.02$$
$$\rho^* = 0.00 \quad r_0 = 14.59\% \quad g_0 = 3.30\%$$

The parameters for the pricing of this contingent claim can be estimated in several ways. The values presented above are for illustrative purposes. However, they can be obtained through least squares estimation procedure or even by the application of the Kalman filter to the state space formulation of the model. We refer to Rogers and Stummer (2000) and Babbs and Nowman (1999) for more details about the fitting of Vasicek models.

For the time series of the term structures of interest rates and inflation rates from January/2002 to December/2005, we have got the following values: $\alpha = 0.022$, $\eta = 0.917$, $\gamma = 0.034$, $\kappa = 0.604$, $\theta = 0.140$, $\sigma = 0.167$, $\rho^* = 0.40$ using Kalman filtering. It is important to note that the year 2002 presented a very high volatility for the term structures of inflation rates.

These term structures were based in swaps contracts data, downloaded from the site of the brazilian futures board of exchange[2] and interpolated to the maturities 1 month, 3 months, 6 months, 1 year and 2 years with cubic spline. The chosen inflation index is the IGP-M and the fixed rate of return is 6%. This is the commonly minimum guaranteed rate offered in brazilian open pension plans. The short and risk-free interest rate is the DI, the most used benchmark for fixed income funds. The market interest rate and inflation rate for the maturity of the option are 15.05% and 6.05%, respectively. Then:

$$B(t,T) = \left(\frac{1}{1+15.05\%}\right)^{\left(\frac{1260}{252}\right)} \text{ and } B^G(t,T) = \left(\frac{1}{1+6.05\%}\right)^{\left(\frac{1260}{252}\right)}.$$

The unitary prices of the options under these conditions are 0.11276 and 0.01615 for the "call" and "put" options, respectively. The sensitivity of the "call" and "put" options to the correlation coefficient is shown in Fig. 1 for everything else constant. Figure 1 (first row) shows the patterns of the values of the options "call" and "put"

[2] www.bmf.com.br

decreasing as the correlation parameter, ρ^*, moves from -1 to $+1$. In such direction, d_2 and d_2^p increase relatively faster than d_1 and d_1^p, respectively. Intuitively, the higher is the correlation, more closely will be the paths of the interest and inflation rates, decreasing the possibility of higher payoffs for both options. The first column refers to the "call" option, and the second to the "put" option.

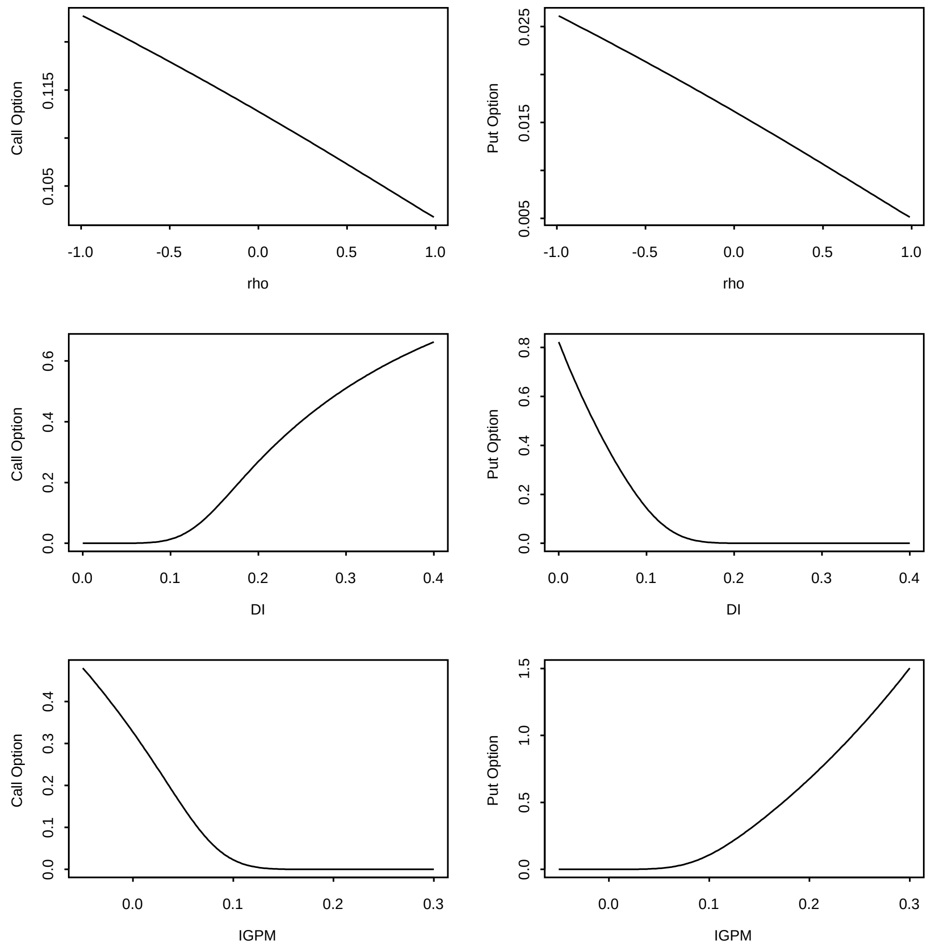

Figure 1. First row: Sensitivity to the linear correlation coefficient. Second row: Sensitivity to the market DI rate for the period (T-t). Third row: Sensitivity to the market IGPM rate for the period (T-t).

On the other hand, if the market value of the interest rate increases, the value of the "call" option increases and the value of the "put" option decreases. The opposite behavior is observed for the market values of the inflation rates. This is obvious because the level of the market interest rate is directly related to the price of the "call" option and inversely related to the price of the "put" option. The market expectation regarding the path of the short interest rate and inflation rate is present in prices of the options.

Figure 2 shows the sensitivity of the values of the "call" and "put" options to movements in the value of the volatilities of the short interest (γ) and inflation (σ) rates for three different correlation coefficients (-0.50, 0.00 and $+0.50$). The first column refers to the "call" option, and the second to the "put" option.

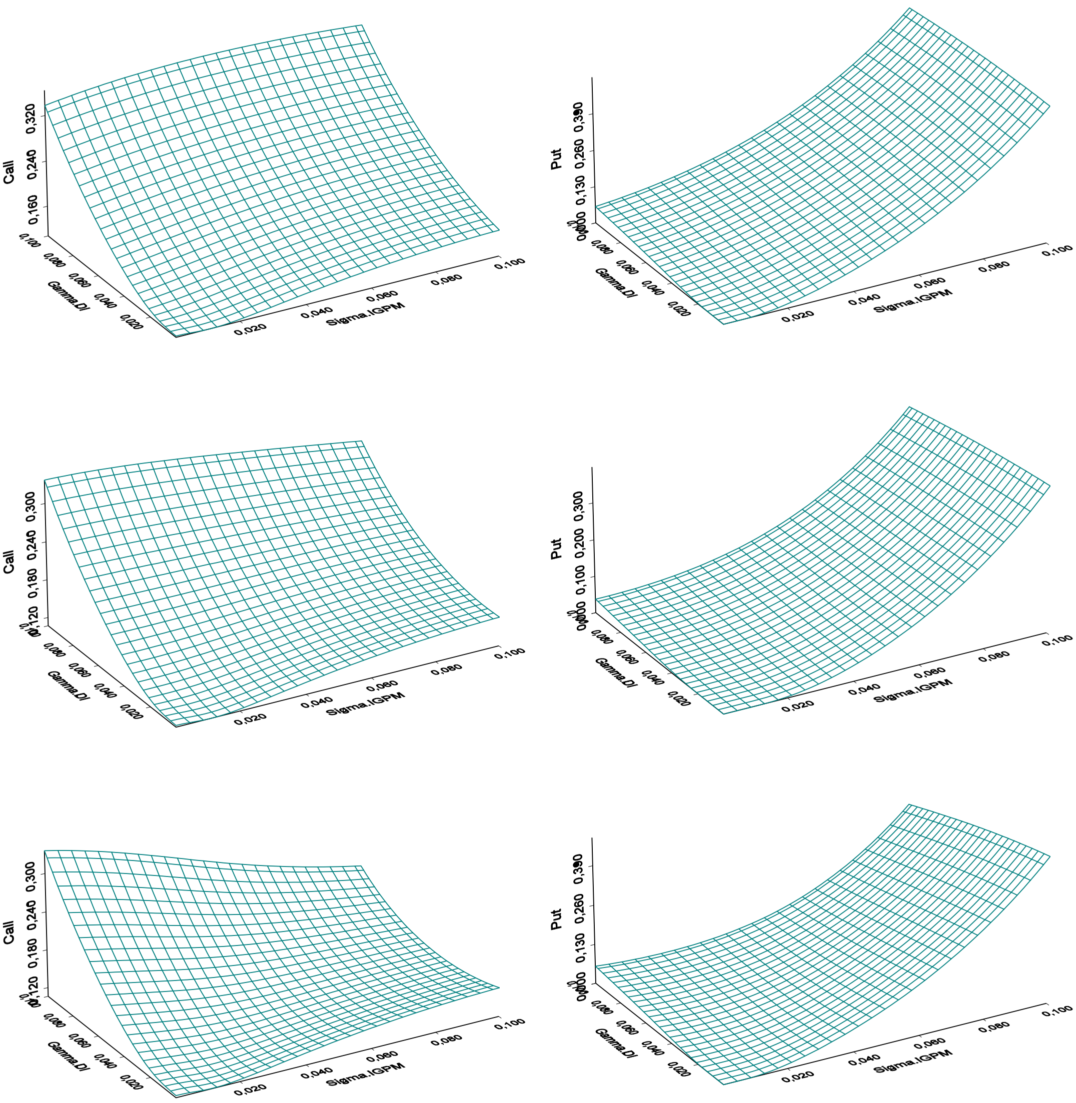

Figure 2. First row: Sensitivity to the volatility of the short interest rate and short inflation rate ($\rho = -0.50$). Second row: Sensitivity to the volatility of the short interest rate and short inflation rate ($\rho = 0.00$). Third row: Sensitivity to the volatility of the short interest rate and short inflation rate ($\rho = +0.50$).

The value of the "call" option increases with the value of the volatility of the short interest rate (DI). Regarding the movements of the price relative to the volatility of the short inflation rate (IGPM), it is shown an ambiguous behavior. If the correlation is negative, an increase in the volatility of the inflation rate will increase the price of the "call" option, since the values of the rates will tend to diverge, rising the possibility of high values of interest rate and low values of inflation rates. When the correlations is positive, the opposite is observed. In this case, the inflation and interest rates will tend to "walk" together, decreasing the value of the "call" option.

When the volatility of the interest rate is very low, it can be considered almost a deterministic rate. In this case, the "call" option can be considered an plain put option over the inflation rate, such that an increase in the value of the volatility will result in an increase in the value of the option.

For the value of the "put" option, it is observed a double-increase behavior. The higher the value of the volatility of the term structure of interest and inflation rates, the higher is the value of the option. Regarding the sensibility with respect to the correlation coefficients, the behavior of the "put" option is different from the one presented by the "call" option, mainly because the "put" option is pretty out of the money with the set of parameters considered.

Figure 3 presents the behavior of the prices of the options according to movements in the parameter of the short interest rate mean reversion speed (first row) and in the short inflation rate mean reversion speed (second row). In the third row, it is presented the sensitivity of the "call" and "put" options to the time to expiration (T-t).

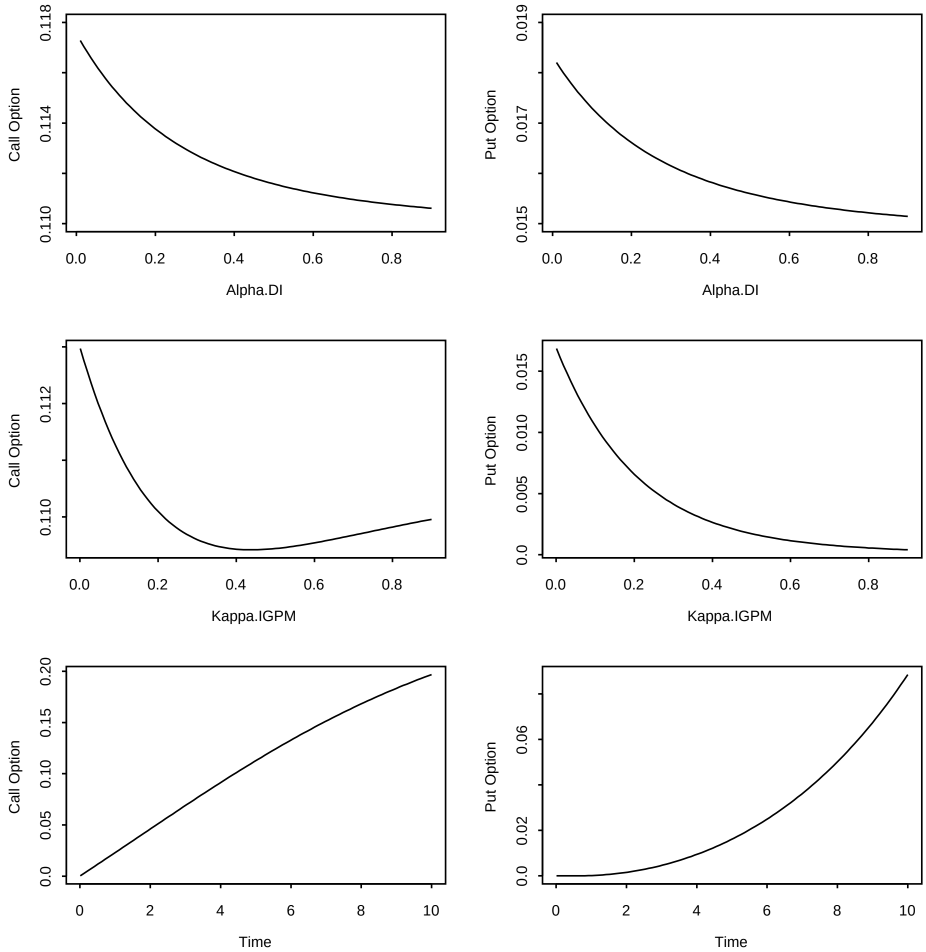

Figure 3. First row: Sensitivity to the mean reversion speed of the short interest rate. Second row: Sensitivity to the mean reversion speed of the short inflation rate. Third row: Sensitivity to the time to expiration (T-t).

The variability of the values of the "call" and "put" options with respect to movements in the value of the mean reversion speeds is quite ambiguous. According to the plots in Figure 3, the higher the short interest rate mean reversion speed, the lower is the price of both options. The bigger the value of the mean reversion speeds, the lower are the values of the variances for $y(t,T)$ and $x(t,T)$ variables. So these plots should be joint analyzed with Figure 2. Most of the cases, lower values for variances have a decreasing impact in the price of the options.

For the short inflation rate mean reversion speed, it is observed a minimal "call" option price near to the value 0.4. This is in line with the behavior of the "call" price shown in Figure 2. For $\rho = 0$, an increase in the value of the mean reversion speed will result in a decrease in the value of the volatility of the inflation rate with a consequent decrease in the price of the option. However, a big decrease in this volatility have a positive impact in the price of the "call" option.

In the third row of Figure 3, we observe an increase in the value of both options when the time to maturity increases. Since the options are European-type ones, it is not possible to state that this behavior will prevail with any set of parameters.

5 Simulations considering others copulas functions

Let $(dW_r; dW_g)$ be a continuous random variable (rv) in $\mathbb{R}^2$ with joint distribution function (cdf) F and margins F_r and F_g. Consider the probability integral transformation of dW_r and dW_g into uniformly distributed rvs on $[0,1]$ (denoted $Uniform(0,1)$), that is, $(U_r; U_g) = (F_r(dW_r); F_g(dW_g))$. The copula C pertaining to F is the joint cdf of $(U_r; U_g)$. As multivariate distributions with $Uniform(0,1)$ margins, copulas provide very convenient models for studying dependence structure with tools that are scale-free.

Consider (x_r, x_g) as a realization of $(dW_r; dW_g)$. As an alternative definition, for every $(x_r, x_g) \in [-\infty, \infty]^2$, consider the point in $[0,1]^3$ with coordinates $(F_r(x_r);$ $F_g(x_g);$ $F(x_r; x_g))$. This mapping from $[0,1]^2$ to $[0,1]$ is a 2-dimensional copula. From Sklar's theorem (Sklar, 1959) we know that for continuous rvs there exists a unique 2-dimensional copula C such that for all $(x_r, x_g) \in [-\infty, \infty]^2$,

$$F(x_r; x_g) = C(F_r(x_r); F_g(x_g)) \tag{15}$$

In order to measure (upper) tail dependence one may use the upper tail dependence coefficient defined as:

$$\lambda_U = \lim_{\alpha \to 0^+} \lambda_U(\alpha) = \lim_{\alpha \to 0^+} .Pr\big(dW_r > F_r^{-1}(1-\alpha) \mid dW_g > F_g^{-1}(1-\alpha)\big)$$

If this limit exists, and where F_i^{-1} is the generalized inverse of F_r or F_g, i.e., $F_i^{-1}(u_i) = sup\{x_i | F_i(x_i) \leq u_i\}$, for $i = r, g$. The lower tail dependence coefficient λ_L is defined in a similar way. Both the upper and the lower tail dependence coefficients may be expressed using the pertaining copula:

$$\lambda_U = \lim_{u \to 1^-} \frac{\overline{C}(u,u)}{1-u}, \ \ where \ \overline{C}(u_r, u_g) = Pr\big(U_r > u_r, U_g > u_g\big) \ and \ \lambda_L = \lim_{u \to 0^+} \frac{C(u,u)}{u}$$

If these limits exist. The measures $\lambda_U \in (0,1]$ (or $\lambda_L \in (0,1]$) quantify the amount of extreme dependence within the class of asymptotically dependent distributions. If $\lambda_U =$

$0(\lambda_L = 0)$, the two variables dW_r and dW_g are said to be asymptotically independent in the upper (lower) tail.

Considering the copula C belonging to the parametric family $\{C_\theta, \theta \in \Theta\}$, the model estimation may be carried on by the maximum likelihood method in two steps: firstly, one estimates the marginal parameters; secondly, one carries on the copula fit, the so called IFM method (Inference for Margins). There are mainly two versions: the fully parametric and the semiparametric approaches (also called CML method, Canonical Maximum Likelihood), detailed in Genest et al. (1995) and Joe and Xu (1996). The fully parametric approach relies on the assumption of parametric marginal distributions. The $Uniform(0,1)$ data, obtained from the estimated marginals, are used to maximize the copula density function with respect to θ. For the CML method, it is used the empirical distribution of the marginals, and then, the θ parameter is obtained through maximum likelihood.

For the derivation of the closed formula in section 3, we assumed a Gaussian copula for the modeling of the dependence structure among dW_r and dW_g. Another possible copula function to be used in this case is the T-Student copula. As main characteristic, depending on the number of degrees of freedom, this copula presents upper and lower tail dependence. The formulas of these two copulas are described in the Appendix.

One should note that the copulas used in this paper are the risk neutral copulas or copulas in the **Q**-measure. For the reader interested in this issue, we refer to Coutant et al. (2001). The authors show that in a Black-Scholes world if the drift, volatility and market price of risk parameters are non-stochastic, the risk neutral copula C^Q and the objective copula C^P are the same. Specifically, for the Vasicek model, the sufficient condition for this is the market price of risk be non-stochastic. Another important result is that the margins of the risk neutral joint distribution are necessarily the univariate risk neutral distributions.

We carried on four simulation experiments with different correlation coefficients in the S-Plus software. Through these experiments it is possible to evaluate the effect of the specification of the marginals and the dependence structure in the price of the options. Many times in the financial markets it is observed the extreme dependence among the variables. This behavior is not modeled by a multivariate normality distribution, as the one assumed in section 3. In these cases, one should use a copula function that presents this characteristic, such as the t-Student copula. Another common situation observed in financial markets is the misspecification of the marginals when normality is assumed. Heavy tails are better modeled by t-Student distributions than Gaussian distributions. In this section, we will test these possibilities in the price of the options.

For the four simulation experiments we used the same set of parameters presented in section 4. In all experiments, we simulated the stochastic differential equations (6) and (7). Through the set of parameters given in section 4, it is possible to derive η (0.134) and θ (1.144). Using the generated short rates, we simulated the two savings accounts, one accrued by the short interest rate, and the other accrued by the short inflation rate plus the fixed rate of return. The price of each option is the mean of the final pay-off simulated discounted by the market interest rate for the maturity of the option.

For all simulations experiments, we consider three different correlation coefficients (-0.50, 0.00 and 0.50). In the first experiment, we simulated a Gaussian copula and Gaussian marginals. This is the assumption of the closed formula derived in section 3. As a base for comparisons, we also evaluated the price of the call and put options using the closed formula.

In the second experiment, we also simulated from a Gaussian copula, but, in this case, we used t-Student marginals with different degrees of freedom (4, 6 and 8). In the third experiment, we simulated from a t-Student copula, considering different degrees of freedom (4, 6 and 8), with Gaussian marginals. Finally, in the fourth experiment, we simulated from a t-Student copula, considering 6 degrees of freedom, with t-Student marginals with different degrees of freedom (4, 6 and 8). The number of simulations was 1000 runs. The results are presented in Table 1.

Table 1: *Prices of the call (put) options obtained through the closed formula and the four simulation experiments considering different Pearson correlation coefficients, Gaussian and t-Student marginals and Gaussian and t-Student copulas. Pearson correlation coefficient $\rho^* = -0.50$, $\rho^* = 0.00$, $\rho^* = 0.50$.*

Copula	Marginals	$\rho^* = -0.50$	$\rho^* = 0.00$	$\rho^* = 0.50$
Closed Formula		0.1179 (0.0213)	0.1128 (0.0162)	0.1073 (0.0107)
Gaussian	Gaussian	0.1179 (0.0225)	0.1130 (0.0156)	0.1066 (0.0109)
Gaussian	t-Student ($\nu = 4$)	0.1166 (0.0220)	0.1143 (0.0211)	0.1144 (0.0199)
Gaussian	t-Student ($\nu = 6$)	0.1137 (0.0149)	0.1109 (0.0165)	0.1107 (0.0168)
Gaussian	t-Student ($\nu = 8$)	0.1046 (0.0140)	0.1038 (0.0112)	0.1040 (0.0123)
t-Student ($\nu = 4$)	Gaussian	0.1339 (0.0343)	0.1289 (0.0339)	0.1274 (0.0352)
t-Student ($\nu = 6$)	Gaussian	0.1172 (0.0242)	0.1211 (0.0276)	0.1162 (0.0312)
t-Student ($\nu = 8$)	Gaussian	0.1094 (0.0234)	0.1136 (0.0201)	0.1119 (0.0184)
t-Student ($\nu = 6$)	t-Student ($\nu = 4$)	0.2334 (0.2307)	0.1582 (0.0991)	0.1418 (0.0580)
t-Student ($\nu = 6$)	t-Student ($\nu = 6$)	0.2274 (0.1528)	0.1558 (0.0772)	0.1359 (0.0504)
t-Student ($\nu = 6$)	t-Student ($\nu = 8$)	0.1853 (0.1378)	0.1369 (0.0650)	0.1289 (0.0450)

As expected the values obtained through the use of a Gaussian copula with Gaussian marginals are very similar to the ones provided by the closed formula. The results do not change so much with the use of t-Student marginals. However, when we introduced t-Student copulas, it is observed a jump in the values of the call and put options for the three levels of correlation coefficients. This jump is even stronger when the t-Student copulas are used with t-Student marginals.

This result can be explained by the upper and lower tail extreme dependence presented by t-Student copulas with a low number of degrees of freedom, as the ones used in the simulation experiment. One should note that the correct specification of such distributions can have a huge impact in the valuation of these insurance liabilities. Another point that should be considered is the number of simulations for the case which it's used t-Student copulas with t-Student marginals. In such cases, the variability of the generated samples can become big, requiring variance reduction techniques.

6 Concluding Remarks

Open pension plans offering minimum performance guarantees are very common worldwide. In Brazil, besides the minimum guaranteed rate, the costumers of some defined contribution plans have the right to receive, over their savings, the positive difference between the return of a specified investment fund and the minimum guaranteed rate, commonly defined as the composition of a fixed interest rate and a floating inflation rate. This instrument can be characterized as an option to exchange one asset, the minimum guaranteed rate, for another, the return of the specified investment fund. This kind of guarantee can be very usual in high inflation markets. In this paper, we provided a closed formula to evaluate this liability.

The model makes use of a one-factor Vasicek framework for the term structures of interest rate and inflation rate. Thus, multivariate normality is assumed. Some adjustments in the specification of the multivariate distribution can be made in order to try to find a closed formula. For this purpose, it can be used copulas functions for the modeling of the dependence structure among the short interest and inflation rates. Along with the specification of copulas, it is also possible to specify different margins for the short rate processes, separately. The challenge in these cases is to solve the integrals and find a closed and friendly formula.

That is the reason why we used simulations for the pricing of the options when we assumed t-Student marginals or t-Student copulas. These distributions present some interesting characteristics such as good fit of heavy tail data and upper and lower tail extreme dependence, respectively. The results shown in Table 1 indicate the importance of the correct specification of the dependence structure among the short rate processes as well as the specification of their marginals. The values of these options suffer a big impact when t-Student copulas are introduced in conjunction with t-Student marginals.

We believe that our analysis is relevant for life insurance and pension funds managers since many real contracts include guarantees similar to the ones treated here, specially in Brazil. The obtained closed formula is easy to implement and can be useful for the design of dynamic hedging strategies. Apparently, current practice among life insurance companies and pension funds does not involve the calculation of explicit market values of such guarantees.

Appendices

Appendix 1 - *Lemma.* The variable $y(t,T) = \int_t^T r_s ds$ has a Gaussian distribution with mean $n = n(r_t, (T-t), \alpha, \eta, \gamma)$ and variance $k^2 = k^2((T-t), \alpha, \gamma)$ under the probability measure $\mathbf{Q}$. Then:

$$y(t,T) \overset{Q}{\sim} N\Big(n, k^2\Big)$$

$$n = n(r_t, (T-t), \alpha, \eta, \gamma) = (T-t)\eta + \frac{1}{\alpha}(r_t - \eta)\big(1 - e^{-\alpha(T-t)}\big)$$

$$k^2 = k^2((T-t), \alpha, \gamma) = \frac{\gamma^2}{2\alpha^3}\big(4e^{-\alpha(T-t)} - e^{-2\alpha(T-t)} + 2\alpha(T-t) - 3\big)$$

Proof. We know from (10):

$$r_t \overset{Q}{\sim} N\Big(r_0.e^{-\alpha.t} + \eta(1 - e^{-\alpha.t}), \frac{\gamma^2}{2\alpha}(1 - e^{-2\alpha.t})\Big)$$

Now, we need to find the mean and the variance of $y(t,T) = \int_t^T r_s ds$. Then:

$$E^Q\Big(\int_t^T r_s ds\Big) = \int_t^T E^Q(r_s)ds = \int_t^T \big(e^{-\alpha.(s-t)}r_t + \eta(1 - e^{-\alpha.(s-t)})\big)ds =$$

$$(T-t)\eta + \frac{1}{\alpha}(r_t - \eta)\big(1 - e^{-\alpha(T-t)}\big) = n(r_t, (T-t), \alpha, \eta, \gamma)$$

$$Var^Q\Big(\int_t^T r_s ds\Big) = E^Q\Bigg(\Big(\int_t^T r_s ds - n\Big)^2\Bigg) = E^Q\Bigg(\Big(\int_t^T \gamma \int_t^s e^{\alpha(u-s)} dW_r(u) ds\Big)^2\Bigg) =$$

$$\gamma^2 E^Q\Bigg(\bigg(\int_t^T\int_u^T e^{\alpha(u-s)}dsdW_r(u)\bigg)^2\Bigg) = \gamma^2\int_t^T\bigg(\int_u^T e^{\alpha(u-s)}ds\bigg)^2 du =$$

$$\frac{\gamma^2}{2\alpha^3}\big(4e^{-\alpha(T-t)} - e^{-2\alpha(T-t)} + 2\alpha(T-t) - 3\big) = k^2((T-t),\alpha,\gamma) \qquad \diamond$$

Appendix 2 - *Copulas formulas*

Gaussian Copula. It is an Elliptical Copula, which are simply the copulas pertaining to elliptical distributions. The Gaussian or Normal copula is the copula pertaining to the multivariate normal distribution. It is given by

$$C_\rho^{Ga}(u,v) = \int_{-\infty}^{\phi^{-1}(u)}\int_{-\infty}^{\phi^{-1}(v)} \frac{1}{2\pi(1-\rho^2)^{1/2}}\exp\Big(-\frac{s^2-2\rho st+t^2}{2(1-\rho^2)}\Big)dsdt$$

where ρ is simply the linear correlation coefficient between the two random variables.

T-Student Copula. The copula of the multivariate standardized t-Student distribution is the t-Student copula and is defined as follows:

$$C_{\rho,\nu}^{Stu}(u,v) = \int_{-\infty}^{t_\nu^{-1}(u)}\int_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi(1-\rho^2)^{1/2}}\exp\Big(1+\frac{r^2-2\rho rs+s^2}{2(1-\rho^2)}\Big)drds$$

where ρ is the linear correlation coefficient between the two random variables, and ν is the number of degrees of freedom.

7 References

Aase, K., Persson, S., 1997. Valuation of the minimum guaranteed return embedded in life insurance products. *Journal of Risk and Insurance* 64 (4), 599-617.

Babbs, S. H., Nowman, K. B., 1999. Kalman filtering of Generalized Vasicek Term Structure Models.*The Journal of Financial and Quantitative Analysis* 34 (1), 115-130.

Bacinello, A., Persson, S., 2002. Design and pricing of equity-linked life insurance under stochastic interest rates. *Journal of Risk Finance* 3 (2), 6-21.

Ballotta, L., Haberman, S., 2003. Valuation of guaranteed annuity conversion options. *Insurance: Mathematics and Economics* 33, 87-108.

Ballotta, L., Haberman, S., 2006. The Fair Valuation Problem of Guaranteed Annuity Options: the Stochastic Mortality Environment Case. *Insurance: Mathematics and Economics* 33, 87-108.

Bauer, D., Kiesel, R., Kling, A., Rub, J., 2006. Risk-neutral valuation of participating life insurance contracts. *Insurance: Mathematics and Economics* 39, 171-183.

Black, F., Scholes, M., 1973. The pricing of options and corporate liabilities. *Journal of Political Economy* 81 (3), 637-654.

Boyle, P., Hardy, M., 1997. Reserving for maturity guarantees: two approaches. *Insurance: Mathematics and Economics* 21, 113-127.

Brennan, M.J., Schwartz, E.S., 1976. The pricing of equity-linked life insurance policies with an asset value guarantee. *Journal of Financial Economics* 3, 195-213.

Chan, T., 1998. Some applications of Lvy processes to stochastic investment models for actuarial use. *Astin Bulletin* 28, 77-93.

Coleman, T. F., Li, Y., Patron, M. C., 2006. Hedging guarantees in variable annuities under both equity and interest rate risks. *Insurance: Mathematics and Economics* 38, 215-228.

Coutant, S., Durrleman, V., Rapuch, G., Roncalli, T., 2001. Copulas, multivariate risk-neutral distributions and implied dependence functions. Working paper. Groupe de Recherche Operationnelle, Credit Lyonnais.

Duffie, D., 1996, *Dynamic Asset Pricing Theory*, second edition, Princeton University Press: New Jersey.

Genest, C., Ghoudi, K., Rivest, L.-P. 1995. "A semiparametric estimation procedure of dependence parameters in multivariate families of distributions". *Biometrika* 82, 543-552.

Grosen, A., Jrgensen, P.L., 1997. Valuation of early exercisable interest rate guarantees. *Journal of Risk and Insurance* 64, 481-503.

Harrison, J.M., Kreps, D.M., 1979. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory* 20, 381-408.

Harrison, J.M., Pliska, S.R., 1981. Martingales and Stochastic Integrals in the Theory of Continuous Trading. *Stochastic Processes and their Applications* 11, 215-260. Addendum: Harrison and Pliska (1983).

Harrison, J.M., Pliska, S.R., 1983. A stochastic calculus model of continuous trading; complete markets. *Stochastic Processes and their Applications* 15, 313-316.

Hipp, C., 1996. Options for Guaranteed Index-Linked Life Insurance in: *Proceedings from AFIR (Actuarial Approach for Financial Risks).* Sixth International Colloquium, vol. 2, Nrnberg, Germany, pp. 1463-1483.

Jamshidian, F., 1989. An exact bond option formula. *Journal of Finance* 44, 205-209.

Joe, H. and Xu, J. 1996. The estimation method of inference function for margins for multivariate models. Technical Report no.166. Vancouver: Univ. of British Columbia, Department of Statistics.

Margrabe, W., 1978. The value of an option to exchange one asset for another. *Journal of Finance* 33, 177-186.

Milevsky, M. A., Kim, Sharon, 1997. The Optimal Choice of Index-Linked GICs: Some Canadian Evidence. *Financial Services Review* 6(4), p. 271-284.

Miltersen, K. R., Persson, S. A., 1999. Pricing rate of return guarantees in a Heath-Jarrow-Morton framework. *Insurance: Mathematics and Economics* 25, 307-25.

Pelsser, A., 2003. Pricing and hedging guaranteed annuity options via static option replication. *Insurance: Mathematics and Economics* 33, 283-296.

Rogers, L. C. G., Stummer, W., 2000. Consistent fitting of one-factor models to interest rate data. *Insurance: Mathematics and Economics* 27, 45-63.

Sklar, A. 1959. Fonctions de rpartition n dimensions et leurs marges. Publ. Inst. Statist. Univ. Paris 8, 229-231.

van den Goorbergh, R., Genest, C., Werker, B., 2005. Bivariate option pricing using dynamic copula models. *Insurance: Mathematics and Economics* 37, 101-114.

Vasicek, O., 1977. An equilibrium characterization of the term structure. *Journal of Financial Economics* 5, 177-188.

Vellekoop, M. H., Kamp, A. A. Vd., Post, B. A., 2006. Pricing and hedging guaranteed returns on mix funds. *Insurance: Mathematics and Economics* 38, 585-598.

Walker, K.L., 1992. *Guaranteed Investment Contracts: Risk Analysis and Portfolio Strategies*, Richard D. Irwin, Homewood, Illinois 60430, USA.