Indifference pricing of a life insurance portfolio with systematic mortality risk in a market with an asset driven by a Lévy process *

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Abstract

In this paper we investigate the problem of pricing and hedging of life insurance liabilities. We consider a financial market consisting of a risk-free asset, with constant rate of return, and a risky asset, which price is driven by a Lévy process. We take into account systematic mortality risk and model mortality intensity as a diffusion process. The principle of equivalent utility is chosen as the valuation rule. In order to solve our optimization problems we apply techniques from the stochastic control theory. An exponential utility is considered in details. We arrive at three pricing equations and investigate some properties of the obtained premiums. The estimate of the finite-time ruin probability is derived. The indifference pricing with respect to a quadratic loss function is also shortly discussed.

Key words: Lévy process, stochastic mortality, counting process, Hamilton-Jacobi-Bellman equation, Lundberg's inequality.

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1 Introduction

It is well-known that the price of any life insurance or pension product depends on demographic and financial assumptions. Traditional actuarial pricing principles, see [1,2], state that an actuary should set prudent (deterministic) estimates of future mortality rates and investment returns. However, insurance practice has shown that the accurate, rather than prudent, estimates of the future trends in mortality rates and investment returns are required. As the precise forecasts are not possible one should develop and apply probabilistic models to arrive at the best estimates.

In this paper we investigate the problem of pricing and hedging of life insurance liabilities in the case when an insurer can trade dynamically in a financial market. It is well-known, see for example [3,4], that the combine insurance and financial market is incomplete, in the sense that there does not exist a unique martingale measure which can be used to price a contract. In incomplete markets one would like to have a selection principle to reduce the class of possible martingale measures. The economically appealing method is the utility indifference principle, which is based on comparing the expected utilities of an investor when taking the risk and without it. We would like to remark that the set of equivalent martingale measures in the combined financial and insurance model, with systematic mortality risk, is studied in [3,4].

The utility indifference principle, in the case when an agent invests its wealth in a financial market, was first introduced in [5] and since then, it has become very popular method of pricing and hedging of financial risks in incomplete markets. This rule was extended in [6,7] to price insurance risks. In [6] the explicit results are derived for an exponential utility function by means of solving Hamilton-Jacobi equations, whereas in [7], and later in [8], mean-variance preferences are investigated and the Galtchouk-Kunita-Watanabe decomposition is applied in order to solve the problem.

In this paper we extend the results from [6]. We consider a financial market with a risky asset which price dynamics is driven by a Lévy process and we model mortality intensity as a stochastic process of diffusion type. Both of this extensions are not only theoretically interesting but are also of great practical relevance, as we discuss later on. To best our knowledge the indifference pricing for life-insurance liabilities in this framework is taken up for the first time. We investigate three different types of indifference arguments. Two of them appear for the first time in the literature. The main goal of this paper is to derive reasonable pricing equations which can be easily applied by an insurer in the valuation process, together with the hedging strategies. The contribution, from the theoretical point, is to find classical solutions of the corresponding Hamilton-Jacobi-Bellman equations.

One of the most important characteristic of financial assets returns is their high variability, resulting from the heavy-tailed nature of empirical returns and observable large sudden movements in stock prices. The so-called six-standard deviation market moves are repeatedly seen in the financial markets around the world. This properties rule out the possibility that the marginal distribution of an asset return is Gaussian. Moreover, all models of the stock price dynamics which generate continuous sample paths are also inadequate. It is now well-known, see chapters 1 and 7 in [9], that Lévy processes can easily reproduce heavy tails, skewness and other distributional properties of asset returns, and, what is very important as well, can generate discontinuities in the price dynamics. As Lévy processes generate more realistic sample paths of stock prices, one should replace, in the celebrated Black-Scholes model, a Gaussian noise by a Lévy noise. One can expect that this would change significantly the position in a risky asset which an investor should hold in order to hedge an issued claim.

In 1980's and 1990's the mortality improvements turned out to be much greater than forecasts and the unexpected decrease in mortality rates effected the solvency of life-insurance companies and pension providers. Over the last 20 years the mortality improvements varied substantially and mortality rates were evolving in a random fashion, see [10]. One can notice the general trend but there is still an unpredictable factor left which cannot be handled by any deterministic model. This is the reason why probabilistic models of the mortality evolution have appeared in the literature, see [3,11,12,13]. The possibility that the mortality intensity curve will evolve in a different way from that anticipated introduces an additional risk for an insurer. This risk, which is called systematic mortality risk, cannot be diversified and an insurer is expected to charge a premium for this risk. Besides systematic mortality risk, which is now gaining much attention, an insurer is facing unsystematic mortality risk, which was recognized long time ago. This risk arises when the actual number of deaths deviates from the anticipated number because of the finite number of lives in a portfolio. In the contrary to systematic mortality risk, unsystematic mortality risk can be diversified by pooling. It is clear that both types of risk must be taken into account in order to avoid mispricing of life-insurance policies which can have far-reaching consequences.

Very recently three papers have appeared dealing with the problem of pricing in the presence of systematic mortality risk, see [4,14,15]. The risk-minimizing criterium is applied in [4] to hedge a general payment process in a financial market consisting of a saving account and a bond. The dynamics of the short rate is given by an affine diffusion process, whereas the evolution of the mortality intensity is described by a Cox-Ross-Ingersoll process. In the same model framework, the mean-variance indifference price for a pure endowment is also derived in [4]. In [14] a pure

endowment contract is priced by assuming that an insurer is compensated for taking the risk via the Sharpe ratio of a portfolio consisting of a bond and the obligation to cover the claim. The hedging strategy is derived by minimizing the variance of the portfolio. The Itô diffusion processes are used to describe the dynamics of the short rate and the mortality intensity. The results from [14] are extended in [15] where a pure endowment with term insurance contract is priced. Let us point out again, that our model framework seems to be a novelty.

This paper is structured as follows. In section 2 we introduce our model of the financial market, the stochastic mortality intensity process and the payment process arising from the life insurance portfolio. The utility indifference pricing principles are discussed in section 3. In section 4 we investigate the optimal control problems and derive the Hamilton-Jacobi-Bellman equations. An exponential utility function is considered in section 5, where we arrive at the explicit solutions. Some properties of the premiums are studied in section 5. A simple numerical example is also given in section 5. The indifference pricing with a quadratic loss function is shortly discussed in section 6. All proofs are given in the appendix.

2 The model

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, where T denotes finite and fixed time horizon. The filtration satisfies the usual hypotheses of completeness ($\mathcal{F}_0$ contains all sets of $\mathbb{P}$-measure zero) and right continuity ($\mathcal{F}_t = \mathcal{F}_{t+}$). The filtration $\mathbb{F}$ consists of three subfiltrations. We set $\mathbb{F} = \mathbb{F}^F \vee \mathbb{F}^M \vee \mathbb{F}^N$, where $\mathbb{F}^F$ contains information about the financial market, $\mathbb{F}^M$ contains information about the mortality intensity and $\mathbb{F}^N$ contains information about the number of survived lives in the portfolio. We assume that the subfiltrations $\mathbb{F}^F$ and $(\mathbb{F}^M, \mathbb{F}^N)$ are independent, which means that the future-life time of an insured person is independent of the financial market. The measure $\mathbb{P}$ is the real-world, objective probability measure. All expected values are taken with respect to the measure $\mathbb{P}$, unless it is stated otherwise. The conditional expectation $\mathbb{E}[\cdot | X(t) = x, \lambda(t) = \lambda, N(t) = n]$ is denoted by $\mathbb{E}^{t,x,\lambda,n}[\cdot]$.

In the following subsections we introduce the financial market, the stochastic mortality intensity process and the life insurance portfolio.

2.1 The financial market

We consider a Lévy diffusion version of a Black-Scholes financial market. The price of a risk-free asset $B := (B(t), 0 \leq t \leq T)$ is described by the ordinary differential

equation

$$\frac{dB(t)}{B(t)} = rdt, \quad B(0) = 1, \tag{2.1}$$

where r denotes a rate of interest. The second tradeable instrument in the market is a risky asset, and the dynamics of its price $S := (S(t), 0 \leq t \leq T)$ is given by the stochastic differential equation

$$\frac{dS(t)}{S(t-)} = \mu dt + \xi dL(t), \quad S(0) = s > 0, \tag{2.2}$$

where μ and ξ denote a drift and volatility, and $L := (L(t), 0 \leq t \leq T)$ denotes a zero-mean Lévy process, $\mathbb{F}^F$-adapted with $\mathbb{P}$-a.s. càdlàg sample paths (paths which are continuous on the right and having limits on the left). Let us recall that a Lévy process is a process with independent and stationary increments.

The process L is assumed to satisfy the following Lévy-Itô decomposition, see chapter 2.4 in [16],

$$L(t) = \sigma W(t) + \int_{(0,t]} \int_{\mathbb{R}} z \big(M(ds \times dz) - \nu(dz)ds \big), \tag{2.3}$$

where $W := (W(t), 0 \leq t \leq T)$ denotes a Brownian motion and $M((s,t] \times A) = \#\{s < u \leq t : (L(u) - L(u-)) \in A\}$ denotes a Poisson random measure, independent of W. We recall that the compensated measure $\tilde{M}((0,t] \times A) = M((0,t] \times A) - t\nu(A)$ is a martingale-valued measure, that is $\tilde{M} := (\tilde{M}((0,t] \times A), 0 \leq t \leq T)$ is a $\mathbb{F}^F$-martingale for all Borel sets $A \in \mathcal{B}(\mathbb{R} - \{0\})$. The compensator ν is called a Lévy measure and verifies $\int_{|z|<1} z^2 \nu(dz) < \infty$. For more information concerning Lévy processes, Lévy measures and Poisson random measures we refer the interested reader to [9,16,17].

Without the loss of generality we may assume that $\xi = 1$. We also need the following assumption concerning the Lévy measure:

(A) the measure ν is defined on $(-1, \infty)$, with $\nu(\{0\}) = 0$, and satisfies the integrability condition $\int_{z \geq 1} z^2 \nu(dz) < \infty$.

Notice that we do not exclude infinite active jump processes, for which the integral $\int_{|z|<1} |z| \nu(dz)$ is infinite.

Under the assumption **(A)** the stochastic differential equation (2.2) has the unique, positive and almost surely finite solution, given explicitly by the Doléans-Dade exponential,

$$\begin{aligned} S(t) &= s_0 \exp \Big\{ \big(\mu - \frac{1}{2}\sigma^2 + \int_{z>-1} (\log(1+z) - z)\nu(dz) \big)t + \sigma W(t) \\ &\quad + \int_{(0,t]} \int_{z>-1} \log(1+z)\tilde{M}(ds \times dz) \Big\} \\ &= s_0 \exp \Big\{ \mu_E t + \sigma W(t) + \int_{(0,t]} \int_{\mathbb{R}} z\tilde{M}_E(ds \times dz) \Big\}, \end{aligned} \tag{2.4}$$

which is the exponential Lévy process with the measure $\nu_E(A) = \nu(\{z : \log(1+z) \in A\})$, see propositions 8.21 and 8.22 in [9]. We would like to point out that in some papers, for example in [18], the exponential price model (2.4) serves as the starting point.

We would like to refer the reader to [9,19], where different topics in financial modelling with Lévy processes are deeply investigated.

2.2 The mortality intensity

In order to capture the unexpected changes in the mortality we assume that the mortality intensity, $\Lambda := (\lambda(t), 0 \leq t \leq T)$, is a stochastic process of the diffusion type (Itô diffusion process), with the dynamics given by the stochastic differential equation

$$d\lambda_x(t) = a(t, \lambda_x(t))dt + b(t, \lambda_x(t))d\bar{W}(t), \quad \lambda_x(0) = \lambda > 0, \tag{2.5}$$

where $\bar{W} := (\bar{W}(t), 0 \leq t \leq T)$ denotes an $\mathbb{F}^M$-adapted Brownian motion, independent of the Brownian motion W and the Poisson random measure $\tilde{M}$, with $\mathbb{P}$-a.s. continuous sample paths. We model, as usual, the mortality intensity as the function of two variables: the age x of an insured person at the moment of the issue of a policy and the time t, which is the duration since the issue. In the sequel the subscript x will be omitted.

We make the following assumptions concerning the stochastic mortality intensity process:

(B1) $a : [0, T] \times (0, \infty) \to \mathbb{R}, b : [0, T] \times (0, \infty) \to (0, \infty)$ are continuous functions, locally Lipschitz continuous in λ, uniformly in t,

(B2) there exists a sequence $(D_n)_{n \in \mathbb{N}}$ of bounded sets with $\bar{D}_n \subseteq (0, \infty)$ and $\bigcup_{n \geq 1} D_n = (0, \infty)$, such that the functions $a(t, \lambda)$ and $b^2(t, \lambda)$ are uniformly Lipschitz continuous on $[0, T] \times \bar{D}_n$,

(B3) $\mathbb{P}(\forall_{s \in [t,T]} \lambda(s) \in (0, \infty) | \lambda(t) = \lambda) = 1$ and $\sup_{s \in [t,T]} \mathbb{E}[|\lambda(s)|^2 | \lambda(t) = \lambda] < \infty$ for all starting points $(t, \lambda) \in [0, T] \times (0, \infty)$.

We would like to point out that the diffusion dynamics of the intensity is used in all papers dealing with the stochastic mortality, see [3,4,11,12,13]. It seems to be very reasonable assumption, as changes in the mortality occur slowly and without sudden jumps. We remark that mortality intensity models, appearing in the literature, satisfy **(B1)-(B3)** and arise as the special cases of (2.5).

Under the assumptions **(B1)** and **(B3)**, for each starting point $(t, \lambda) \in [0, T] \times (0, \infty)$, the mortality intensity process is nonexplosive on $[t, T]$ and there exists the

unique strong solution to the stochastic differential equation (2.5), such that the mapping $(t, \lambda, s) \mapsto \lambda^{t,\lambda}(s)$ is $\mathbb{P}$-a.s. continuous, see [20,21]. The assumption (**B2**) and the integrability condition in (**B3**) are required in order to verify the optimality of the derived investment strategy and to show that our candidate value function satisfies the Hamilton-Jacobi-Bellman equation in the classical sense.

We assume that the process Λ is $\mathbb{F}^M$-adapted which is not a realistic assumption, since the mortality intensity cannot be observed like the price of a tradeable asset. In this context, measurability means that it is possible to estimate the "true" intensity based on available data from a reference population. In this way we can treat the mortality intensity as an observable quantity, see [3] for discussion.

2.3 The life insurance portfolio

We deal with the portfolio consisting of the same life insurance policies issued at time 0 to a group of n_0 persons. Each policyholder is entitled to three types of payments. Firstly, there are amounts payable continuously at the rate c, as long as an insured person is alive, but no longer than T years. This could be benefits, in the case of positive cashflows, or premiums, in the case of negative cashflows. Secondly, there is a benefit payable immediately at the moment of the death of an insured person, in the amount of b, within T years. Thirdly, there are two types of endowments: a certain, initial premium $B(0)$ and a terminal, survival benefit in the amount of $B(T)$, payable provided that an insured person is still alive at time T. The payment scheme, which we consider, is very general and it arises in all traditional life insurance and pension products. Notice that we can deal not only with term and differed life products but also with whole life products by considering the time T as the maximum possible future life-time of an insured person.

Let $T_1, T_2, ..., T_{n_0}$ denote the future life-times of insured persons who are all at the same age x at the moment of the issue of the policies. We assume that the random variables $T_1, T_2, ..., T_{n_0}$ are identically distributed with the survival function

$$\mathbb{P}(T_i > t | \mathcal{F}_t^M) = e^{-\int_0^t \lambda(s)ds}, \quad i = 1, 2, ..., n_0. \tag{2.6}$$

The censored life-times $\big((T_1 \wedge T, \mathbf{1}\{T_1 \leq T\}), ..., (T_{n_0} \wedge T, \mathbf{1}\{T_{n_0} \leq T\})\big)$ are assumed to be $\mathbb{F}^N$-measurable. Notice that due to the process Λ, which effects the future life-times of all persons in the portfolio, the random variables $T_1, T_2, ..., T_{n_0}$ are correlated. For more information concerning survival models we refer the interested reader to chapters 1 and 2 in [22].

Let us define the counting process $N := (N(t), 0 \leq t \leq T)$

$$N(t) = n_0 - \sum_{i=1}^{n_0} \mathbf{1}\{T_i \leq t\}, \tag{2.7}$$

which counts the number of survivors in the portfolio. The main object of our interest is the cumulative payment process $P := (P(t), 0 \leq t \leq T)$ with the dynamics

$$dP(t) = N(t-)cdt - bdN(t) + N(T)B(T)d\mathbf{1}\{t \geq T\}, \qquad (2.8)$$

with $P(0) = n_0 B(0)$. We would like to point out that the process P is $\mathbb{F}^N$-adapted with $\mathbb{P}$-a.s cádlág sample paths of finite variations.

We remark that in [4] the same payment process is considered, whereas in [14,15], a portfolio consisting of only pure endowments and term insurance policies is investigated.

3   Indifference pricing principle

In this section we discuss the indifference valuation rules which may be applied when dealing with pricing of life insurance liabilities. Before we state the principles we first introduce the wealth process and some notations which we will use in the sequel.

Consider the wealth process of the insurer $X^\pi := (X^\pi(t), 0 \leq t \leq T)$ who handles the payment process P arising from the issued policies. The dynamics of the process X^π is given by the stochastic differential equation

$$
\begin{aligned}
dX^\pi(t) &= \pi(t)\Big(\mu dt + \sigma dW(t) + \int_{z>-1} z\tilde{M}(dt \times dz)\Big) \\
&\quad + \big(X^\pi(t-) - \pi(t)\big)rdt - dP(t), \\
X(0) &= x + P(0), \qquad (3.1)
\end{aligned}
$$

where $\pi(t)$ denotes an amount of wealth invested in the risky asset and x denotes the initial wealth the insurer.

For $n = 0, 1, ..., n_0$ let $X_n^{\pi_n} := (X_n^{\pi_n}(t), 0 \leq t \leq T)$ denote the process with the dynamics given by the stochastic differential equation

$$
\begin{aligned}
dX_n^{\pi_n}(t) &= \pi_n(t)\Big(\mu dt + \sigma dW(t) + \int_{z>-1} z\tilde{M}(dt \times dz)\Big) \\
&\quad + \big(X_n^{\pi_n}(t-) - \pi_n(t)\big)rdt - ncdt. \qquad (3.2)
\end{aligned}
$$

Notice that the equation (3.2), with the control π_n, describes the evolution of the process X^π, when there is n policies in the portfolio. By X_n^{s,x,π_n} we denote the process which starts at $X(s) = x$ and whose dynamics is given by (3.2) for $t > s$. Let the stopping time τ_i, for $i = 0, 1, ..., n_0 - 1$, denote the moment of $(n_0 - i)$th death. We

set $\tau_{n_0} = 0$. The process X^π can be defined recursively as

$$X^\pi(t) = \begin{cases} x + n_0 B(0), & t = 0 \\ X_n^{\tau_n, X^\pi(\tau_n), \pi_n}(t), & \tau_n < t < \tau_{n-1} \wedge T, \tau_n < T, \\ X_n^{\tau_n, X^\pi(\tau_n), \pi_n}(\tau_{n-1}) - b, & t = \tau_{n-1}, \tau_{n-1} < T, \\ X_n^{\tau_n, X^\pi(\tau_n), \pi_n}(T) - b - (n-1)B(T), & t = T, \tau_{n-1} = T, \\ X_n^{\tau_n, X^\pi(\tau_n), \pi_n}(T) - nB(T), & t = T, \tau_{n-1} > T, \\ X_0^{\tau_0, X^\pi(\tau_0), \pi_0}(t), & \tau_0 < t \leq T, \tau_0 < T. \end{cases} \qquad (3.3)$$

The above recursion starts at $n = n_0$.

We will also deal with the discounted wealth process $Y^\pi := (e^{-\rho t} X^\pi(t), 0 \leq t \leq T)$. The following relation holds:

$$Y^\pi(t) = \begin{cases} y + n_0 B(0), & t = 0 \\ Y_n^{\tau_n, Y^\pi(\tau_n), \pi_n}(t), & \tau_n < t < \tau_{n-1} \wedge T, \tau_n < T, \\ Y_n^{\tau_n, Y^\pi(\tau_n), \pi_n}(\tau_{n-1}) - be^{-\rho \tau_{n-1}}, & t = \tau_{n-1}, \tau_{n-1} < T, \\ Y_n^{\tau_n, Y^\pi(\tau_n), \pi_n}(T) - be^{-\rho T} - (n-1)B(T)e^{-\rho T}, & t = T, \tau_{n-1} = T, \\ Y_n^{\tau_n, Y^\pi(\tau_n), \pi_n}(T) - nB(T)e^{-\rho T}, & t = T, \tau_{n-1} > T \\ Y_0^{\tau_0, Y^\pi(\tau_0), \pi_0}(t), & \tau_0 < t \leq T, \tau_0 < T, \end{cases} \qquad (3.4)$$

where the dynamics of $Y_n^{\pi_n}$ is described by the equation

$$\begin{aligned} dY_n^{\pi_n}(t) &= e^{-\rho t} \pi_n(t) \left((\mu - r)ds + \sigma dW(t) + \int_{z > -1} z \tilde{M}(dt \times dz) \right) \\ &\quad + Y_n^{\pi_n}(t-)(r - \rho)dt - nce^{-\rho t}dt. \end{aligned} \qquad (3.5)$$

Let u denote a function describing the preferences of the insurer. In the classical decision-making theory under uncertainty u is assumed to be a utility function (concave and increasing). However, other objective functions are also reasonable, for example in [4,7,8] the mean-variance preferences are considered. Under the indifference principle the insurer prices the payment process in such a way, that it is indifferent, with respect to its preferences u, between writing and not writing n_0 life insurance contracts. According to this rule the insurer arranges its payment process such that

$$\begin{aligned} \sup_\pi \mathbb{E}\big[u(X^\pi(T)) | X(0) = x + n_0 B(0), \lambda(0) = \lambda, N(0) = n_0\big] \\ = \sup_{\pi_0} \mathbb{E}\big[u(X_0^{\pi_0}(T)) | X_0(0) = x\big]. \end{aligned} \qquad (3.6)$$

Under the valuation rule (3.6) the insurer maximizes the utility of its wealth at the fix terminal time T. However, the life insurance portfolio may terminate before time T and the insurer might be interested in valuating the utility of the return on its insurance portfolio only up to the time when the portfolio terminates, without any

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additional financial gains which may occur after covering the final payment. In [6] the indifference pricing with respect to random time, which is the moment of the death of an insured person, is proposed. We continue to investigate this principle and deal with the following valuation rule

$$\sup_\pi \mathbb{E}\big[u(Y^\pi(\tau_0 \wedge T))|Y(0) = y + n_0 B(0), \lambda(0) = \lambda, N(0) = n_0\big]$$
$$= u(y). \tag{3.7}$$

Under this principle the insurer prices the payment process such that the expected utility of the wealth, at the moment of the termination of the portfolio, discounted at the time when the portfolio is issued, equals the utility of the wealth available before issuing the policies. We would like to point out that the indifference arguments with respect to a discounted wealth process have already been applied in [4,7,8]. We can as well consider the following pricing principle

$$\sup_\pi \mathbb{E}\big[u(Y^\pi(\tau_0 \wedge T))|Y(0) = y + n_0 B(0), \lambda(0) = \lambda, N(0) = n_0\big]$$
$$\sup_{\pi_0} \mathbb{E}\big[u(Y_0^{\pi_0}(\tau_0 \wedge T))|Y(0) = y, \lambda(0) = \lambda\big], \tag{3.8}$$

which extends the right side of (3.7) by taking into account a trading strategy. We remark that the principle (3.8), with $\rho = r = 0$, is considered in [6].

We believe that the valuation rules (3.7) and (3.8) are reasonable. In section 5 we show that for an exponential utility, the pricing equations (3.7) and (3.8) are interesting generalizations of the equation (3.6).

4   The optimal stochastic control problems

In this section we derive the Hamilton-Jacobi-Bellman equations and the classical verification theorem.

First we investigate the pricing principle (3.6). Let us define the optimal value functions

$$V_n(t, x, \lambda) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\big[u(X^\pi(T))|X(t) = x, \lambda(t) = \lambda, N(t) = n\big], \tag{4.1}$$

for $n = 0, 1, ..., n_0$. Notice that $V_0(t, x, \lambda) = V_0(t, x)$ is the value function for the standard investment problem of maximizing the expected utility of the wealth at the terminal time, without any payment process. This optimization problem is well-understood in the financial literature, see chapter 3 in [23]. We are left with finding the value function V_n for $n > 0$.

The following lemma, which is intuitively clear, turns out to be very useful in the derivation of the solution.

Lemma 4.1. *For $n = 1, 2, ..., n_0$ and all $(t, x, \lambda) \in [0, T] \times \mathbb{R} \times (0, \infty)$ the value function $V_n(t, x, \lambda)$ has the following representation*

$$\begin{aligned}
V_n(t, x, \lambda) & = \sup_{\pi_n \in \mathcal{A}} \mathbb{E}^{t,x,\lambda,n}\big[u(X_n^{\pi_n}(T) - nB(T))\mathbf{1}\{\tau_{n-1} > T\} \\
& \quad + V_{n-1}(\tau_{n-1}, X_n^{\pi_n}(\tau_{n-1}) - b)\mathbf{1}\{\tau_{n-1} \le T\}\big],
\end{aligned} \tag{4.2}$$

provided that the value function V_{n-1} is well-defined.

This lemma is very interesting as it gives the method of solving the optimization problem (4.1) for the portfolio consisting of $n = n_0$ policies. First, one has to find the value function V_0 and then, solve the optimization problems (4.2) starting with $n = 1$ and ending with $n = n_0$. Each optimization problem in this iterative procedure involves the value function from the previous step. The optimal investment strategy $\hat{\pi}$ consists of the sequence of $n_0 + 1$ controls, $\hat{\pi} = (\hat{\pi}_{n_0}, ..., \hat{\pi}_0)$, where $\hat{\pi}_0$ is the optimal investment strategy for the standard optimization problem, whereas $\hat{\pi}_n$ are the optimal investment strategies derived by solving the problems (4.2) iteratively.

Let us introduce the set of admissible strategies and two operators.

Definition 4.1. *The sequence of controls $\pi_n := (\pi_n(t), 0 < t \le T)$ is an admissible, $\pi_n \in \mathcal{A}$, if it satisfies the following assumptions:*

1. *$\pi_n : (0, T] \times \Omega \mapsto \mathbb{R}$ is a predictable mapping with respect to filtration $\mathbb{F}$,*

2. *$\int_0^T \pi_n^2(t)dt < \infty$ $\mathbb{P}$-a.s.,*

3. *the stochastic differential equation (3.2) has a unique solution $X_n^{\pi_n}$,*

for all $n = 0, 1, ..., n_0$.

We can conclude that for any $\pi_n \in \mathcal{A}$ the process $X_n^{\pi_n}$ is a semimartingale with càdlàg sample paths, see chapter 4.3.3 in [16].

Definition 4.2. *Let $\mathcal{L}_{F,\rho}$ denote the integro-differential operator given by*

$$\begin{aligned}
\mathcal{L}_{F,\rho}^{\pi} h(t, x) & = \big(e^{-\rho t}\pi(\mu - r) + x(r - \rho)\big)\frac{\partial h}{\partial x}(t, x) + \frac{1}{2}e^{-2\rho t}\pi^2\sigma^2\frac{\partial^2 h}{\partial x^2}(t, x) \\
& \quad + \int_{z > -1}\big(h(t, x + e^{-\rho t}\pi z) - \phi(t, x) - e^{-\rho t}\pi z\frac{\partial h}{\partial x}(t, x)\big)\nu(dz),
\end{aligned} \tag{4.3}$$

and let $\mathcal{L}_M$ denote the differential operator given by

$$\mathcal{L}_M h(t, \lambda) = a(t, \lambda)\frac{\partial h}{\partial \lambda}(t, \lambda) + \frac{1}{2}b^2(t, \lambda)\frac{\partial^2 h}{\partial \lambda^2}(t, \lambda). \tag{4.4}$$

This two operators are defined for functions h such that $\mathcal{L}h$ are well-defined pointwise and all derivatives appearing in $\mathcal{L}h$ exist and are continuous functions.

Below we state the classical verification theorem. Let us simply denote the operator $\mathcal{L}_{F,0}$ by $\mathcal{L}_F$.

Theorem 4.1. *Assume that v_{n-1} is a candidate function, which coincides with the optimal value function V_{n-1}, such that*

$$\mathbb{E}^{0,x,\lambda,n}\Big[\int_0^T \big|v_{n-1}(t, X_n^{\pi_n}(t), \lambda(t))\big|^2 dt\Big] < \infty, \tag{4.5}$$

for $\pi_n \in \mathcal{A}$. Let $v_n \in \mathcal{C}^{1,2,2}([0,T) \times \mathbb{R} \times (0,\infty)) \cap \mathcal{C}([0,T] \times \mathbb{R} \times (0,\infty))$ satisfies for $\pi_n \in \mathcal{A}$

$$\begin{aligned} 0 \;\geq\; &\frac{\partial v_n}{\partial t}(t,x,\lambda) + \mathcal{L}_F^{\pi_n} v_n(t,x,\lambda) - nc\frac{\partial v_n}{\partial x}(t,x,\lambda) + \mathcal{L}_M v_n(t,x,\lambda) \\ &+ n\lambda\big(v_{n-1}(t, x-b, \lambda) - v_n(t,x,\lambda)\big), \end{aligned} \tag{4.6}$$

for all $(t,x,\lambda) \in [0,T) \times \mathbb{R} \times (0,\infty)$, with

$$v_n(T,x,\lambda) = u(x - nB(T)), \quad \forall (x,\lambda) \in \mathbb{R} \times (0,\infty). \tag{4.7}$$

Assume also that for $\pi_n \in \mathcal{A}$

$$\begin{aligned} \mathbb{E}^{0,x,\lambda,n}\Big[\int_0^T \int_{z>-1} \big|&v_n(t, X_n^{\pi_n}(t-) + \pi_n(t)z, \lambda(t)) \\ &-v_n(t, X_n^{\pi_n}(t-), \lambda(t))\big|^2 \nu(dz)dt\Big] < \infty, \end{aligned} \tag{4.8}$$

$$\mathbb{E}^{0,x,\lambda,n}\Big[\int_0^T \big|v_n(t, X_n^{\pi_n}(t), \lambda(t))\big|^2 dt\Big] < \infty, \tag{4.9}$$

and

$\{v_n^-(\mathcal{T}, X_n^{\pi_n}(\mathcal{T}), \lambda(\mathcal{T}))\}_{0<\mathcal{T}\leq T}$ *is uniformly integrable for all stopping times $\mathcal{T}$*(4.10)

Then

$$v_n(t,x,\lambda) \geq V_n(t,x,\lambda), \quad \forall (t,x,\lambda) \in [0,T] \times \mathbb{R} \times (0,\infty). \tag{4.11}$$

If additionally there exists an admissible control $\hat{\pi}_n \in \mathcal{A}$ such that

$$\begin{aligned} 0 \;=\; &\frac{\partial v_n}{\partial t}(t,x,\lambda) + \mathcal{L}_F^{\hat{\pi}_n} v_n(t,x,\lambda) - nc\frac{\partial v_n}{\partial x}(t,x,\lambda) + \mathcal{L}_M v_n(t,x,\lambda) \\ &+ n\lambda\big(v_{n-1}(t, x-b, \lambda) - v_n(t,x,\lambda)\big), \end{aligned} \tag{4.12}$$

holds for all $(t,x,\lambda) \in [0,T) \times \mathbb{R} \times (0,\infty)$, and

$\{v_n(\mathcal{T}, X_n^{\hat{\pi}_n}(\mathcal{T}), \lambda(\mathcal{T}))\}_{0<\mathcal{T}\leq T}$ *is uniformly integrable for all stopping times $\mathcal{T}$*(4.13)

then

$$v_n(t,x,\lambda) = V_n(t,x,\lambda), \quad \forall (t,x,\lambda) \in [0,T] \times \mathbb{R} \times (0,\infty), \tag{4.14}$$

and $\hat{\pi}_n$ is the optimal control for the optimization problem (4.2).

In order to claim that a solution, which one finds, is optimal, one has to verify that this solution satisfies all conditions stated in the verification theorem. The conditions from the theorem 4.1 can be relaxed and one can consider a solution in the viscosity sense. As we are able to find smooth solutions we do not investigate this concept.

Consider now the pricing principles (3.7) and (3.8). Let us define the optimal value functions

$$W_n(t, x, \lambda) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\big[u(Y^\pi(\tau_0 \wedge T))|Y(t) = y, \lambda(t) = \lambda, N(t) = n\big], \qquad (4.15)$$

for $n = 1, 2, ..., n_0$. In the view of the equations (3.4) and (3.5) and the lemma 4.1 we can arrive at the Hamilton-Jacobi-Bellman equation corresponding to the optimization problem (4.15):

$$
\begin{aligned}
0 \;=\; & \frac{\partial w_n}{\partial t}(t, y, \lambda) + \sup_{\pi_n \in \mathbb{R}} \{\mathcal{L}_{F,\rho}^{\pi_n} w_n(t, y, \lambda)\} - nce^{-\rho t}\frac{\partial w_n}{\partial x}(t, y, \lambda) + \mathcal{L}_M w_n(t, y, \lambda) \\
& + n\lambda\big(w_{n-1}(t, y - be^{-\rho t}, \lambda) - w_n(t, y, \lambda)\big),
\end{aligned}
\qquad (4.16)
$$

with the terminal condition

$$w_n(T, y, \lambda) = u(y - nB(T)e^{-\rho T}). \qquad (4.17)$$

Notice that this time $w_0(t, x, \lambda) = u(x)$, as the optimization procedure stops at the time $\tau_0 \wedge T$. The reader can modify easily the conditions in the verification theorem 4.1 for a candidate value function w_n.

To best our knowledge the optimization problems (4.1), (4.2) and (4.15), together with the lemma 4.1 and the verification theorem 4.1, are new in the financial and the actuarial literature. In the life-time portfolio selection theory the representation (4.2) means that an investor faces the terminal utility u and the bequest utility v_{n-1}. We would like to point out that the problem of maximizing the utility with respect to random time is investigated in [24]. Most of the results, stated there, deal with a deterministic intensity of exiting the market. One example considers a stochastic density process, modelled as a geometric Brownian motion, and the solution is derived for power utility functions.

We find the explicit solutions of the Hamilton-Jacobi-Bellman equations (4.12) and (4.16) for an exponential utility and a quadratic loss function.

5   Exponential utility function

In this section we assume that the insurer applies the exponential utility function of the form $u(x) = -\frac{1}{\alpha}e^{-\alpha x}, \alpha > 0$. We point out that the exponential indifference pricing is well-known in the insurance theory, see [1,2,6,18].

5.1 Indifference pricing with respect to terminal time T

The first step in the iterative procedure (4.2) is to find the value function v_0. The following lemma is taken from [25].

Lemma 5.1. *Consider the Hamilton-Jacobi-Bellman equation associated with the optimization problem (4.1) for $n = 0$:*

$$0 = \frac{\partial v_0}{\partial t}(t, x) + \sup_{\pi_0 \in \mathbb{R}} \{\mathcal{L}_F^{\pi_0} v_0(t, x)\}, \quad (t, x) \in [0, T) \times \mathbb{R},$$

$$v_0(T, x) = -\frac{1}{\alpha} e^{-\alpha x}, \quad x \in \mathbb{R}. \tag{5.1}$$

The function v_0 defined as

$$v_0(t, x) = -\frac{1}{\alpha} e^{-\alpha f(t)x + g(t)} \tag{5.2}$$

satisfies the above equation in the classical sense, with $f(t) = e^{r(T-t)}$ and $g(t) = G(\hat{\kappa})(T - t)$, where $\hat{\kappa}$ is the unique minimizer of the convex function

$$G(\kappa) = -\kappa(\mu - r) + \frac{1}{2}\kappa^2 \sigma^2 + \int_{z > -1} (e^{-\kappa z} - 1 + \kappa z)\nu(dz). \tag{5.3}$$

The optimal investment strategy is $\hat{\pi}_0(t) = \frac{\hat{\kappa}}{\alpha f(t)}$.

If the price dynamics (2.2) contains the diffusion part and the jump part, then one can show that $\hat{\kappa}$ is smaller in the model with jumps. An investor who applied the investment strategy, which is optimal for the lognormally distributed returns, in the market in which jumps can occur, would take too much financial risk. This result is intuitively clear. In [18] the sensitivity of the optimal investment strategy with respect to a jump size, a jump activity and a jump asymmetry is investigated. The conclusion is that jumps can significantly change the optimal amount which should be invested in the risky asset.

In this section we assume that $\mu > r$. Under this assumption $\hat{\kappa} > 0$ and $\hat{\pi}_0 > 0$, so the risky asset is never short-sold. Notice that the function G is strictly decreasing in the right neighborhood of zero, which implies that $G(\hat{\kappa}) < 0$.

We postulate the following relation

$$v_n(t, x, \lambda) = v_0(t, x)\phi_n(t, \lambda), \quad \forall (t, x, \lambda) \in [0, T] \times \mathbb{R} \times (0, \infty), \tag{5.4}$$

with $\phi_0(t, \lambda) = 1$ for all $(t, \lambda) \in [0, T] \times (0, \infty)$. Assume that we have found the value function v_{n-1} (or equivalently ϕ_{n-1}), together with the optimal investment strategy $\hat{\pi}_{n-1}$ applied when there is $n - 1$ policies in the portfolio. The next step is to solve the optimization problem (4.2) and find the function ϕ_n and the optimal investment

strategy $\hat{\pi}_n$ which is applied when there is n policies in the portfolio. By substituting (5.4) into (4.12) we arrive at

$$
\begin{aligned}
0 \;=\; & v_0(t,x)\frac{\partial \phi_n}{\partial t}(t,\lambda) + \phi_n(t,\lambda)\left(\frac{\partial v_0}{\partial t}(t,x) + \sup_{\pi_n \in \mathbb{R}} \mathcal{L}_F^{\pi_n} v_0(t,x)\right) \\
& + v_0(t,x)\alpha f(t)nc\phi_n(t,\lambda) + v_0(t,x)\mathcal{L}_M\phi_n(t,\lambda) \\
& + v_0(t,x)n\lambda\big(e^{\alpha f(t)b}\phi_{n-1}(t,\lambda) - \phi_n(t,\lambda)\big).
\end{aligned}
\tag{5.5}
$$

Notice that the strategy π_n, which realizes the supremum in (5.5), is exactly the same as the strategy π_0 realizing the supremum in (5.1). We can state very important result that the optimal investment strategy, which should be applied when there is n policies in the portfolio, is independent of n and equals the optimal investment strategy when there is no insurance risk:

$$
\hat{\pi}_n(t) = \frac{\hat{\kappa}}{\alpha}e^{-r(T-t)}, \quad n = 0,1,...,n_0.
\tag{5.6}
$$

One could have expected such result. As our objective is to maximize the expected wealth by investing in the financial market, it seems that there is no point why the strategy, which achieves this goal, should depend on the payment process which is independent of the financial market.

From the equation (5.5) we derive the partial differential equation for the function $\phi_n(t,\lambda)$

$$
\begin{aligned}
0 = & \frac{\partial \phi_n}{\partial t}(t,\lambda) + \mathcal{L}_M\phi_n(t,\lambda) + (\alpha f(t)nc - n\lambda)\phi_n(t,\lambda) \\
& + n\lambda e^{\alpha f(t)b}\phi_{n-1}(t,\lambda), \quad (t,\lambda) \in [0,T) \times (0,\infty), \\
& \phi_n(T,\lambda) = e^{\alpha n B(T)}.
\end{aligned}
\tag{5.7}
$$

One can prove the following lemma.

Lemma 5.2. *Assume that $\phi_{n-1}(t,\lambda) \in \mathcal{C}^{1,2}([0,T) \times (0,\infty)) \cap \mathcal{C}_b([0,T] \times (0,\infty))$. Then the equation (5.7) has the unique solution $\phi_n(t,\lambda) \in \mathcal{C}^{1,2}([0,T) \times (0,\infty)) \cap \mathcal{C}_b([0,T] \times (0,\infty))$. The following probabilistic representation holds*

$$
\begin{aligned}
\phi_n(t,\lambda) \;=\; & \mathbb{E}^{t,\lambda}\Big[e^{\alpha n B(T)}e^{\int_t^T (\alpha f(s)nc - n\lambda(s))ds} \\
& + \int_t^T e^{\alpha f(s)b}n\lambda(s)\phi_{n-1}(s,\lambda(s))e^{\int_t^s(\alpha f(u)nc - n\lambda(u))du}ds\Big].
\end{aligned}
\tag{5.8}
$$

The representation (5.8) is known as the Feynman-Kac formula. Notice that $\phi_0 = 1$ satisfies the assumption of the lemma 5.2 so that we can solve the partial differential equations recursively to arrive at the sequence of functions $\phi_n(t,\lambda) \in \mathcal{C}^{1,2}([0,T) \times (0,\infty)) \cap \mathcal{C}_b([0,T] \times (0,\infty))$, for all $n = 1,2,...,n_0$. This completes the derivation of the solution to our optimization problem.

The recursion (5.8) is not very appealing from the computational point. The next lemma not only gives the explicit representation of the function ϕ_{n_0} but also provides very interesting insight into the structure of our solution.

Lemma 5.3. *The function $\phi_{n_0}(0, \lambda)$ can be represented as*

$$\phi_{n_0}(0, \lambda) = \mathbb{E}^{0,\lambda}\big[\exp\big(\alpha \int_{(0,T]} e^{r(T-t)}dP(t)\big)\big]. \tag{5.9}$$

We would like to remark that this simple representation has not been noticed in [6] in the case of deterministic mortality.

By recalling the indifference pricing principle (3.6) we are now ready to state the equation

$$n_0 B(0) = \frac{1}{\alpha}e^{-rT}\log \mathbb{E}^{0,\lambda}\big[\exp\big(\alpha \int_{(0,T]} e^{r(T-t)}dP(t)\big)\big], \tag{5.10}$$

according to which the payment process should be arranged. We remark that the pricing equation (5.10) derived for the insurer investing in the market with the risk-free and the risky asset is the same as the equation for the insurer who can only invest the risk-free asset. If $n_0 B(0)$ is the collective single premium to be paid to cover the benefits P, then the insured persons don't profit from the premium reduction which seems to be possible due to the higher return in the risky asset. Notice that the whole gain is taken by the insurer as the optimal value function in the market with the risk-free and the risky asset ($g < 0$) is greater than the corresponding value function in the market with the risk-free asset ($g = 0$).

We have derived the pricing equation in the case when the insurer is considering issuing n policies. However, the insurer may also be interested in the marginal price of a n-th additional policy when it has already $n - 1$ policies in its portfolio, see [15]. The indifference arguments yields the following pricing equation

$$v_0(0, x + \Delta B(0))\phi_n(0, \lambda) = v_0(0, x)\phi_{n-1}(0, \lambda), \tag{5.11}$$

from which we can calculate the marginal indifference price

$$\Delta B(0) = \frac{1}{\alpha}e^{-rT}\log\big(\frac{\phi_n(0, \lambda)}{\phi_{n-1}(0, \lambda)}\big). \tag{5.12}$$

We point out that the price (5.12), which an individual has to pay, equals the difference of the collective premiums for the portfolios consisting of n and $n - 1$ policies.

5.2 Indifference pricing with respect to a random date

In this subsection we deal with the pricing principles (3.7) and (3.8). In order to arrive at the explicit solutions we additionally assume that the discount factor equals

the risk-free rate of return, $\rho = r$.

First, let us have a look at the pricing equation which arises from the principles (3.7) and (3.8) in the market consisting of the risk-free asset only. Assume that the insurer issues one policy paying P at the moment of the death, if it occurs within T years, or at the end of the contract. The principles yield

$$-\frac{1}{\alpha}e^{-\alpha x} = \mathbb{E}\Big[-\frac{1}{\alpha}e^{-\alpha(x+B(0)-e^{-r\tau_0}P\mathbf{1}\{\tau_0 \leq T\}+e^{-rT}P\mathbf{1}\{\tau_0 > T\})}\Big], \qquad (5.13)$$

from which we can calculate the premium

$$B(0) = \frac{1}{\alpha}\log\mathbb{E}\big[e^{\alpha e^{-r(\tau_0 \wedge T)}P}\big]. \qquad (5.14)$$

The derived premium is very reasonable, as it corresponds to the classical exponential indifference price of the claim discounted at the moment of the issue of the contract. This gives motivation for investigating the indifference principles (3.7) and (3.8) in the dynamic setting. Notice that if we applied the arguments (3.8) with respect to the undiscounted wealth process, as in [6], then the price, even in our simple example, would be more complicated and would depend on the initial wealth x. This problem is not discussed in [6], as the case of $r = 0$ is only considered.

Let us deal with the main problem. We only state our results, as the method of arriving at the solution is the same as in the previous section. We postulate the following relations

$$w_n(t, y, \lambda) = u(y)\varphi_n(t, \lambda), \quad \forall (t, y, \lambda) \in [0, T] \times \mathbb{R} \times (0, \infty), \qquad (5.15)$$

with $\varphi_0(t, \lambda) = 1$ for all $(t, \lambda) \in [0, T] \times (0, \infty)$. By substituting (5.15) into (4.16) we arrive at

$$\begin{aligned}
0 &= u(y)\frac{\partial \varphi_n}{\partial t}(t, \lambda) + \varphi_n(t, \lambda)\sup_{\pi_n \in \mathbb{R}} \mathcal{L}^{\pi_n}_{F, \rho} u(y) + u(y)\alpha nce^{-rt}\varphi_n(t, \lambda) \\
&\quad + u(y)\mathcal{L}_M\varphi_n(t, \lambda) + u(y)n\lambda\big(e^{\alpha be^{-rt}}\varphi_{n-1}(t, \lambda) - \varphi_n(t, \lambda)\big), \qquad (5.16)
\end{aligned}$$

from which we can conclude that the optimal investment strategy is the same for all n and equals

$$\tilde{\pi}_n(t) = \frac{\hat{\kappa}}{\alpha}e^{rt}, \quad n = 1, 2, ..., n_0. \qquad (5.17)$$

Notice that due to discounting, the strategy (5.17) is independent of terminal time T. We remark that both optimal investment strategies are increasing functions of time t.

The sequence of functions φ_n satisfies the system of partial parabolic differential equations

$$\begin{aligned}
0 &= \frac{\partial \varphi_n}{\partial t}(t, \lambda) + \mathcal{L}_M\varphi_n(t, \lambda) + (G(\hat{\kappa}) + \alpha nce^{-rt} - n\lambda)\varphi_n(t, \lambda) \\
&\quad + n\lambda e^{\alpha be^{-rt}}\varphi_{n-1}(t, \lambda), \quad (t, \lambda) \in [0, T) \times (0, \infty), \qquad (5.18) \\
\varphi_n&(T, \lambda) = e^{\alpha nB(T)e^{-rT}}.
\end{aligned}$$

We conclude that $\varphi_n \in \mathcal{C}^{1,2}([0,T] \times (0,\infty)) \cap \mathcal{C}_b([0,T] \times (0,\infty))$, for $n = 1, 2, ..., n_0$, and that the probabilistic representation

$$
\begin{aligned}
\varphi_n(t,\lambda) &= \mathbb{E}^{t,\lambda}\big[e^{\alpha n B(T)e^{-rT}}e^{\int_t^T (\alpha n c e^{-rs} - n\lambda(s))ds} \\
&\quad + \int_t^T e^{\alpha b e^{-rs}}n\lambda(s)\varphi_{n-1}(s,\lambda(s))e^{\int_t^s (\alpha c e^{-ru} - n\lambda(u))du}ds\big],
\end{aligned} \quad (5.19)
$$

holds. Similarly to the lemma 5.3 we have

$$
\varphi_{n_0}(0,\lambda) = \mathbb{E}^{0,\lambda}\big[\exp\big(G(\hat{\kappa})(\tau_0 \wedge T) + \alpha \int_{(0,T]} e^{-rt}dP(t)\big)\big]. \quad (5.20)
$$

The indifference principle (3.7) yields the pricing equation

$$
n_0 B(0) = \frac{1}{\alpha}\log \mathbb{E}^{0,\lambda}\big[\exp\big(G(\hat{\kappa})(\tau_0 \wedge T) + \alpha \int_{(0,T]} e^{-rt}dP(t)\big)\big], \quad (5.21)
$$

which takes into account the financial market in which the insurer invests. Notice that under the principle (3.7), the insured persons can profit from the premium reduction when the insurer is allowed to invest in the risky asset ($G(\hat{\kappa}) < 0$). Moreover, the expected utility of the insurer's discounted wealth is also greater when the risky asset is available in the market comparing with the case when the insurer can only invest in the risk-free asset. It turns out that both sides of the contract can profit from the higher gains in the financial market. It is also interesting to note that the function $\kappa \mapsto G(\kappa)$ depends on the investment strategy, as $\kappa := \alpha e^{-rt}\pi$. Applying the strategy (5.17) reduces the premium by the maximum amount, while choosing $\pi = 0$ recovers the premium (5.14).

We can also state the pricing equation corresponding to the indifference principle (3.8):

$$
n_0 B(0) = \frac{1}{\alpha}\log \frac{\mathbb{E}^{0,\lambda}\big[e^{G(\hat{\kappa})(\tau_0 \wedge T)}e^{\alpha \int_{(0,T]} e^{-rt}dP(t)}\big]}{\mathbb{E}^{0,\lambda}\big[e^{G(\hat{\kappa})(\tau_0 \wedge T)}\big]}, \quad (5.22)
$$

which can be rewritten as

$$
n_0 B(0) = \frac{1}{\alpha}\log \mathbb{E}^{\mathbb{P}^G}\big[\exp\big(\alpha \int_{(0,T]} e^{-rt}dP(t)\big)\big], \quad (5.23)
$$

where the equivalent measure $\mathbb{P}^G \sim \mathbb{P}$ is defined through the Radon-Nikodym derivative

$$
\frac{d\mathbb{P}^G}{d\mathbb{P}} = \frac{e^{G(\hat{\kappa})(\tau_0 \wedge T)}}{\mathbb{E}^{0,\lambda}\big[e^{G(\hat{\kappa})(\tau_0 \wedge T)}\big]}. \quad (5.24)
$$

The effect of G on the price (5.22) will be investigated in the next section.

We believe that both pricing equations, derived in this section, are reasonable and economically sensible. They can be viewed as the generalizations of the traditional pricing equation (5.10).

5.3 Properties of the premiums

In this section we assume that $n_0 B(0)$ is the single premium which the insurer has to collect from the portfolio in order to cover the benefits $c, b, B(T)$ to be paid to the insured persons. We state some properties of $B(0)$, which is the single premium an individual has to pay. Numerical example is also considered.

The following lemma gives the important properties of the derived exponential utility indifference premiums. It is interesting to note that all three premiums, that we consider, share the well-known properties.

Lemma 5.4. *Assume that the individual premium $B(0)$ is set according to (5.10). Then*

1. *$B^\alpha(0)$ is (strictly) increasing in α,*

2. *$\lim_{\alpha \to 0} B^\alpha(0) = \mathbb{E}\big[\int_0^{T_i \wedge T} ce^{-rt}dt + be^{-rT_i}\mathbf{1}\{T_i \le T\} + B(T)e^{-rT}\mathbf{1}\{T_i > T\}\big]$,*

3. *$\lim_{\alpha \to \infty} B^\alpha(0) = \mathbb{P}$-ess sup $\{\int_0^{T_i \wedge T} ce^{-rt}dt + be^{-rT_i}\mathbf{1}\{T_i \le T\}$*
   $+ B(T)e^{-rT}\mathbf{1}\{T_i > T\}\big]\}$,

4. *if $c^1 \le c^2, b^1 \le b^2, B^1(T) \le B^2(T)$ then $B^1(0) \le B^2(0)$.*

If the premium is set according to (5.21) or (5.23), then the points 1,3,4 hold as well. For the premium (5.21) we have

2'. *$\lim_{\alpha \to 0} B^\alpha(0) = -\infty$,*

while for the premium (5.23) we have

2''. *$\lim_{\alpha \to 0} B^\alpha(0) = \mathbb{E}^{\mathbb{P}^G}\big[\int_0^{T_i \wedge T} ce^{-rt}dt + be^{-rT_i}\mathbf{1}\{T_i \le T\} + B(T)e^{-rT}\mathbf{1}\{T_i > T\}\big]$.*

We believe that the property $2'$ is not a disadvantage of the applied pricing principle. Notice that due to the monotonicity and the continuity of the mapping $\alpha \mapsto B^\alpha(0)$, there always exists $\alpha^* < \infty$ such that the premium is well-defined for all $\alpha > \alpha^*$. The premium is meant to be well-defined if it is greater than the expected value of the benefits or, simply, if it is positive. We are allowed only to consider some set of parameters α, however, within this set, we can arrive at all possible prices. We remark that in [14,15] the price is also well-defined only for some values of the Sharpe ratio.

We continue to investigate properties of the derived premiums by means of a numerical example. We assume that the mortality intensity follows the exponential Ornstein-Uhlenbeck process

$$\lambda(t) = 0,02e^{0,08t+0,1Y(t)}, \quad dY(t) = -0,2Y(t) + d\bar{W}(t), \tag{5.25}$$

whereas the price of the stock follows the exponential Variance Gamma process

$$S(t) = e^{\mu_E t + L(t)}, \quad L(t) = -0,2h(t) + 0,2W(h(t)), \quad (5.26)$$

where $h(t)$ denotes a Gamma distributed random variable with the density function

$$g_{h(t)}(y) = \frac{1}{\Gamma(t/0,003)(0,003)^{t/0,003}} y^{\frac{t}{0,003}-1} e^{-\frac{y}{0,003}}. \quad (5.27)$$

For the subordinated Brownian motion representation of Variance Gamma processes we refer the reader to chapter 2.3 in [19]. The processes (5.25) and (5.26) appear in numerical examples in the literature, see for example [26,18].

Let us consider a pure endowment contract with the term of $T = 10$ years and the benefit $B(T) = 100$. We investigate the impact of α, μ and n_0 on the premiums $B(0)$ calculated according to (5.10), (5.21) and (5.23). We also calculate the premium assuming deterministic mortality evolution over time $\bar{\lambda}(t) = \mathbb{E}[\lambda(t)]$. In order to arrive at the prices we have applied Monte-Carlo simulation.

The obtained numerical results, presented in tables 1-3, obviously agree with

Table 1: The effect of α on the premiums; $\mu = 0,1, n_0 = 1$.

α	Premium for $\bar{\lambda}$	Premium (5.10)	Premium (5.21)	Premium (5.23)
0,001	49,564	50,606	-342,199	49,103
0,005	51,916	53,384	-28,686	49,964
0,01	54,295	55,801	12,283	51,585
0,02	58,134	59,295	34,780	54,516
0,04	61,399	62,792	48,472	58,255
0,05	62,807	63,901	52,514	60,426
0,06	63,154	64,465	54,945	61,418
0,1	64,085	64,932	59,832	63,753
0,15	64,876	65,767	61,980	64,558
0,2	65,293	66,183	63,283	65,112
0,6	66,659	66,648	65,892	66,441

the properties stated in the lemma 5.4. Notice that the premium which eliminates the financial and the insurance risk ($\alpha \to \infty$) is equal to 67.

Let us first deal with the investment strategy. For $\mu_E = 0,28$, which corresponds to $\mu = 0,1$, the coefficient $\hat{\kappa}$, which determines the optimal investment strategy, equals to $\hat{\kappa} = 1,49$. If we replace the Variance Gamma process by the Brownian motion and keep the drift and the volatility at the same level, then $\hat{\kappa}$ increases to

Table 2: The reduction in the premiums due to investment in the risky asset; $\alpha = 0,05, n_0 = 1$.

μ	Reduction for the premium (5.21)	Reduction for the premium (5.23)
0,05	4,09%	3,75%
0,06	5,85%	3,91%
0,07	7,22%	4,13%
0,08	9,73%	4,29%
0,09	12,90%	4,31%
0,1	17,82%	5,44%
0,11	22,54%	6,18%
0,12	29,64%	7,89%
0,13	34,11%	8,68%
0,14	42,08%	10,09%

1,51. In the model with jumps it is optimal to invest smaller amount in the risky asset. The nice feature of the exponential indifference pricing is that the optimal strategy does not only depend on the first two moments of the distribution of the asset return but also takes into account the heavy-tailed nature of the underlying distribution.

Notice that the premium (5.10) is always greater than the premium calculated in the case of deterministic mortality. It means that the indifference principle (3.6) puts the price on systematic mortality risk. In our simple example the systematic mortality risk premium is about $2\% - 3\%$.

It has already been stated in section 4 that the premium can be reduced if the insurer applies the principle (3.7). It is interesting to note that the reduction is also possible under the principle (3.8). In table 2 we give the percentages of the premium reduction. The reductions, resulting from our two indifference principles (3.7) and (3.8), are significant and we believe that charging lower premium, when investing in the risky asset, is justified from the point of the utility theory. We would like to point out that the premium (5.21) is greater than the expected value of the liability for $\alpha > 0,045$.

Finally, if we take a look at the price which an individual has to pay for a contract in the portfolio, we notice that the price decreases when the number of policies increases. However, the difference between prices calculated for deterministic and stochastic intensity does not converge to zero when increasing the number of policies but stabilizes at the level around 2%. It supports commonly known fact

Table 3: The premiums per individual in the portfolio; $\alpha = 0,05, \mu = 0,1$.

n_0	Premium for $\bar{\lambda}$	Premium (5.10)
1	62,807	63,901
5	61,887	63,289
10	61,108	62,592
20	60,404	61,750
50	59,653	60,931
70	58,507	59,647
100	57,971	59,102
200	57,841	58,958

that unsystematic mortality risk can be diversified, whereas systematic mortality risk cannot be diversified.

We finish this section with the result concerning the probability of the insurer's financial insolvency. We believe that it is worth presenting the following simple estimate.

Lemma 5.5. *Assume that $r = 0$ and that the single premium is calculated according to (5.10). The ruin probability decreases to zero exponentially and the inequality holds*

$$\mathbb{P}\left(\inf_{t \in [0,T]} X^{\hat{\pi}}(t) < 0 | X(0) = x + n_0 B(0) \right) \leq e^{-\beta x + \beta n_0 \left(B^\beta(0) - B^\alpha(0) \right)}, \qquad (5.28)$$

where $\beta > \alpha$ is the unique solution of the equation

$$G(\beta \frac{\hat{\kappa}}{\alpha}) = 0. \qquad (5.29)$$

The coefficient β would be called an adjustment coefficient in the classical ruin theory and the estimate (5.28) can be viewed as the new type of the Lundberg's inequality, which is very common in the ruin theory, see [27].

6   Quadratic loss function

In this section we assume that the insurer applies the quadratic loss function of the form $u(x) = -(x - \alpha)^2, \alpha > 0$. Under this preferences the insurer arranges its payment process with the aim of reaching the desired level of the wealth α, while minimizing the mean-square error. This optimization criterium is similar to the criteria applied in [4,14,15]. Notice that our financial market differs from the markets considered in [4,14,15].

We need the stronger assumption concerning the measure ν in order to arrive at smooth solutions of the Hamilton-Jacobi-Bellman equations:

(A) $\int_{z>-1} z^4 \nu(dz), \infty$.

We do not present details of the calculations and only give same remarks. One can guess that the optimal value functions are given in the form $A_n(t, \lambda)x^2 + B_n(t, \lambda)x + C_n(t, \lambda)$, where the functions A_n, B_n and C_n solve partial parabolic differential equations which are different for each optimization problem. The optimal investment strategy is given by

$$\pi_n(t) = -\frac{\mu - r}{\sigma^2 + \int_{z>-1} z^2 \nu(dz)} \left(X(t-) + \frac{B_n(t, \lambda(t))}{2A_n(t, \lambda(t))} \right). \tag{6.1}$$

We would like to point out that the optimal investment strategy takes into account the current number of survived persons in the portfolio. Moreover, the optimal strategy depends not only on the currently available wealth of the insurer but also on the current value of the mortality intensity. This differs significantly from the optimal strategy for exponential utility functions.

7 Conclusions

In this paper we have investigated the problem of pricing and hedging of life insurance liabilities, in the presence of systematic mortality risk, in the financial market with an asset driven by the Lévy process. The indifference arguments were applied in order to derive pricing equations. The explicit solutions were found for exponential and quadratic utility/loss functions. Numerical methods can be used to arrive at solutions in the case of other loss functions. It would also be interesting to investigate how the systematic mortality risk effects quantiles of the distribution of the terminal wealth.

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8 Appendix

This appendix contains proofs of lemmas and theorems appearing in the paper.

Proof of lemma 4.1.
Let $\pi \in \mathcal{A}$ denote an arbitrary admissible control for the problem (4.1). The following relation holds

$$
\begin{aligned}
\mathbb{E}^{t,x,\lambda,n}\big[u(X^\pi(T))\big] \;=\; & \mathbb{E}^{t,x,\lambda,n}\big[u(X_n^{\pi_n}(T) - nB(T))\mathbf{1}\{\tau_{n-1} > T\} \\
& + u(X^\pi(T))\mathbf{1}\{\tau_{n-1} \le T\}\big].
\end{aligned}
\tag{8.1}
$$

We deal with the second factor. The property of conditional expectations implies that

$$
\begin{aligned}
&\mathbb{E}^{t,x,\lambda,n}\big[u(X^{\pi}(T))\mathbf{1}\{\tau_{n-1} \leq T\}\big] \\
&= \mathbb{E}^{t,x,\lambda,n}\big[\mathbb{E}\big[u(X^{\pi}(T))\mathbf{1}\{\tau_{n-1} \leq T\}|\mathcal{F}_{\tau_{n-1}\wedge T}\big]\big] \\
&= \mathbb{E}^{t,x,\lambda,n}\big[\mathbf{1}\{\tau_{n-1} \leq T\}\mathbb{E}\big[u(X^{\pi}(T))|\mathcal{F}_{\tau_{n-1}\wedge T}\big]\big].
\end{aligned} \tag{8.2}
$$

Notice that the following inequality holds $\mathbb{P}$-a.s.

$$
\begin{aligned}
&\mathbf{1}\{\tau_{n-1} \leq T\}\mathbb{E}\big[u(X^{\pi}(T)|\mathcal{F}_{\tau_{n-1}\wedge T}\big] \\
&\leq \mathbf{1}\{\tau_{n-1} \leq T\}V_{n-1}(\tau_{n-1}, X_n^{\pi_n}(\tau_{n-1}) - b, \lambda(\tau_{n-1})),
\end{aligned} \tag{8.3}
$$

due to the representation (3.3), the strong Markov property of the process X and the definition of the value function V_{n-1}. The equality in (8.3) is obtained by applying the optimal control on $(\tau_{n-1}, T]$ corresponding to the optimization problem with the value function V_{n-1}. $\qquad\square$

Proof of theorem 4.1

The similar verification theorem, for $n = 1$, is proved in [28], to which the interested reader is referred for more explanations.

Fix $t \in [0, T]$ and take an arbitrary admissible control $\pi_n \in \mathcal{A}$. Let v_n denote a function which satisfies the conditions of the theorem 4.1.

We start with establishing the following relation

$$
\begin{aligned}
&\mathbb{E}^{t,x,\lambda,n}\big[v_{n-1}(\tau_{n-1}, X_n^{\pi_n}(\tau_{n-1}) - b, \lambda(\tau_n))\mathbf{1}\{\tau_{n-1} \leq T\} + u(X_n^{\pi_n}(T) - nB(T))\mathbf{1}\{\tau_{n-1} > T\}\big] \\
&\quad - v_n(t, x, \lambda) \\
&= \mathbb{E}^{t,x,\lambda,n}\big[\big\{v_{n-1}(\tau_{n-1}, X_n^{\pi_n}(\tau_{n-1}) - b, \lambda(\tau_{n-1})) - v_n(\tau_{n-1}, X_n^{\pi_n}(\tau_{n-1}), \lambda(\tau_{n-1}))\big\} \\
&\qquad \times \mathbf{1}\{\tau_{n-1} \leq T\}\big] \\
&\quad + \mathbb{E}^{t,x,\lambda,n}\big[v_n(\tau_{n-1} \wedge T, X_n^{\pi_n}(\tau_{n-1} \wedge T), \lambda(\tau_{n-1} \wedge T)) - v_n(t, x, \lambda)\big].
\end{aligned} \tag{8.4}
$$

The first term can be rewritten as

$$\mathbb{E}^{t,x,\lambda,n}\Big[\big\{v_{n-1}(\tau_{n-1}, X_n^{\pi_n}(\tau_{n-1}) - b, \lambda(\tau_{n-1})) - v_n(\tau_{n-1}, X_n^{\pi_n}(\tau_{n-1}), \lambda(\tau_{n-1}))\big\}$$
$$\times \mathbf{1}\{\tau_{n-1} \leq T\}\Big]$$
$$= \mathbb{E}^{t,x,\lambda,n}\Big[\int_t^T \mathbb{E}^{t,x,\lambda,n}\big[\big(v_{n-1}(s, X_n^{\pi_n}(s-) - b, \lambda(s)) - v_n(s, X_n^{\pi_n}(s-), \lambda(s))\big)\big|\mathcal{F}_T^M\big]$$
$$\times n\lambda(s)e^{-\int_t^s n\lambda(w)dw}ds\Big]$$
$$= \mathbb{E}^{t,x,\lambda,n}\Big[\int_t^T \mathbb{E}^{t,x,\lambda,n}\big[\big(v_{n-1}(s, X_n^{\pi_n}(s-) - b, \lambda(s)) - v_n(s, X_n^{\pi_n}(s-), \lambda(s))\big)$$
$$\times n\lambda(s)\mathbf{1}\{\tau_{n-1} \geq s\}|\mathcal{F}_T^M\big]ds\Big]$$
$$= \mathbb{E}^{t,x,\lambda,n}\Big[\int_t^T n\lambda(s)\big(v_{n-1}(s, X_n^{\pi_n}(s-) - b, \lambda(s)) - v_n(s, X_n^{\pi_n}(s-), \lambda(s))\big)$$
$$\times \mathbf{1}\{\tau_{n-1} \geq s\}ds\Big] \tag{8.5}$$

where we have applied: the property that $X_n^{\pi_n}(\tau_{n-1}-) = X_n^{\pi_n}(\tau_{n-1})$ holds $\mathbb{P}$-a.s., the property of conditional expectations, the distribution of τ_{n-1} given the filtration $\mathcal{F}_T^M$, the independence of the random variables $\mathbf{1}\{\tau_{n-1} \geq s\}$, $X_n^{\pi_n}(s-)$ and the Fubini's theorem.

In order to handle the second term in (8.4) we introduce the localizing sequence $t_m = \inf\{s \in (t, T]; |X_n^{\pi_n}(s) - x| + |\lambda(s) - \lambda| + |\pi_n(s)| > m\}$. By choosing an arbitrary $\varepsilon \in (0, T - t)$ and applying the Itô formula for semimartingales, see the theorem 4.4.7 in [16], we arrive at

$$\mathbb{E}^{t,x,\lambda,n}\Big[v_n(\tau_{n-1} \wedge t_m \wedge (T - \varepsilon), X_n^{\pi_n}(\tau_{n-1} \wedge t_m \wedge (T - \varepsilon)), \lambda(\tau_{n-1} \wedge t_m \wedge (T - \varepsilon)))\Big]$$
$$- v_n(t, x, \lambda)$$
$$= \mathbb{E}^{t,x,\lambda,n}\Big[\int_t^{T-\varepsilon}\big(\frac{\partial v_n}{\partial t}(s, X_n^{\pi_n}(s-), \lambda(s)) + \mathcal{L}_F^{\pi_n(s)}v(s, X_n^{\pi_n}(s-), \lambda(s))$$
$$- \frac{\partial v_n}{\partial x}(s, X_n^{\pi_n}(s), \lambda(s))nc + \mathcal{L}_M v_n(s, X_n^{\pi_n}(s-), \lambda(s))\big)\mathbf{1}\{\tau_{n-1} \geq s, t_m \geq s\}ds\Big]. \tag{8.6}$$

We point out that the expected values of the stochastic integrals with respect to the Brownian motions and the Poisson random measure are equal to zero due to the condition (4.8) and the localizing sequence.

The next steps are rather standard, see for example the theorem 3.1 in [23]. In order to prove (4.11), first combine (8.5) with (8.6), then apply the inequality (4.6) and finally, take the limit $m \to \infty$ and $\varepsilon \to 0$. In order to arrive at the equality (4.14) apply the strategy (4.12). $\qquad\square$

We would like to point out that, due to our localizing sequence, we can get rid off some of the conditions stated in the theorem 3.1 in [23], which would be difficult

to check when verifying the optimality of the solution.

Proof of lemma 5.2

We follow the proof of the proposition 2.3 in [21]. Choose $\varepsilon > 0$ and consider the partial differential equation (5.7) on the time interval $[0, T - \varepsilon]$. The function

$$
\begin{aligned}
\phi_n(t, \lambda) \;=\; & \mathbb{E}^{t,\lambda}\big[\phi_n(T - \varepsilon, \lambda(T - \varepsilon))e^{\int_t^{T-\varepsilon}(\alpha f(s)nc - n\lambda(s))ds} \\
& + \int_t^{T-\varepsilon} e^{\alpha f(s)b}n\lambda(s)\phi_{n-1}(s, \lambda(s))e^{\int_t^s(\alpha f(u)nc - n\lambda(u))du}ds\big] \qquad (8.7)
\end{aligned}
$$

is the unique solution of the equation (5.7) on the time interval $[0, T - \varepsilon] \times (0, \infty)$, $\phi_n \in \mathcal{C}^{1,2}([0, T - \varepsilon) \times (0, \infty)) \times \mathcal{C}_b([0, T - \varepsilon] \times (0, \infty))$, provided that we show that the conditions from the theorem 1 in [20] are satisfied. By the assumption, the function ϕ_{n-1} is uniformly Hölder continuous on compact subsets of $[0, T - \varepsilon] \times \bar{D}_n$, so we only have to show that the mapping $(t, \lambda) \mapsto \phi_n(t, \lambda)$, defined in (5.8), is continuous.

Notice that the mapping

$$
\begin{aligned}
(t, \lambda) \mapsto\; & e^{\alpha n B(T)}e^{\int_t^T(\alpha f(s)nc - n\lambda^{t,\lambda}(s))ds} \\
& + \int_t^T e^{\alpha f(s)b}n\lambda^{t,\lambda}(s)\phi_{n-1}(s, \lambda^{t,\lambda}(s))e^{\int_t^s(\alpha f(u)nc - n\lambda^{t,\lambda}(u))du}ds, \qquad (8.8)
\end{aligned}
$$

is $\mathbb{P}$-a.s. uniformly bounded in (t, λ), with the bound depending on the bound of ϕ_{n-1}. The continuity of the mapping $(t, \lambda) \mapsto \phi_n(t, \lambda)$ now follows from the Lebesgue's dominated convergence theorem and the continuity of the mapping $(t, \lambda) \mapsto \lambda^{t,\lambda}(s)$.

As ε is arbitrary, the existence of the solution ϕ_n, its probabilistic representation (5.8) and its smoothness on $[0, T] \times (0, \infty)$ follows. $\qquad\square$

Proof of lemma 5.3

Notice that

$$
\begin{aligned}
\phi_n(t, \lambda) \;=\; & \mathbb{E}^{t,\lambda,n}\big[e^{\int_t^T \alpha f(s)ncds + \alpha n B(T)}\mathbf{1}\{\tau_{n-1} > T\} \\
& + e^{\int_t^{\tau_{n-1}} \alpha f(u)ncdu + \alpha f(\tau_{n-1})b}\phi_{n-1}(\tau_{n-1}, \lambda(\tau_{n-1}))\mathbf{1}\{\tau_{n-1} \leq T\}\big], \;\; (8.9)
\end{aligned}
$$

holds due to the property of conditional expectations and the distribution of the random variable τ_{n-1} given the filtration $\mathcal{F}_T^M$. By applying again the property of

conditional expectations and the strong Markov property we can arrive at

$$
\begin{aligned}
&\phi_n(t,\lambda) \\
&= \mathbb{E}^{t,\lambda,n}\Big[\mathbb{E}\big[e^{\int_t^T \alpha f(s)ncds+\alpha nB(T)}\mathbf{1}\{\tau_{n-1}>T\} \\
&\quad + e^{\int_t^{\tau_{n-1}} \alpha f(u)ncdu+\alpha f(\tau_{n-1})b}\phi_{n-1}(\tau_{n-1},\lambda(\tau_{n-1}))\mathbf{1}\{\tau_{n-1}\le T\}\big|\mathcal{F}_{\tau_{n-1}\wedge T}\big]\Big] \\
&= \mathbb{E}^{t,\lambda,n}\Big[e^{\int_t^T \alpha f(s)ncds+\alpha nB(T)}\mathbf{1}\{\tau_{n-1}>T\} + e^{\int_t^{\tau_{n-1}}\alpha f(u)ncdu+\alpha f(\tau_{n-1})b} \\
&\quad \times \mathbb{E}\big[e^{\int_{\tau_{n-1}}^T \alpha f(s)(n-1)cds+\alpha(n-1)B(T)}\mathbf{1}\{\tau_{n-2}>T\} + e^{\int_{\tau_{n-1}}^{\tau_{n-2}}\alpha f(u)(n-1)cdu+\alpha f(\tau_{n-2})b} \\
&\quad \times \phi_{n-2}(\tau_{n-2},\lambda(\tau_{n-2}))\mathbf{1}\{\tau_{n-2}\le T\}\big|\mathcal{F}_{\tau_{n-1}\wedge T}\big]\mathbf{1}\{\tau_{n-1}\le T\}\Big] \\
&= \mathbb{E}^{t,\lambda,n}\Big[e^{\int_t^T \alpha f(s)ncds+\alpha nB(T)}\mathbf{1}\{\tau_{n-1}>T\} \\
&\quad + e^{\int_t^{\tau_{n-1}}\alpha f(u)ncdu+\alpha f(\tau_{n-1})b+\int_{\tau_{n-1}}^T \alpha f(s)(n-1)cds+\alpha(n-1)B(T)}\mathbf{1}\{\tau_{n-1}\le T,\tau_{n-2}>T\} \\
&\quad + e^{\int_t^{\tau_{n-1}}\alpha f(u)ncdu+\alpha f(\tau_{n-1})b+\int_{\tau_{n-1}}^{\tau_{n-2}}\alpha f(u)(n-1)cdu+\alpha f(\tau_{n-2})b}\phi_{n-2}(\tau_{n-2},\lambda(\tau_{n-2}))\mathbf{1}\{\tau_{n-2}\le T\}\Big]
\end{aligned}
$$

$$(8.10)$$

By continuing the calculations we can arrive at the representation (5.9) for $n=n_0$ and $t=0$. $\qquad\square$

Proof of lemma 5.4

The properties 1-4 of the premium (5.10) are well-known, see for example the theorem 3.1.1 in [27]. These properties can also be extended for the premium (5.23). As the expectation in (5.23) is taken under the measure $\mathbb{P}^G$, the expected value, arising in the limit $\alpha \to 0$, has also to be taken under the measure $\mathbb{P}^G$. The property 3 remains unchanged due to the equivalence of the measures.

We can now deal with the premium (5.21). The properties 2 and 4 clearly hold true. To prove the property 1 take $\alpha_1 \le \alpha_2$ and apply Jensen's inequality to arrive at

$$
\begin{aligned}
&\Big(\mathbb{E}\big[e^{G(\hat\kappa)(\tau_0\wedge T)+\alpha_1\int_0^T e^{-rt}dP(t)}\big]\Big)^{\frac{\alpha_2}{\alpha_1}} \\
&\le \mathbb{E}\big[e^{\frac{\alpha_2}{\alpha_1}G(\hat\kappa)(\tau_0\wedge T)+\alpha_2\int_0^T e^{-rt}dP(t)}\big] \le \mathbb{E}\big[e^{G(\hat\kappa)(\tau_0\wedge T)+\alpha_2\int_0^T e^{-rt}dP(t)}\big],
\end{aligned}
$$

$$(8.11)$$

where the second inequality follows from the negativity of $G(\hat\kappa)$. To prove the property 3 notice that

$$
\begin{aligned}
&\frac{G(\hat\kappa)T}{\alpha} + \frac{1}{\alpha}\log\mathbb{E}\big[e^{\alpha\int_0^T e^{-rt}dP(t)}\big] \\
&\le \frac{1}{\alpha}\log\mathbb{E}\big[e^{G(\hat\kappa)(\tau_0\wedge T)+\alpha\int_0^T e^{-rt}dP(t)}\big] \le \frac{1}{\alpha}\log\mathbb{E}\big[e^{\alpha\int_0^T e^{-rt}dP(t)}\big],
\end{aligned}
$$

$$(8.12)$$

holds. The proof is completed by taking the limit $\alpha\to\infty$. $\qquad\square$

Proof of lemma 5.5

Notice that the process

$$X_0^{x,\hat{\pi}_0}(t) = x + \frac{\hat{\kappa}}{\alpha}\left(\mu t + \sigma W(t) + \int_0^T \int_{z>-1} z\tilde{M}(dz \times dt)\right) \qquad (8.13)$$

is a Lévy process. It follows from chapter 5.4.5 in [16] that the process $\exp\big(-\beta X_0^{x,\hat{\pi}_0}(t) - tG(\beta\frac{\hat{\kappa}}{\alpha})\big)$ is a martingale. As $X^{x,\hat{\pi}}(t) = X_0^{x,\hat{\pi}_0}(t) + n_0 B(0) - \int_0^t dP(s)$ holds $\mathbb{P}$-a.s. and the payment process is increasing, we can conclude that the process $\exp\big(-\beta X^{x,\hat{\pi}}(t) - tG(\beta\frac{\hat{\kappa}}{\alpha})\big)$ is a submartingale.

Choose β such that $G(\beta\frac{\hat{\kappa}}{\alpha}) = 0$. The existence of $\beta > \alpha$ follows from the properties of the function G. By applying the submartingale inequality, see the theorem 10.2.1 in [27], we arrive at

$$\mathbb{P}\big(\inf_{t\in[0,T]} X^{x,\hat{\pi}}(t) \leq 0\big) = \mathbb{P}\big(\sup_{t\in[0,T]} e^{-\beta X^{x,\hat{\pi}}(t)} \geq 1\big)$$

$$\leq \mathbb{E}\big[e^{-\beta X^{x,\hat{\pi}}(T)}\big] = \mathbb{E}\big[e^{-\beta(X_0^{x,\hat{\pi}_0}(T)+n_0 B(0)-\int_0^T dP(t))}\big]$$

$$= \mathbb{E}\big[e^{-\beta X_0^{x,\hat{\pi}_0}(T)}\big]e^{-\beta n_0 B(0)}\mathbb{E}\big[e^{\beta\int_0^T dP(t)}\big] = e^{-\beta x - \beta n_0 B^\alpha(0) + n_0\beta B^\beta(0)}, \quad (8.14)$$

which completes the proof. $\qquad\qquad\square$

The proofs that the investment strategy and our candidate value function satisfy the verification theorem is left to the reader. The details of the calculations can be obtained from the author upon request.