The Aggregate Claim Amount Discrete Time Semi-Markov Model

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Abstract
In this paper a semi-Markov model useful for following the time evolution of the aggregate claim amount is presented. The paper firstly presents a short introduction to homogeneous and non-homogeneous discrete time semi-Markov processes and the related reward processes with finite state number. In a second section it is explained how it is possible to follow the time evolution of the aggregate claim amount by means of a discrete time semi-Markov processes with a denumerable state number in both homogeneous and non-homogeneous environments. The following step given in the paper is the introduction of rewards in the model. This is the fundamental step because, by means of both models, in this way it is possible to study the time evolution of the system and also the mean amount of claims. Indeed, by semi-Markov solution the mass probability function is reconstructed for each period and by means of reward processes it is possible to know the mean of the claim reserve for each year of forecasting. After it will be shown how changing the reward values it is possible to study also the claim number process.

Keywords: risk processes, semi-Markov processes, reward processes, homogeneous, non-homogeneous

1. Introduction

As well known the three basic risk models:

i) the claim number process,
ii) the aggregated claim amount process,
iii) the premium process,

are studied, starting from the beginning of the last century by means of renewal processes.
The seminal paper of Lundberg (1909) was the one that presented this approach. After many authors gave important contribution to the theory; we recall Cramér (1930), Cramér (1955), Andersen (1967), Andersen et al (1993). More recently the main results were reported in Asmussen (2000), Rolski et al (1999) and Janssen and Manca (2007) books.
Semi-Markov processes are a strong generalization of the renewal processes (see Limnios and Oprişan (2001), Janssen and Manca (2006)) and it is quite natural to think about the generalization of the renewal risk models in a semi-Markov environment. This idea was introduced by Miller (1962) and fully developed in Janssen (1969, 1970, 1977).
The original idea was to introduce m possible types of claims that constitute the set of states of the semi-Markov process. This set was considered as an environment parameter having influences on the three basic risk processes.

Later Janssen and Reinhard (1982) generalized this approach presenting a two time dimensional semi-Markov process.
See Janssen and Manca (2007) for a complete presentation of semi-Markov risk models.
In this paper the semi-Markov approach is given from another point of view. In fact, we present a straightforward semi-Markov generalization of the renewal aggregate claim amount risk process and of the renewal claim number process.
The state sets are denumerable and represent, in the aggregate claim amount process the value of the total claims paid from the insurance company in the aggregated claim amount process, and in the claim number process the number of claims.
To get the mean accumulated claim value it is necessary to introduce also the semi-Markov reward processes. The reward processes are also used to obtain the mean total number of the claims.
The model is presented in both homogeneous and non-homogeneous discrete time environment. The choice between the two models depends on the quantity of data that are available. Non-homogeneous environment is closer to the real world problems but the number of data that is necessary is by far greater than the one necessary in the homogeneous case. So if the number of data is sufficient, the best approach should be the non-homogeneous. Instead, if the set of data is not enough sufficient for the non-homogeneous approach, then it could be better to work in a homogeneous environment.
The paper is structured in the following way. In the next section the semi-Markov environment is shortly introduced; in section three the aggregate claim amount is presented. In the fourth section the claim number processes are presented. In the last section some conclusive remarks are given.

2. The homogeneous and non-homogeneous discrete time semi-Markov and semi-Markov reward processes

2.1 Semi-Markov processes

In SMP environment, two random variables (r.v.) run together. $J_n \quad n\in \mathbb{N}$, with finite state space $E=\{1,\ldots,m\}$ represents the state at the n-th transition. T_n, $n\in \mathbb{N}$ with state space equal to $\mathbb{N}$ represents the time of the nth transition, $J_n:\Omega\to E \quad T_n:\Omega\to\mathbb{N}$.
We suppose that the processes (J_n, T_n) are homogeneous or non-homogeneous Markov renewal processes.
The kernels $\mathbf{Q}=[Q_{ij}(t)]$, $\mathbf{Q}=[Q_{ij}(s,t)]$ associated respectively to the homogeneous and the non-homogeneous processes are defined in the following way:

$$Q_{ij}(t)=\mathrm{P}\left[J_{n+1}=j,T_{n+1}-T_n\le t\mid J_n=i,\,J_{n-1},T_{n-1},\ldots,J_1,T_1,J_0,T_0\right]=\mathrm{P}\left[J_{n+1}=j,T_{n+1}-T_n\le t\mid J_n=i\right],$$
$$Q_{ij}(s,t)=\mathrm{P}\left[J_{n+1}=j,T_{n+1}\le t\mid J_n=i,T_n=s,J_{n-1},T_{n-1},\ldots,J_1,T_1,J_0,T_0\right]=\mathrm{P}\left[J_{n+1}=j,T_{n+1}\le t\mid J_n=i,T_n=s\right]$$

and it results:

$$p_{ij}=\lim_{t\to\infty}Q_{ij}(t);\ i,j\in E,\ t\in\mathbb{N},$$
$$p_{ij}(s)=\lim_{t\to\infty}Q_{ij}(s,t);\ i,j\in E,\ s\in\mathbb{N},\ s\le t\,,$$

where respectively $\mathbf{P}=[p_{ij}]$ and $\mathbf{P}(s)=[p_{ij}(s)]$ are the transition matrices of the embedded homogeneous and non-homogeneous Markov chains of the two processes.
In the discrete environment it is necessary to define also the following probabilities:

$$b_{ij}(t)=\mathrm{P}\left[J_{n+1}=j,T_{n+1}-T_n=t\mid J_n=i\right],$$

$b_{ij}(s,t) = \mathrm{P}\left[J_{n+1} = j, T_{n+1} = t \mid J_n = i, T_n = s\right].$

Now it is necessary to introduce the probability that the processes will leave state *i* in a time *t*, homogeneous case, and from time *s* up to time *t*, non-homogeneous case:

$$H_i(t) = \mathrm{P}\left[T_{n+1} - T_n \le t \mid J_n = i\right] = \sum_{j=1}^{m} Q_{ij}(t) \tag{2.1}$$

$$H_i(s,t) = \mathrm{P}\left[T_{n+1} \le t \mid J_n = i,\, T_n = s\right] = \sum_{j=1}^{m} Q_{ij}(s,t)\,. \tag{2.2}$$

The distribution functions of the waiting time in each state *i*, given that the state successively occupied is known are the following:

$$F_{ij}(t)\left[T_{n+1} - T_n \le t \mid J_{n+1} = j, J_n = i\right] = \begin{cases} Q_{i,j}(t) / p_{i,j} & \text{if} \quad p_{i,j} \ne 0 \\ 1 & \text{if} \quad p_{i,j} = 0, \end{cases}$$

$$F_{ij}(s,t)\left[T_{n+1} \le t \mid J_{n+1} = j, J_n = i, T_n = s\right] = \begin{cases} Q_{i,j}(s,t) / p_{i,j}(s) & \text{if} \quad p_{i,j}(s) \ne 0 \\ 1 & \text{if} \quad p_{i,j}(s) = 0. \end{cases}$$

Now let be $N(t) = \sup\{n \in \mathbb{N} \mid T_n \le t\}$ then the HSMP and NHSMP $Z(t) = J_{N(t)},\ t \in \mathbb{N}$ can be defined.
They represent, for each waiting time, the state occupied respectively by the homogeneous and non homogenous processes.
The semi-Markov transition probabilities are defined respectively by

$$\phi_{i,j}(t) = \mathrm{P}\left[Z(t) = j \mid Z(0) = i\right],$$
$$\phi_{i,j}(s,t) = \mathrm{P}\left[Z(t) = j \mid Z(s) = i\right].$$

The evolution equations of DTHSMP and DTNHSMP are:

$$\phi_{i,j}(t) = \left(1 - H_i(t)\right)\delta_{ij} + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{i,k}(\vartheta)\phi_{k,j}(t-\vartheta), \tag{2.3}$$

$$\phi_{i,j}(s,t) = \left(1 - H_i(s,t)\right)\delta_{ij} + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{i,k}(s,\vartheta)\phi_{k,j}(\vartheta,t). \tag{2.4}$$

In matrix form (2.3) and (2.4) become:

$$\mathbf{\Phi}(t) = \mathbf{D}(t) + \sum_{\vartheta=1}^{t} \mathbf{B}(\vartheta) * \mathbf{\Phi}(t-\vartheta), \tag{2.5}$$

$$\mathbf{\Phi}(s,t) = \mathbf{D}(s,t) + \sum_{\vartheta=s+1}^{t} \mathbf{B}(s,\vartheta) * \mathbf{\Phi}(\vartheta,t), \tag{2.6}$$

where $\mathbf{D}(t)$ and $\mathbf{D}(s,t)$ are diagonal matrices respectively with $d_{ii}(t) = 1 - H_i(t),\ d_{ii}(s,t) = 1 - H_i(s,t)$.

These two matrix equations can be written in block matrix form in the following way:

$$\begin{bmatrix}\mathbf{\Phi}(0)\\ \mathbf{\Phi}(1)\\ \mathbf{\Phi}(2)\\ \vdots\end{bmatrix}=\begin{bmatrix}\mathbf{I}\\ \mathbf{D}(1)\\ \mathbf{D}(2)\\ \vdots\end{bmatrix}+\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots\\ \mathbf{B}(1) & \mathbf{0} & \mathbf{0} & \cdots\\ \mathbf{B}(2) & \mathbf{B}(1) & \mathbf{0} & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}*\begin{bmatrix}\mathbf{\Phi}(0)\\ \mathbf{\Phi}(1)\\ \mathbf{\Phi}(2)\\ \vdots\end{bmatrix}, \tag{2.7}$$

$$\begin{aligned}\begin{bmatrix}\mathbf{\Phi}(0,0) & \mathbf{\Phi}(0,1) & \mathbf{\Phi}(0,2) & \cdots\\ \mathbf{0} & \mathbf{\Phi}(1,1) & \mathbf{\Phi}(1,2) & \cdots\\ \mathbf{0} & \mathbf{0} & \mathbf{\Phi}(2,2) & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}&=\begin{bmatrix}\mathbf{I} & \mathbf{D}(0,1) & \mathbf{D}(0,2) & \cdots\\ \mathbf{0} & \mathbf{I} & \mathbf{D}(1,2) & \cdots\\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}\\ &+\begin{bmatrix}\mathbf{0} & \mathbf{B}(0,1) & \mathbf{B}(0,2) & \cdots\\ \mathbf{0} & \mathbf{0} & \mathbf{B}(1,2) & \cdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}*\begin{bmatrix}\mathbf{\Phi}(0,0) & \mathbf{\Phi}(0,1) & \mathbf{\Phi}(0,2) & \cdots\\ \mathbf{0} & \mathbf{\Phi}(1,1) & \mathbf{\Phi}(1,2) & \cdots\\ \mathbf{0} & \mathbf{0} & \mathbf{\Phi}(2,2) & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}\end{aligned} \tag{2.8}$$

where **I** is the identity matrix and **0** the null matrix.
(2.7) can be solved very easily by means of a forward algorithm. Indeed, $\mathbf{\Phi}(0)=\mathbf{I}$ because in a time 0 it is impossible to move from a state; known $\mathbf{\Phi}(0)$ it is possible to compute $\mathbf{\Phi}(1)$, known $\mathbf{\Phi}(0)$ and $\mathbf{\Phi}(1)$ it is possible to compute $\mathbf{\Phi}(2)$ and so on. Also (2.8) can be computed by means of a forward algorithm. It results that $\mathbf{\Phi}(t,t)=\mathbf{I}$ then it is possible to compute $\mathbf{\Phi}(t,t-1)$ known these two blocks it is possible to compute $\mathbf{\Phi}(t,t-2)$ and so on. To have a forward algorithm we can start from $t=0$ then find the two blocks with $t=1\ldots$ In the non-homogeneous case it is possible also to work with a diagonal and a backward approach (see Stenberg et al (2007)).

2.2 Semi-Markov reward processes

Semi-Markov reward processes, as specified in Janssen Manca (2006, 2007), can be seen as a class of stochastic processes because there are a lot of different evolution equations depending on the problem that should be solved.
Rewards can be money amounts but also other things. The evolution equations that will be reported are the only ones that we need in this paper.
In general it is possible to have permanence rewards or transition rewards (in literature they are also called respectively rate rewards and impulse rewards see Qureshi and Sanders (1994)). The first is paid or received because of the permanence inside a state, the second because of a transition.
With $\psi_i(t)$ and $\gamma_{ij}(t)$ we denote respectively the rewards paid or received for the permanence inside the state *i* and for transition from the state *i* to the state *j*.
The homogeneous and the non-homogeneous semi-Markov reward evolution equations that we report are the following:

$$V_i(t)=\left(1-H_i(t)\right)\sum_{\tau=1}^{t}\psi_i(\tau)\nu(\tau)+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{i,k}(\vartheta)\left(\nu(\vartheta)\gamma_{ik}(\vartheta)+\sum_{\tau=1}^{\vartheta}\psi_i(\tau)\nu(\tau)\right)+\sum_{k=1}^{m}\sum_{\vartheta=1}^{t}b_{i,k}(\vartheta)\nu(\vartheta)V_k(t-\vartheta), \tag{2.9}$$

$$V_i(s,t) = \left(1 - H_i(s,t)\right) \sum_{\tau=s+1}^{t} \psi_i(\tau)\nu(s,\tau) + \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{i,k}(s,\vartheta)\left(\nu(s,\vartheta)\gamma_{ik}(\vartheta) + \sum_{\tau=s+1}^{\vartheta} \psi_i(\tau)\nu(s,\tau) \right)$$
$$+ \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{i,k}(s,\vartheta)\nu(s,\vartheta)V_k(\vartheta,t). \tag{2.10}$$

$V_i(t)$ and $V_i(s,t)$ represent the mean present values of all the rewards (RMPV) that were paid and/or received starting in the state *i* respectively from time *s* up to time *t* in non-homogenous case and in a time *t* in homogeneous one; $\nu(t)$ is the discount factor for a time *t* while $\nu(s,t)$ is the discount factor from time *s* up to the time *t*.
The two evolution equations are compound by three parts:

$$\left(1 - H_i(t)\right) \sum_{\tau=1}^{t} \psi_i(\tau)\nu(\tau), \quad \left(1 - H_i(s,t)\right) \sum_{\tau=s+1}^{t} \psi_i(\tau)\nu(s,\tau), \tag{2.11}$$

$$\sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{i,k}(\vartheta)\left(\nu(\vartheta)\gamma_{ik}(\vartheta) + \sum_{\tau=1}^{\vartheta} \psi_i(\tau)\nu(\tau) \right), \ \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\left(\nu(s,\vartheta)\gamma_{ik}(\vartheta) + \sum_{\tau=s+1}^{\vartheta} \psi_i(\tau)\nu(s,\tau) \right), \tag{2.12}$$

$$\sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{i,k}(\vartheta)\nu(\vartheta)V_k(t-\vartheta), \ \sum_{k=1}^{m} \sum_{\vartheta=s+1}^{t} b_{i,k}(s,\vartheta)\nu(s,\vartheta)V_k(\vartheta,t). \tag{2.13}$$

(2.11) represents the present values of the rewards gotten remaining in the state *i* without any state change in a time *t*, for the homogeneous environment, and from *s* to *t* in non-homogeneous case.
(2.12) gives the present values of the rewards paid and/or received because of the transition in the state *k* and remaining inside the state *i* up to the first transition into the state *k*.
At last (2.13) represents the mean present values of all the rewards gotten in the state *k* after the transition. All these amounts are at time ϑ so, in homogeneous case, we must discount at time 0 and in non-homogeneous case at time *s*.
The Matrix form of (2.9) and (2.10) are the following:

$$\mathbf{V}(t) = \left(\mathbf{D}(t) \cdot \mathbf{A}(t)\right) * \mathbf{1} + \sum_{\vartheta=1}^{t} \mathbf{B}(\vartheta) \circ \overline{\mathbf{A}}(\vartheta) + \sum_{\vartheta=1}^{t} \nu(\vartheta)\mathbf{B}(\vartheta) * \mathbf{V}(t-\vartheta), \tag{2.14}$$

$$\mathbf{V}(s,t) = \left(\mathbf{D}(s,t) \cdot \mathbf{A}(s,t)\right) * \mathbf{1} + \sum_{\vartheta=s+1}^{t} \mathbf{B}(s,\vartheta) \circ \overline{\mathbf{A}}(s,\vartheta) + \sum_{\vartheta=s+1}^{t} \nu(s,\vartheta)\mathbf{B}(s,\vartheta) * \mathbf{V}(\vartheta,t), \tag{2.15}$$

where $\cdot$ represents the elements by elements matrix product, $\mathbf{c} = \mathbf{B} \circ \mathbf{A}$ is a column vector in which it results that $c_i = \sum_{k=1}^{m} b_{ik}a_{ik}$ and

$$a_{ij}(t) = \begin{cases} \sum_{\tau=1}^{t} \psi_i(\tau)\nu(\tau) & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases} \quad \text{and} \quad a_{ij}(s,t) = \begin{cases} \sum_{\tau=s+1}^{t} \psi_i(\tau)\nu(s,\tau) & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases}$$

$$\bar{a}_{ij}(t)=\begin{cases}\nu(t)\gamma_{ii}+\sum_{\tau=1}^{t}\psi_i(\tau)\nu(\tau) & \text{if} \quad i=j,\\ \nu(t)\gamma_{ij} & \text{if} \quad i\neq j,\end{cases} \quad \text{and} \quad \bar{a}_{ij}(s,t)=\begin{cases}\nu(s,t)\gamma_{ii}(t)+\sum_{\tau=s+1}^{t}\psi_i(\tau)\nu(s,\tau) & \text{if} \quad i=j,\\ \nu(s,t)\gamma_{ij}(t) & \text{if} \quad i\neq j.\end{cases}$$

The (2.14) and (2.15) can be written in block matrix form in the following way:

$$\begin{bmatrix}\mathbf{V}(0)\\ \mathbf{V}(1)\\ \mathbf{V}(2)\\ \vdots\end{bmatrix}=\left(\begin{bmatrix}\mathbf{I}\\ \mathbf{D}(1)\\ \mathbf{D}(2)\\ \vdots\end{bmatrix}\cdot\begin{bmatrix}\mathbf{0}\\ \mathbf{A}(1)\\ \mathbf{A}(2)\\ \vdots\end{bmatrix}\right)\odot\begin{bmatrix}\mathbf{1}\\ \mathbf{1}\\ \mathbf{1}\\ \vdots\end{bmatrix}+\begin{bmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots\\ \mathbf{B}(1)&\mathbf{0}&\mathbf{0}&\cdots\\ \mathbf{B}(1)&\mathbf{B}(2)&\mathbf{0}&\cdots\\ \vdots&\vdots&\vdots&\ddots\end{bmatrix}\otimes\begin{bmatrix}\mathbf{0}\\ \bar{\mathbf{A}}(1)\\ \bar{\mathbf{A}}(2)\\ \vdots\end{bmatrix}+\begin{bmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots\\ \nu(1)\mathbf{B}(1)&\mathbf{0}&\mathbf{0}&\cdots\\ \nu(2)\mathbf{B}(2)&\nu(1)\mathbf{B}(1)&\mathbf{0}&\cdots\\ \vdots&\vdots&\vdots&\ddots\end{bmatrix}*\begin{bmatrix}\mathbf{V}(0)\\ \mathbf{V}(1)\\ \mathbf{V}(2)\\ \vdots\end{bmatrix},$$

$$\begin{bmatrix}\mathbf{V}(0,0)&\mathbf{V}(0,1)&\mathbf{V}(0,2)&\cdots\\ \mathbf{0}&\mathbf{V}(1,1)&\mathbf{V}(1,2)&\cdots\\ \mathbf{0}&\mathbf{0}&\mathbf{V}(2,2)&\cdots\\ \vdots&\vdots&\vdots&\ddots\end{bmatrix}=\left(\begin{bmatrix}\mathbf{I}&\mathbf{D}(0,1)&\mathbf{D}(0,2)&\cdots\\ \mathbf{0}&\mathbf{I}&\mathbf{D}(1,2)&\cdots\\ \mathbf{0}&\mathbf{0}&\mathbf{I}&\cdots\\ \vdots&\vdots&\vdots&\ddots\end{bmatrix}\cdot\begin{bmatrix}\mathbf{0}&\mathbf{A}(0,1)&\mathbf{A}(0,2)&\cdots\\ \mathbf{0}&\mathbf{0}&\mathbf{A}(1,2)&\cdots\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots\\ \vdots&\vdots&\vdots&\ddots\end{bmatrix}\right)\odot\begin{bmatrix}\mathbf{1}&\mathbf{1}&\mathbf{1}&\cdots\\ \mathbf{0}&\mathbf{1}&\mathbf{1}&\cdots\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\cdots\\ \vdots&\vdots&\vdots&\ddots\end{bmatrix}$$

$$+\begin{bmatrix}\mathbf{0}&\mathbf{B}(0,1)&\mathbf{B}(0,2)&\cdots\\ \mathbf{0}&\mathbf{0}&\mathbf{B}(1,2)&\cdots\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots\\ \vdots&\vdots&\vdots&\ddots\end{bmatrix}\oplus\begin{bmatrix}\mathbf{0}&\bar{\mathbf{A}}(0,1)&\bar{\mathbf{A}}(0,2)&\cdots\\ \mathbf{0}&\mathbf{0}&\bar{\mathbf{A}}(1,2)&\cdots\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots\\ \vdots&\vdots&\vdots&\ddots\end{bmatrix}$$

$$+\begin{bmatrix}\mathbf{0}&\nu(0,1)\mathbf{B}(0,1)&\nu(0,2)\mathbf{B}(0,2)&\cdots\\ \mathbf{0}&\mathbf{0}&\nu(1,2)\mathbf{B}(1,2)&\cdots\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots\\ \vdots&\vdots&\vdots&\ddots\end{bmatrix}*\begin{bmatrix}\mathbf{V}(0,0)&\mathbf{V}(0,1)&\mathbf{V}(0,2)&\cdots\\ \mathbf{0}&\mathbf{V}(1,1)&\mathbf{V}(1,2)&\cdots\\ \mathbf{0}&\mathbf{0}&\mathbf{V}(2,2)&\cdots\\ \vdots&\vdots&\vdots&\ddots\end{bmatrix},$$

where **1** is the sum vector, $\odot$ represents the elements by elements matrix product at block level and the usual row column product between the blocks.
$\otimes$ is defined in the following way:

$$\begin{bmatrix}\mathbf{0}\\ \mathbf{c}(1)\\ \mathbf{c}(2)\\ \vdots\end{bmatrix}=\begin{bmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots\\ \mathbf{B}(1)&\mathbf{0}&\mathbf{0}&\cdots\\ \mathbf{B}(1)&\mathbf{B}(0,2)&\mathbf{0}&\cdots\\ \vdots&\vdots&\vdots&\ddots\end{bmatrix}\otimes\begin{bmatrix}\mathbf{0}\\ \bar{\mathbf{A}}(1)\\ \bar{\mathbf{A}}(2)\\ \vdots\end{bmatrix},$$

$$\mathbf{c}(t)=\sum_{\vartheta=1}^{t}\mathbf{B}(\vartheta)\circ\bar{\mathbf{A}}(\vartheta).$$

$\oplus$ is defined in the following way:

$$\begin{bmatrix} \mathbf{0} & \mathbf{c}(0,1) & \mathbf{c}(0,2) & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{c}(1,2) & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{B}(0,1) & \mathbf{B}(0,2) & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{B}(1,2) & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \oplus \begin{bmatrix} \mathbf{0} & \bar{\mathbf{A}}(0,1) & \bar{\mathbf{A}}(0,2) & \cdots \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}(1,2) & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad (2.16)$$

$$\mathbf{c}(s,t) = \sum_{\vartheta=s+1}^{t} \mathbf{B}(s,\vartheta) \circ \bar{\mathbf{A}}(s,\vartheta).$$

The construction of the algorithms in reward cases is similar to one described for the general homogeneous and non-homogeneous semi-Markov processes. Indeed, in reward cases the main change is related to the construction of the known term matrices. The structure of the coefficient matrices is similar. The difference is that, after the construction of the block matrices the blocks will be constituted by vector and not matrices. The forward algorithm for the homogeneous case and the three different approaches for the non-homogeneous case hold also for the reward processes (Stenberg et al. (2007)).

3. The homogeneous and non-homogeneous aggregate claim amount

3.1 The semi-Markov model

In the proposed model the two elements of the couple (J_n, T_n) represent respectively the accumulation value at *n*th transition and the time of the *n*th transition. In this case it results:

$$(J_n, T_n) \in \mathbb{N} \times \mathbb{N}.$$

The state set is given by $\mathbb{N}$ and the most part of the relations of the previous section are unchanged. We report, for the sake of clarity, only the (2.1), (2.2) and the two evolution equations that become respectively:

$$H_i(t) = \sum_{j\in\mathbb{N}} Q_{ij}(t), \quad H_i(s,t) = \sum_{j\in\mathbb{N}} Q_{ij}(s,t)$$

and

$$\begin{aligned} \phi_{ij}(t) &= \left(1 - H_i(t)\right)\delta_{ij} + \sum_{k\in\mathbb{N}} \sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\phi_{kj}(t-\vartheta), \\ \phi_{ij}(s,t) &= \left(1 - H_i(s,t)\right)\delta_{ij} + \sum_{k\in\mathbb{N}} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\phi_{kj}(\vartheta,t). \end{aligned} \quad (3.1)$$

The matrix equations of the two evolution equations (3.1) are the same of (2.5) and (2.6), the unique difference is given by the fact that the state set is denumerable.
The two matrix equations can be solved, as already explained in section two, by a forward process. In this case we have that also the states are denumerable and this fact could be a serious problem for the model application. But we are studying the aggregate claim process and the embedded Markov chains in homogeneous case and non-homogeneous case have a very special topological structure.
Indeed, we study an accumulation process and J_n represents the total amount of claims gotten up to the *n*th transition. It is possible to consider the following two different hypotheses:

i) $J_{n-1} < J_n$
ii) $J_{n-1} \leq J_n$

In the second case we suppose that it is possible to have some claim without any payment, this can be, for example, the case of precautionary letter (for example in motorcar insurance the insurer write to the insurance company that there was an accident; he is the damaged person and does not have any responsibility for the accident).
The embedded Markov chains in homogenous and non-homogeneous environments in the two cases will be respectively:

$$\text{i)}\ \mathbf{P} = \begin{bmatrix} 0 & p_{12} & p_{13} & p_{14} & \cdots \\ 0 & 0 & p_{23} & p_{24} & \cdots \\ 0 & 0 & 0 & p_{34} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad \mathbf{P}(s) = \begin{bmatrix} 0 & p_{12}(s) & p_{13}(s) & p_{14}(s) & \cdots \\ 0 & 0 & p_{23}(s) & p_{24}(s) & \cdots \\ 0 & 0 & 0 & p_{34}(s) & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\text{ii)}\ \mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & \cdots \\ 0 & p_{22} & p_{23} & p_{24} & \cdots \\ 0 & 0 & p_{33} & p_{34} & \cdots \\ 0 & 0 & 0 & p_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad \mathbf{P}(s) = \begin{bmatrix} p_{11}(s) & p_{12}(s) & p_{13}(s) & p_{14}(s) & \cdots \\ 0 & p_{22}(s) & p_{23}(s) & p_{24}(s) & \cdots \\ 0 & 0 & p_{33}(s) & p_{34}(s) & \cdots \\ 0 & 0 & 0 & p_{44}(s) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Now we report the evolution equations of (2.3) for the case i) and of (2.4) for the case ii).

$$\phi_{ij}(t) = \left(1 - H_i(t)\right)\delta_{ij} + \sum_{k=i+1}^{j} \sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\phi_{kj}(t-\vartheta),$$

$$\phi_{ij}(s,t) = \left(1 - H_i(s,t)\right)\delta_{ij} + \sum_{k=i}^{j} \sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\phi_{kj}(\vartheta,t).$$

The related exploded matrix evolution equations are:

$$\begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) & \phi_{13}(t) & \phi_{14}(t) & \cdots \\ 0 & \phi_{22}(t) & \phi_{23}(t) & \phi_{24}(t) & \cdots \\ 0 & 0 & \phi_{33}(t) & \phi_{34}(t) & \cdots \\ 0 & 0 & 0 & \phi_{34}(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} d_{11}(t) & 0 & 0 & 0 & \cdots \\ 0 & d_{22}(t) & 0 & 0 & \cdots \\ 0 & 0 & d_{33}(t) & 0 & \cdots \\ 0 & 0 & 0 & d_{34}(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$+ \sum_{\vartheta=s+1}^{t} \begin{bmatrix} 0 & b_{12}(\vartheta) & b_{13}(\vartheta) & b_{14}(\vartheta) & \cdots \\ 0 & 0 & b_{23}(\vartheta) & b_{24}(\vartheta) & \cdots \\ 0 & 0 & 0 & b_{34}(\vartheta) & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} * \begin{bmatrix} \phi_{11}(t-\vartheta) & \phi_{12}(t-\vartheta) & \phi_{13}(t-\vartheta) & \phi_{14}(t-\vartheta) & \cdots \\ 0 & \phi_{22}(t-\vartheta) & \phi_{23}(t-\vartheta) & \phi_{24}(t-\vartheta) & \cdots \\ 0 & 0 & \phi_{33}(t-\vartheta) & \phi_{34}(t-\vartheta) & \cdots \\ 0 & 0 & 0 & \phi_{34}(t-\vartheta) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\begin{bmatrix} \phi_{11}(s,t) & \phi_{12}(s,t) & \phi_{13}(s,t) & \phi_{14}(s,t) & \cdots \\ 0 & \phi_{22}(s,t) & \phi_{23}(s,t) & \phi_{24}(s,t) & \cdots \\ 0 & 0 & \phi_{33}(s,t) & \phi_{34}(s,t) & \cdots \\ 0 & 0 & 0 & \phi_{44}(s,t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} d_{11}(s,t) & 0 & 0 & 0 & \cdots \\ 0 & d_{22}(s,t) & 0 & 0 & \cdots \\ 0 & 0 & d_{33}(s,t) & 0 & \cdots \\ 0 & 0 & 0 & d_{44}(s,t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$+ \sum_{\vartheta=s+1}^{t} \begin{bmatrix} b_{11}(s,\vartheta) & b_{12}(s,\vartheta) & b_{13}(s,\vartheta) & b_{14}(s,\vartheta) & \cdots \\ 0 & b_{22}(s,\vartheta) & b_{23}(s,\vartheta) & b_{24}(s,\vartheta) & \cdots \\ 0 & 0 & b_{33}(s,\vartheta) & b_{34}(s,\vartheta) & \cdots \\ 0 & 0 & 0 & b_{44}(s,\vartheta) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} * \begin{bmatrix} \phi_{11}(\vartheta,t) & \phi_{12}(\vartheta,t) & \phi_{13}(\vartheta,t) & \phi_{14}(\vartheta,t) & \cdots \\ 0 & \phi_{22}(\vartheta,t) & \phi_{23}(\vartheta,t) & \phi_{24}(\vartheta,t) & \cdots \\ 0 & 0 & \phi_{33}(\vartheta,t) & \phi_{34}(\vartheta,t) & \cdots \\ 0 & 0 & 0 & \phi_{44}(\vartheta,t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Given the upper triangular structure of each matrix block the system (3.1) can be solved by means of a forward-forward algorithm. It will work in this way; we decide to solve our model up to the time t and the state k. If we think that we need to go for more time we can advance on the times, if we think that we need more information on the states we can go ahead on the states; we can decide also to advance in both the directions. We will stop when we are satisfied on the obtained results. This decision can be function of some parameter that we could verify at each forward step using for example criteria similar to the ones used in the truncation method (Riesz (1913)).
The important fact is that also inside the blocks there is a triangular structure and this permits to obtain the exact results working by a forward approach up to a given state k and a given time t.

3.2 The semi-Markov reward model

In semi-Markov reward environment the (2.9) and (2.10) with denumerable state set become:

$$V_i(t) = \left(1 - H_i(t)\right)\sum_{\tau=1}^{t}\psi_i(\tau)\nu(\tau) + \sum_{k\in\mathbb{N}}\sum_{\vartheta=1}^{t} b_{i,k}(\vartheta)\left(\nu(\vartheta)\gamma_{ik}(\vartheta) + \sum_{\tau=1}^{\vartheta}\psi_i(\tau)\nu(\tau)\right) + \sum_{k\in\mathbb{N}}\sum_{\vartheta=1}^{t} b_{i,k}(\vartheta)\nu(\vartheta)V_k(t-\vartheta), \quad (3.2)$$

$$\begin{aligned} V_i(s,t) = {} & \left(1 - H_i(s,t)\right)\sum_{\tau=s+1}^{t}\psi_i(\tau)\nu(s,\tau) + \sum_{k\in\mathbb{N}}\sum_{\vartheta=s+1}^{t} b_{i,k}(s,\vartheta)\left(\nu(s,\vartheta)\gamma_{ik}(\vartheta) + \sum_{\tau=s+1}^{\vartheta}\psi_i(\tau)\nu(s,\tau)\right) \\ & + \sum_{k\in\mathbb{N}}\sum_{\vartheta=s+1}^{t} b_{i,k}(s,\vartheta)\nu(s,\vartheta)V_k(\vartheta,t). \end{aligned} \quad (3.3)$$

In the aggregate claim process semi-Markov reward model only transition rewards are present because the accumulation of claims. This fact means that when there is a new claim there is a sum to be paid from the insurance company to the insured subject. Only in the case of the model in which it is considered the precautionary letter the virtual transition are allowed but in any case there will be nothing to be paid; the distinction done for the evolution equations of semi-Markov processes is not necessary in reward case. So (3.2) and (3.3) respectively become:

$$V_i(t) = \sum_{k\in\mathbb{N}}\sum_{\vartheta=1}^{t} b_{i,k}(\vartheta)\nu(\vartheta)\gamma_{ik}(\vartheta) + \sum_{k\in\mathbb{N}}\sum_{\vartheta=1}^{t} b_{i,k}(\vartheta)\nu(\vartheta)V_k(t-\vartheta), \quad (3.4)$$

$$V_i(s,t) = \sum_{k\in\mathbb{N}}\sum_{\vartheta=s+1}^{t} b_{i,k}(s,\vartheta)\nu(s,\vartheta)\gamma_{ik}(\vartheta) + \sum_{k\in\mathbb{N}}\sum_{\vartheta=s+1}^{t} b_{i,k}(s,\vartheta)\nu(s,\vartheta)V_k(\vartheta,t). \quad (3.5)$$

As specified before, the state set corresponds to the accumulated claim value. So if the process was in the state i a new claim of value j implies an increasing of the claim sum of j and the system will go to the state $k = i + j$. The process has a transition from the state i to the state k. The sum $j = k - i$ that is reimbursed to the insured person is just equal to the jump that the system does going from i to k. Furthermore, j represents the transition reward value.
Taking into account the upper triangularity of matrices, (3.4) and (3.5) can be rewritten in this way:

$$V_i(t) = \sum_{k>i}\sum_{\vartheta=1}^{t} b_{i,k}(\vartheta)\nu(\vartheta)(k-i) + \sum_{k>i}\sum_{\vartheta=1}^{t} b_{i,k}(\vartheta)\nu(\vartheta)V_k(t-\vartheta),$$

$$V_i(s,t) = \sum_{k>i}\sum_{\vartheta=s+1}^{t} b_{i,k}(s,\vartheta)\nu(s,\vartheta)(k-i) + \sum_{k>i}\sum_{\vartheta=s+1}^{t} b_{i,k}(s,\vartheta)\nu(s,\vartheta)V_k(\vartheta,t),$$

that in matrix form become:

$$\mathbf{V}(t) = \sum_{\vartheta=1}^{t} \mathbf{B}(\vartheta) \circ \bar{\mathbf{A}}(\vartheta) + \sum_{\vartheta=1}^{t} \nu(\vartheta)\mathbf{B}(\vartheta) * \mathbf{V}(t-\vartheta), \tag{3.6}$$

$$\mathbf{V}(s,t) = \sum_{\vartheta=s+1}^{t} \mathbf{B}(s,\vartheta) \circ \bar{\mathbf{A}}(s,\vartheta) + \sum_{\vartheta=s+1}^{t} \nu(s,\vartheta)\mathbf{B}(s,\vartheta) * \mathbf{V}(\vartheta,t), \tag{3.7}$$

where $\bar{a}_{ij}(t) = (j-i)\nu(t)$ and $\bar{a}_{ij}(s,t) = (j-i)\nu(s,t)$.

Remark 3.1 In the aggregate claim process reward model it is not necessary to do the distinction between i) and ii) because virtual transition from i into itself will give no contribution in the reward computation.

Exploding (3.6) and (3.7) it results:

$$\begin{bmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \\ V_4(t) \\ V_5(t) \\ \vdots \end{bmatrix} = \sum_{\vartheta=1}^{t} \begin{bmatrix} b_{11}(\vartheta) & b_{12}(\vartheta) & b_{13}(\vartheta) & b_{14}(\vartheta) & b_{15}(\vartheta) & \cdots \\ 0 & b_{22}(\vartheta) & b_{23}(\vartheta) & b_{24}(\vartheta) & b_{25}(\vartheta) & \cdots \\ 0 & 0 & b_{33}(\vartheta) & b_{34}(\vartheta) & b_{35}(\vartheta) & \cdots \\ 0 & 0 & 0 & b_{44}(\vartheta) & b_{45}(\vartheta) & \cdots \\ 0 & 0 & 0 & 0 & b_{55}(\vartheta) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \circ \nu(\vartheta) \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & \cdots \\ 0 & 0 & 1 & 2 & 3 & \cdots \\ 0 & 0 & 0 & 1 & 2 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$+ \sum_{\vartheta=s+1}^{t} \begin{bmatrix} b_{11}(\vartheta) & b_{12}(\vartheta) & b_{13}(\vartheta) & b_{14}(\vartheta) & b_{15}(\vartheta) & \cdots \\ 0 & b_{22}(\vartheta) & b_{23}(\vartheta) & b_{24}(\vartheta) & b_{25}(\vartheta) & \cdots \\ 0 & 0 & b_{33}(\vartheta) & b_{34}(\vartheta) & b_{35}(\vartheta) & \cdots \\ 0 & 0 & 0 & b_{44}(\vartheta) & b_{45}(\vartheta) & \cdots \\ 0 & 0 & 0 & 0 & b_{55}(\vartheta) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \circ \nu(\vartheta) \begin{bmatrix} V_1(t-\vartheta) \\ V_2(t-\vartheta) \\ V_3(t-\vartheta) \\ V_4(t-\vartheta) \\ V_5(t-\vartheta) \\ \vdots \end{bmatrix},$$

$$\begin{bmatrix} V_1(s,t) \\ V_2(s,t) \\ V_3(s,t) \\ V_4(s,t) \\ V_5(s,t) \\ \vdots \end{bmatrix} = \sum_{\vartheta=s+1}^{t} \begin{bmatrix} b_{11}(s,\vartheta) & b_{12}(s,\vartheta) & b_{13}(s,\vartheta) & b_{14}(s,\vartheta) & b_{15}(s,\vartheta) & \cdots \\ 0 & b_{22}(s,\vartheta) & b_{23}(s,\vartheta) & b_{24}(s,\vartheta) & b_{25}(s,\vartheta) & \cdots \\ 0 & 0 & b_{33}(s,\vartheta) & b_{34}(s,\vartheta) & b_{35}(s,\vartheta) & \cdots \\ 0 & 0 & 0 & b_{44}(s,\vartheta) & b_{45}(s,\vartheta) & \cdots \\ 0 & 0 & 0 & 0 & b_{55}(s,\vartheta) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \circ \nu(s,\vartheta) \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & \cdots \\ 0 & 0 & 1 & 2 & 3 & \cdots \\ 0 & 0 & 0 & 1 & 2 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$+ \sum_{\vartheta=s+1}^{t} \begin{bmatrix} b_{11}(s,\vartheta) & b_{12}(s,\vartheta) & b_{13}(s,\vartheta) & b_{14}(s,\vartheta) & b_{15}(s,\vartheta) & \cdots \\ 0 & b_{22}(s,\vartheta) & b_{23}(s,\vartheta) & b_{24}(s,\vartheta) & b_{25}(s,\vartheta) & \cdots \\ 0 & 0 & b_{33}(s,\vartheta) & b_{34}(s,\vartheta) & b_{35}(s,\vartheta) & \cdots \\ 0 & 0 & 0 & b_{44}(s,\vartheta) & b_{45}(s,\vartheta) & \cdots \\ 0 & 0 & 0 & 0 & b_{55}(s,\vartheta) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \circ \nu(s,\vartheta) \begin{bmatrix} V_1(\vartheta,t) \\ V_2(\vartheta,t) \\ V_3(\vartheta,t) \\ V_4(\vartheta,t) \\ V_5(\vartheta,t) \\ \vdots \end{bmatrix}.$$

In this case could be more difficult to get the solution of the systems because in the construction of the known terms it is necessary to consider all the infinite elements that increase with the time running. But it could be done using the truncation method (Riestz 1913). Indeed it is reasonable to suppose that the probability to have a big money amount claim should be very low. Another reasonable hypothesis could be to suppose that the aggregate claim amount process could not be larger than a given sum. In this way the state set will be no more denumerable and there will be an absorbing state given by the largest reachable claim amount.

4. The mean number claim processes

By means of small changes in the semi-Markov reward processes it is possible also to get the models useful for the computation of mean claim numbers.
It is to precise that reward processes can be discounted and not discounted; for a complete classification of reward processes see Janssen Manca (2006).
In the previous sections we reported the evolution equation of discounted cases. In non-discounted cases (3.2) and (3.3) become:

$$V_i(t) = \left(1 - H_i(t)\right)\sum_{\tau=1}^{t}\psi_i(\tau) + \sum_{k\in\mathbb{N}}\sum_{\vartheta=1}^{t} b_{i,k}(\vartheta)\left(\gamma_{ik}(\vartheta) + \sum_{\tau=1}^{\vartheta}\psi_i(\tau)\right) + \sum_{k\in\mathbb{N}}\sum_{\vartheta=1}^{t} b_{i,k}(\vartheta)V_k(t-\vartheta), \tag{4.1}$$

$$V_i(s,t) = \left(1 - H_i(s,t)\right)\sum_{\tau=s+1}^{t}\psi_i(\tau) + \sum_{k\in\mathbb{N}}\sum_{\vartheta=s+1}^{t} b_{i,k}(s,\vartheta)\left(\gamma_{ik}(\vartheta) + \sum_{\tau=s+1}^{\vartheta}\psi_i(\tau)\right) + \sum_{k\in\mathbb{N}}\sum_{\vartheta=s+1}^{t} b_{i,k}(s,\vartheta)V_k(\vartheta,t). \tag{4.2}$$

In this case (4.1) and (4.2) represent the mean total rewards (MTR) gotten within the time t in homogeneous case and from time s to time t in non-homogenous case.
(3.4) and (3.5) in the non-discounted case become:

$$V_i(t) = \sum_{k\in\mathbb{N}}\sum_{\vartheta=1}^{t} b_{i,k}(\vartheta)\gamma_{ik}(\vartheta) + \sum_{k\in\mathbb{N}}\sum_{\vartheta=1}^{t} b_{i,k}(\vartheta)V_k(t-\vartheta), \tag{4.3}$$

$$V_i(s,t) = \sum_{k\in\mathbb{N}}\sum_{\vartheta=s+1}^{t} b_{i,k}(s,\vartheta)\gamma_{ik}(\vartheta) + \sum_{k\in\mathbb{N}}\sum_{\vartheta=s+1}^{t} b_{i,k}(s,\vartheta)V_k(\vartheta,t). \tag{4.4}$$

Now it is possible to take into account also the claims in which there will be no payments. So the matrix rewards can be of two types, more precisely:

$$\Gamma = [\gamma_{ik}] = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ or } \Gamma = [\gamma_{ik}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

As for the semi-Markov evolution equations, we report the homogenous reward evolution equation for the first case and the non-homogeneous for the second case. Respectively (4.3) and (4.4) become:

$$V_i(t) = \sum_{k>1} \sum_{\vartheta=1}^{t} b_{i,k}(\vartheta) + \sum_{k>1} \sum_{\vartheta=1}^{t} b_{i,k}(\vartheta) V_k(t-\vartheta), \tag{4.5}$$

$$V_i(s,t) = \sum_{k\geq 1} \sum_{\vartheta=s+1}^{t} b_{i,k}(s,\vartheta) + \sum_{k\geq 1} \sum_{\vartheta=s+1}^{t} b_{i,k}(s,\vartheta) V_k(\vartheta,t). \tag{4.6}$$

(4.5) and (4.6) represent the mean total rewards gotten in the considered time interval. Each reward is equal to 1 and it is given at each transition, but for the model construction each transition represents a claim in (4.5) only with related payments and in (4.6) also without payments. The two relations represent the mean number of claims that will happen in the considered time interval.

5. Conclusions

In this paper a new semi-Markov approach to classical risk processes was presented. It was shown how it is possible to solve the aggregate claim amount and the claim number processes. In both cases the semi-Markov reward processes were applied. The aggregate claim amount process was solved by means of a discounted semi-Markov reward process. The claim number process was solved by a non-discounted approach.

In both cases the homogeneous and non-homogeneous relations were presented.

No applicative examples were presented.

In the near future the authors would:

- firstly construct the algorithm useful to solve this special kind of semi-Markov reward processes,
- after get real data to apply their model and to compare their results with the results of the classical risk models,
- generalize the model in the way to obtain the relations useful for the computation of higher order moments as presented in Stenberg et al (2006) for homogeneous case and in Stenberg et al (2007) in non-homogeneous case.

Doing all these steps the authors could get a model that forecasts the random evolution of the two studied risk processes and the related variance, skewness and kurtosis.

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