Longevity Risk Pricing*

Jiajia Cui†

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Abstract

The uncertainty about the future mortality developments is referred to as longevity risk. This paper quantifies the size of longevity risk premium which should be priced in various longevity-linked securities and annuity contracts. The goal of this project is to tackle the pricing difficulty emerged during the market innovation of the potential longevity-linked securities. Based on the equivalent utility pricing principle, we obtain the minimum risk premium required by the longevity insurance seller and the maximum acceptable risk premium by the longevity insurance buyer. Our estimated risk premium is consistent with the limited observations in the market. We show that the size of the risk premium depends on the financial position of the seller and buyer, and the availability of the natural hedges. One important implication for the market development of longevity-linked securities is that multiple sellers are required instead of a single seller. The paper also shows that the availability of natural hedge has significant impact on the longevity risk premium required. Longevity risk is modelled by Lee-Carter (1992) model, and estimated according to the U.K. and the Dutch mortality data.

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JEL codes: G13, G22

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†Tilburg University and ABP Pension Fund; E-mail: j.cui@uvt.nl

1 Introduction

In the past century remarkable improvements in human life expectancy have been observed. However the future improvements of life expectancy are uncertain and difficult to be predicted accurately[1]. The general opinion from the experts tends to be that the trend of longevity improvements is certain, but the deviations to both sides are possible. The uncertainty around the mortality trend is referred to as the longevity risk. Longevity risk is a macro risk, or systematic risk, which can not be reduced by diversification or increasing the size of the pool. Since Lee and Carter (1992) mortality rates are clearly modeled as stochastic processes. Longevity risk thus imposes a challenge on annuity providers.

In practice, life insurers, also pension funds, claim that their annuity businesses are losing money due to the unexpected longevity improvements over years. Yet, in their premium pricing system, longevity risk loading is either ignored or set by rule of thumb. Reinsurance contracts do exist, but the capacity of reinsurance is limited (OECD (2005). By a reinsurance contract, the longevity risk is concentrated on one large reinsurance company. The reinsurance approach works best when the underlying risks are diversifiable. However, the fact that longevity risk is a systematic risk weakens the diversification principle that the reinsurance requires.

The longevity risk could be transferred to financial market, also known as securitization. By transferring longevity risk to financial markets, the risk is distributed among a (large) number of market participants. The first longevity bond was announced by the EIB and BNP[2] Paribas in November 2004, but it has been undersubscribed, and withdrawn for redesign in late 2005. The EIB/BNP survivor bond is a 'coupon-based' bond, in which the floating annual coupon is indexed to a cohort survivor index. The EIB/BNP survivor bond required a longevity risk premium of 20 basis points, which it was regarded too high for many annuity providers. The EIB/BNP bond, although linked to the British survivor index, was also marketed among Dutch pension funds. Is 20 basis points a good deal or not for a Dutch pension funds? There is a lack of clear view on how longevity risk should be charged.

Blake, Cairns and Dowd (2006) address the associated obstacles in current market development of longevity-linked securities. The obstacles are categorized into 'design issues' (regarding the payoff structure, maturity, choice of survivor index, nominal or real payments, etc), 'pricing issues' and 'institutional issues'. As to the pricing issues, the authors comment that "even if the (survivor) bond provides a perfect hedge, there will be uncertainty over what the right price to pay or charge should be." Before the longevity linked securities are traded in the market, the financial market is incomplete. Indeed, pricing in incomplete market is the origin of the pricing obstacle for the market participants.

[1] Brown and Orszag (2006) discuss the difficulties in making an accurate mortality projection.

[2] EIB/BNP stands for the European Investment Bank (EIB) and Banque Nationale de Paris (BNP).

The goal of this project is to tackle the pricing difficulty emerged during the market innovation of the potential longevity-linked securities. Based on the equivalent utility pricing principle, we obtain the minimum risk premium required by the longevity insurance seller and the maximum acceptable risk premium by the longevity insurance buyer. This paper shows how the risk premiums are affected by the different designs in payoff structures and maturities. Furthermore, the risk premium can be reduced by distributing the longevity risk among more market participants. The market calls for more longevity bond issuers in order to reduce the longevity risk premium. One important implication for the market development of longevity-linked securities is that multiple sellers, instead of a single seller, are required .

Apart from securitization, there are three other possibilities of managing longevity risk (see also Brown and Orszag (2006), Blake, Cairns and Dowd (2006)), namely hedgeing, reserving, and risk sharing. The longevity risk could be partly hedged using natural hedgeing, for example between life annuity and term insurance. This paper illustrates the effect of the natural hedgeing on longevity risk premiums. The impact of natural hedgeing is potentially significant.

The contribution of this paper is that we quantify the longevity risk premiums in various longevity-linked securities by applying the equivalent utility pricing principle. This paper shows a range of possible prices for the designed longevity-linked securities before the market opens up. Using indifference pricing, this paper finds important implications about product design, market development and pricing. The results provide design implications for longevity-linked securities and longevity risk management.

Recently, a few approaches to price longevity risk were proposed in the literature. Friedberg and Webb (2005) apply CAPM and CCAPM to estimate the longevity premium, by constructing sympathetic longevity bonds returns. The result based on CAPM is that the risk premium should be roughly 75 basis points, with confidence interval ranging from -75 to 230 basis points, due to inaccuracy in estimating the beta. The result based on CCAPM is merely 2 basis points, due to the low variation in consumption data. Milevsky, Promislow and Young (2005, 2006) proposed a Sharpe ratio approach, which is based on mean and volatility of payoffs instead of returns. The result is not known yet.

The methodology used in this paper is the equivalent utility pricing principle, which is a well known pricing principle in actuarial sciences (see e.g. a summary by Young (2004)). It reveals the minimum compensation required by the insurance seller and the maximum price acceptable to the insurance buyer. Here we need to distinguish the minimum premium required by a seller, who sells an insurance product (e.g. the longevity bond), from the maximum risk premium accepted by a buyer, who buys the insurance. In the context of a longevity bond, the bond issuance company acts as the insurer, whereas the pension funds, who buy the longevity bond, act as the insured. The seller's minimum risk premium depends on the financial position of the insurer, her risk aversion, maturity, payoff structure and the correlation with the rest of her portfolio. Similarly, the buyer's maximum risk premium depends on the financial position of the insured, his risk aversion, maturity, payoff structure and the comovements with the rest of his portfolio.

Based on the equivalent utility pricing principle, we quantify the sizes of the potential longevity risk premiums in various longevity-linked securities, including longevity bonds and derivatives. In this paper, the longevity risk is modeled as proposed by Lee-Carter (1992), and estimated according to the U.K. and the Dutch mortality data. This paper focuses on the immediate annuity contracts for retirees, because before retirement, people may adjust their contributions and retirement age to react to the changes in life expectancies. However, after retirement the remaining longevity risk is borne by the annuity provider. Given plausible ranges of risk aversion, financial position and other assumptions of the seller and the buyer, we calculate risk premiums for longevity-linked securities. The resulting risk premiums found in this paper are consistent with the limited market observations.

The organization of this proposal is the following: Three building blocks for longevity risk pricing will be introduced in Section 2, including stochastic mortality modeling, mortality linked securities, and incomplete-market pricing principles. In Section 3 we quantify the (seller's minimum) longevity risk premium using the equivalent utility pricing principle. Section 4 we compare the longevity risk premiums in different longevity linked securities, including swaps, deferred bonds, floors and caps. Section 5 we introduce the correlated risks, which induce natural hedgeing into our pricing framework. Section 6 considers buyer's maximum premium, together with hedgeing and basis risk. Section 7 concludes.

2 Related literatures

Three building blocks are needed for pricing the longevity risk, namely a model quantifying longevity risk, longevity linked securities and the pricing principle. We first introduce a model quantifying the longevity risk, then we describe the financial instruments that are linked to the longevity risk. Finally several incomplete-market pricing principles are discussed.

2.1 Stochastic mortality models

It is important to address that the pricing framework (and the resulting formula) in this paper is independent from which stochastic mortality model is used. The numerical results presented in this paper are based on Lee and Carter (1992) model. However, other stochastic mortality models also fit in our pricing framework.

The literature of stochastic mortality trend starts from Lee and Carter (1992)[3]. According

[3] Various extensions of the Lee-Carter model can be found in Cairns, Blake and Dowd (2005b), Hari, De Waegenaere, Melenberg and Nijman (2006).

to Deaton and Paxson (2004) the Lee-Carter model has become the 'leading statistical model of mortality in the demographic literature'. Dahl (2004) and Schrager (2006) advocate the affine stochastic mortality models to capture the birth cohort mortality dynamics over one's life cycle instead of the time series of an age group over time. We leave the affine stochastic mortality approach as a future work for robustness analysis.

Lee and Carter (1992) model the time series behavior of log central death rate movements between age groups by using a single latent factor. Formally, the log mortality rate of the x-year-old, $\ln\left(\mu_{x,t}\right)$, is determined by a common latent factor, γ_t, with an age specific sensitivity parameter β_x and an age specific level parameter α_x

$$\ln\left(\mu_{x,t}\right) = \alpha_x + \beta_x \gamma_t + \delta_{x,t} \tag{1}$$

where $\delta_{x,t} \overset{iid}{\sim} N\left(0, \sigma_\delta^2\right)$ is a white noise, representing transitory shocks. The time dependent latent factor γ_t drives the mortality rates of all age cohorts, which satisfies a random walk with drift process as

$$\gamma_t = c + \gamma_{t-1} + \varepsilon_t \tag{2}$$

where $\varepsilon_t \overset{iid}{\sim} N\left(0, \sigma_\varepsilon^2\right)$ is white noise, representing permanent shocks. $\delta_{x,t}$ and ε_t are independent.

Conditioning on the information available up to time t, the survival probability of the x-year-old over one year is given by $p_x\left(t\right) = p_{[x_0]+t}\left(t\right) = \exp\left(-\mu_{x,t}\right)$, assuming that the force of mortality is constant during the year $\mu_{x+u,t+u} = \mu_{x,t}$ $(0 \leq u < 1)$. The conditional probability of an x-year-old surviving next τ years is given by

$${}_\tau p_x\left(t\right) = \exp\left(-\sum_{i=1}^{\tau} \mu_{x+i,t+i}\right) \tag{3}$$

The expected remaining life time of an x-year-old in year t is given by

$$E_t\left[T\right] = E_t\left[\sum_{\tau=1}^{\omega-x} \left({}_\tau p_x\left(t\right)\right)\right] \tag{4}$$

where ω denotes the maximum obtainable age, e.g. $\omega = 110$.

The price of an immediate annuity paying 1 Euro in each surviving year, assuming a fixed and flat yield curve, is given by

$$E_t\left[L\right] = E_t\left[\sum_{\tau=1}^{110-x} e^{-rt}\left({}_\tau p_x\left(t\right)\right)\right]. \tag{5}$$

The model is estimated using the yearly U.K. (England and Wales) and Dutch male mortality data from 1880 to 2003, downloaded from the Human Mortality Database[4]. Appendix A provides a more detailed treatment of the model, together with its estimation and simulation procedures. The figures below shows the estimation results using the British data. Figure 1 visualizes the estimated results of the model, with the estimated β_x (left upper panel), γ_t (right upper panel), α_x (left lower panel) and $\ln \mu_x$ (right lower panel, $x = 65$ and 35). The estimated latent process (in U.K.) is the following, including two temporary shocks captured by a 'WWI' dummy and a 'WWII' dummy[5]:

$$\gamma_t = -0.0725 + \gamma_{t-1} + 0.65 * WWI_t + 1.9 * WWII_t + \varepsilon_t \tag{6}$$

with $\widehat{\sigma}_\varepsilon = 0.169$. The dummy variables do not change the drift but reduce the volatility of the innovations.

Assuming that the estimated model (6) is the 'true' process, and taking the estimated parameters as of year 2003, we could simulate the survival probabilities of 65-yaer-old male in 2004. Figure 2 describe the distribution of a set of simulated survival probabilities of the 65-year0old male. From Figure 2(b) we see that the volatility of the survival probabilities exhibits a hump shape, which means that the uncertainty over a longer horizon (up to 20 years) first increases and then decreases. The distribution is also skewed. The skewness of the survival probabilities does not close to zero.

Using the estimation results, we can show the size of the uncertainties involved in life expectancy and annuity prices. According to the estimated Lee-Carter model, the remaining life time of a 65-year-old British male in year 2004, is 16 years with standard error of [± 0.2] years,as given by (4). An immediate annuity paying 1 Euro in each surviving year on average worthies[6] 13.1 euro, assuming a fixed and flat yield curve at 2%, as given by (5). The standard error of the value of this annuity is [± 0.15] euro, or [± 1.1%] in relative terms. The longevity risk in such immediate annuities is not negligeable.

[4]Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de (data downloaded on 27/03/2006).

[5]'WWI' dummy takes value of 1 for year { } and zero elsewhere. 'WWII' dummy takes value of 1 for year { }.

[6]Money's worth of annuity is the expected discounted value of future payments, without risk loadings.

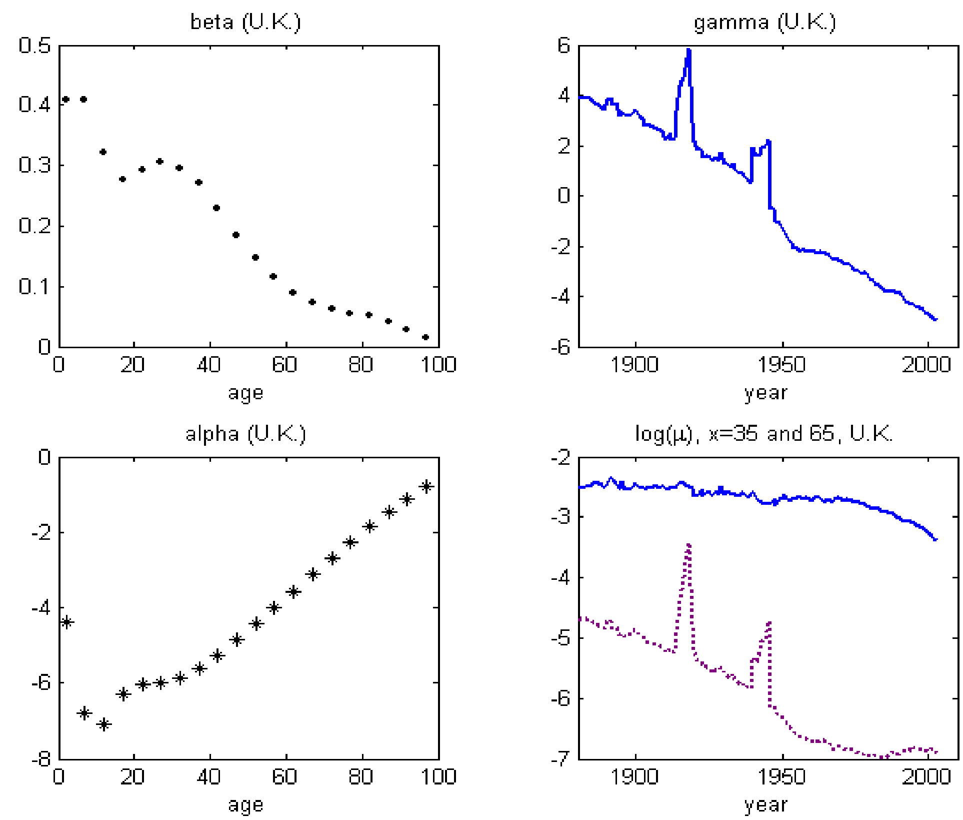

Figure 1: The estimated Lee-Carter (1992) model parameters, with β_x (left upper panel), γ_t (right upper panel), a_x (left lower panel) and $\ln \mu_x$ (right lower panel, the top curve for $x = 65$, and the bottom dashed curve for $x = 35$).

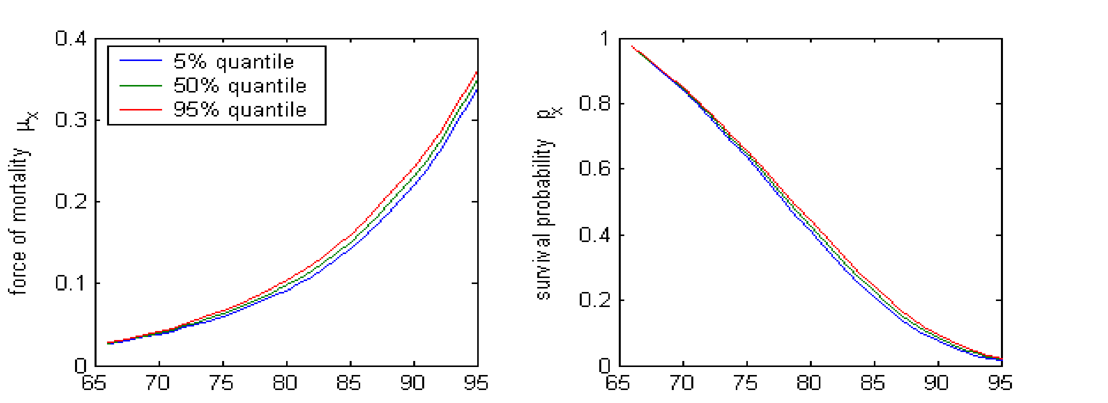

Figure 2 (a): The forecasted force of mortality μ_{x+t} (left panel) and the forecasted survival probabilities ${}_tp_x$ (right panel) of the 65-year-old cohort retiring in year 2004.

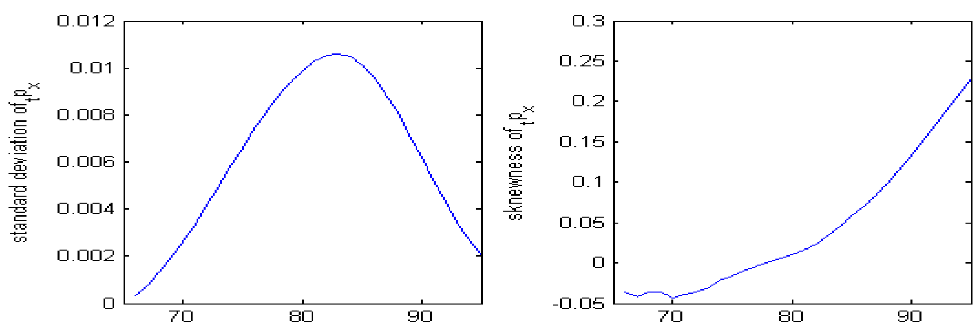

Figure 2 (b): The standard deviation (left) and skewness (right) of the simulated survival probability, ${}_tp_{65}$.

2.2 Longevity linked securities

Given the potential size of the longevity trend uncertainty, financial markets proposed longevity linked securities. The first longevity bond was announced by the EIB and BNP[7] Paribas in November 2004, but withdrawn for redesign in late 2005. The EIB/BNP bond is a 'coupon-based' bond, in which the notional annual coupon is indexed to a cohort survivor index. This cohort retires at age 65 in 2004. The maturity of this bond is 25 years. Blake, Cairns and Dowd (2006) address the lessons learned from the failure of EIB/BNP survivor bond and provides instructive suggestions for future developments of the flourishing new market. The main lessons are the following:

1) The designed 25-year horizon is perhaps too short for an effective hedge, since longevity risk

[7]EIB/BNP stands for the European Investment Bank (EIB) and Banque Nationale de Paris (BNP).

in the near future (<10 years) is small.

2) The required upfront capital is large, especially a major part of these capital is taken by the ineffective hedge coupons.

3) The coupons are indexed to 65-year-old males, but annuity providers worry about longevity risk of younger cohorts and females.

4) Large uncertainty about what the right price should be charged.

5) Hedge failure or basis risks are large, due to a number of ways: the reference population is different from that of an annuity provider, the survivor index is not timely available, etc.

6) The payments are nominal, whereas most pension schemes aim at inflation linked real payments.

Their paper introduces some innovative hypothetical mortality-linked securities as potential solutions to the aforementioned problems. These securities include Zero-Coupon Longevity Bonds, Longevity Bull Spread Bonds, Deferred Longevity Bonds, Vanilla Mortality Swaps, Survivor caps and Floors, Mortality Swaptions, Longevity Future. The paper illustrates how these mortality-linked securities could be used to manage longevity risk and boost market development.

Enlightened by these discussions, our paper present the required longevity risk premiums in different longevity-linked security designs, including zeros, EIB/BNP bonds, swaps, caps and floors and deferred longevity bonds. This paper also considers the natural hedge and basis risk in Section 6.

2.3 Longevity risk pricing principles

2.3.1 CAPM and CCAPM based approach

Friedberg and Webb (2005) apply CAPM and CCAPM to estimate the premium for longevity risk. The authors construct a pseudo-EIB mortality bond. $R_{b,t}$ denotes the returns of such pseudo bond. Following the CAPM, the longevity risk premium is its beta, which is defined by $\beta_b = cov\left(R_b, R_m\right)/\sigma_m^2$, times the market risk premium:

$$E_t(R_b) - R_f = \beta_b\left[E\left(R_m\right) - R_f\right]$$

The authors claim that the beta on pseudo-EIB/BNP bond is 0.15 with 95% confidence interval of [-0.15, 0.46]. Therefore, if market risk premium is 5%, the longevity risk premium on this bond is 75 basis points, with confidence interval of [-75, 230] basis points. Given the wide confidence interval, the authors suggest CCAPM as a better alternative.

Following the CCAPM, the longevity risk premium is determined by the relationship between the expected return on the asset and the marginal utility of consumption.

$$E_t(R_{b,t+1}) - R_f = -\frac{Cov_t(U'(C_{t+1}), R_{b,t+1})}{E_t(U'(C_{t+1}))}$$

The paper shows that the correlation between consumption growth and mortality bond returns is -0.1958 and is significantly. However since the standard deviation of mortality bond returns is small, as a result, the covariance between mortality bond returns and consumption growth is extremely small at -0.0015 percent. Applying the CCAPM, the risk discount is only 2 basis points when the coefficient of risk aversion equals 10. This result is far below the 20 bp risk premium in the EIB/BNP bonds.

2.3.2 Sharpe Ratio approach

Cochrane and Saa-Requejo (2000) suggest that the absolute value of the Sharpe ratio on any unhedgeable portfolio should be bounded, so that too 'good deals' are ruled out. Milevsky, Promislow and Young (2005, 2006) propose a so-called instantaneous Sharpe ratio to determine the mortality risk premiums. Using the analogy to the Sharpe ratio in financial market, $SR^{Market} := (E[R] - R_f)/\sigma[R]$, the Sharpe ratio in the insurance context could be defined as the excess payoff above the expected payment, divided by the standard deviation of the risky payment, $SR^{Insur} := (N(1+L) - E[W_N])/\sigma[W_N]$. The authors argue that the longevity risk loading L will be set so that the Sharpe Ratio is consistent with other asset classes in the economy. For example, if the sharpe ratio for large cap equities is roughly 0.25, then the Sharpe ratio of the insurance policy should also be bounded within a similar magnitude[8].

2.3.3 Equivalent utility based approach

Equivalent utility based approach is a popular pricing methodology for incomplete market setting. The related literature includes Svensson and Werner (1993), Young and Zariphopoulou (2002), Young (2004), Musela and Zariphopoulou (2004), De Jong (2006), and other references listed in the bibliography. As pointed out by Svensson and Werner (1993), the shadow value of a non-traded or non-hedgeable asset can be interpreted as the additional wealth added to investor's budget so that the investor is indifferent between holding a non-hedgeable asset and hedgeable asset. Furthermore, the shadow value is investor-specific, depending on investor's preference. In the context of longevity-linked securities, equivalent utility pricing principle reveals the minimum compensation required by the seller and the maximum price acceptable to the buyer. Therefore, this paper shows the range of possible prices for the designed longevity-linked securities before the market opens up.

[8] The authors are still working on the estimation of the Sharpe ratio using annuity rate quotes. The results are not available yet.

Assuming exponential utility function, Musela and Zariphopoulou (2004) show a simple analytical formula for pricing an non-traded claim. As we shall see, the pricing formula of the longevity-linked claims derived in this paper is similar to the result found by Musela and Zariphopoulou (2004).

De Jong (2006) applies the principle of equivalent utility in pricing wage-linked securities, in incomplete market setting. In the context of DB pension fund liability valuation, the main source of unhedgeable risk is the real wage growth. The pension fund is modeled as a potential buyer of the wage-linked bonds. Hence the equivalent utility pricing gives the maximum risk premium that the pension fund is willing to pay in order to obtain the insurance against the wage rate fluctuations. The paper shows the risk premium is determined by the additional wealth needed to be invested in the financial market in order to provide the participants the same level of utility as a fully wage-indexed pension.

The next sub-section illustrates the idea of equivalence pricing principle using a very simple model. The complete model is treated in Section 3.

2.3.4 A simple illustration

Before going into the pricing, let's fix some notations. Let S_t denote a cohort survival index in year t, $S_t = NK_tp_x$. In the context of annuities, N denotes the initial size of the x-year-old cohort at time zero, and K is the agreed amount of annuity payment per annual. In the context of longevity bonds with varying coupons, NK denotes the notional coupon and NK_tp_x is the amount of coupon due in year t. In this paper, K is normalized to 1.

Now we illustrate the idea by pricing the zero coupon longevity bond, with maturity of t years. Such a zero coupon bond is effectively a group of single premium endowment contract which pays an agreed amount (normalized to $K = 1$) at a future time t to the survivors of the current x-year-old cohort. The survival index in t years' time is a random variable, with mean $E[S_t]$, and variance $Var[S_t]$. The longevity risk can be described as the deviation from the expected survival rate, $S_t - E[S_t]$. The single premium is paid at time zero, and consists of two parts. One part is the expected loss $E[S_t]$, the other part is a premium loading P. Let's call the longevity bond issuance company (and the pure endowment seller) the 'seller', since the 'seller' provides insurance against longevity risk. First we take the seller's point of view by considering the seller's utility function. We assume that the longevity risk is the only risk factor in this simple illustration.

The seller invests her initial wealth W_0 and the received total premium $(E[S_t] + P)$ in risk free asset. Further assume the risk free rate is zero, therefore, $W_t = W_0$. The minimum premium loading for this single premium endowment contract is the lowest amount that the seller asks for bearing the longevity risk $S_t - E[S_t]$. The minimum premium loading P^- equalizes the expected utility of underwriting the risk S_t with a compensation $E[S_t] + P^-$, and the utility of not underwriting the risk, from the seller's viewpoint. Let $U(\cdot)$ denote the utility function of the seller, we have

$$E\left[U\left(W_t + E[S_t] + P^- - S_t\right)\right] = U(W_t)$$

Case 1: CARA

First we assume the seller has a constant absolute risk averse utility: $U(w) = -\frac{1}{\overline{\alpha}}\exp(-\overline{\alpha}w)$, where $\overline{\alpha}$ is the absolute risk aversion coefficient.

$$\begin{aligned} E\left[U\left(W_t + E[S_t] + P^- - S_t\right)\right] &= -\frac{1}{\overline{\alpha}}\exp(-\overline{\alpha}W_t) \\ e^{\left(-\overline{\alpha}(E[S_t]+P^-)\right)}E[e^{(\overline{\alpha}S_t)}] &= 1 \\ P^- &= \frac{1}{\overline{\alpha}}\ln E\left[\exp\left(\overline{\alpha}\left(S_t - E[S_t]\right)\right)\right] \qquad (7) \end{aligned}$$

The last equation is the so-called exponential risk premium (Kaas, et al (2001), p7). The 'best estimate' plus the (macro) risk premium equals the logarithm of the moment generating function of risk S_t at argument $\overline{\alpha}$ divided by the CARA coefficient $\overline{\alpha}$. Notice that the risk loading is not affected by the initial wealth, W_0, for the CARA preference.

In a special case, if S_t is normally distributed, then the minimum loading P^- is proportional to the variance of the survival index. However, Figure 2(b) shows that the distribution of S_t is skewed. Therefore (8) is not accurate.

$$\begin{aligned} \text{if} \qquad S_t &\sim N\left(E[S_t],\ Var[S_t]\right), \\ \text{then} \qquad \overline{\alpha}\left(S_t - E[S_t]\right) &\sim N\left(0,\ \overline{\alpha}^2 Var[S_t]\right) \\ E\left[\exp\left(\overline{\alpha}\left(S_t - E[S_t]\right)\right)\right] &= \exp\left(\frac{1}{2}\overline{\alpha}^2 Var[S_t]\right) \\ P^- &= \frac{1}{2}\overline{\alpha}Var[S_t] \qquad (8) \end{aligned}$$

Case 2: CRRA

Alternatively we assume the seller has a constant relative risk averse utility: $U(w) = w^{1-\gamma}/(1-\gamma)$. Notice that the risk loading does depend on the initial wealth and risk aversion parameter for the CRRA preference. The minimum loading P^- is the one that solves equation (9).

$$
\begin{aligned}
E\left[U\left(W_t + E\left[S_t\right] + P^- - S_t\right)\right] &= \frac{W_t^{1-\gamma}}{1-\gamma} \\
E\left[\left(W_t + E\left[S_t\right] + P^- - S_t\right)^{1-\gamma}\right] &= W_t^{1-\gamma} \\
E\left[\left(1 + \frac{E\left[S_t\right] + P^- - S_t}{W_t}\right)^{1-\gamma}\right] &= 1 \qquad (9)
\end{aligned}
$$

Numerical results:

A numerical quantification of P^- of both cases can be done by means of simulation. Using the estimation and simulation procedures presented in Appendix A, we calculate the expected loss $E\left[S_t\right]$ and the risk loading P^- for the endowment contract. The total premium paid per person is $\frac{E[S_t]+P^-}{N}$, since the insured amount $K = 1$. We can express P^- in terms of risk premium, R_p, which is a discount rate above the risk free rate (i.e. 0 as assumed) and an actuarial discount rate:

$$
\frac{1}{\left(1 + R_a + R_p\right)^t} = \frac{E\left[S_t\right] + P^-}{N}
$$

where the actuarial discount rate R_a is defined by

$$
\frac{1}{\left(1 + R_a\right)^t} = \frac{E\left[S_t\right]}{N} = E[{}_tp_x]
$$

The risk premiums for CARA and CRRA utility specifications are presented in the Table 1 and 2 below.

	N = 10			N = 100			N = 1000		
maturity	CARA = 1	CARA = 3	CARA = 5	CARA = 1	CARA = 3	CARA = 5	CARA = 1	CARA = 3	CARA = 5
5	0	0	0	-1	-2	-3	-5	-12	-15
10	0	-1	-1	-2	-7	-11	-18	-24	-26
15	-1	-2	-3	-6	-16	-24	-31	-38	-39
20	-1	-3	-5	-10	-28	-39	-48	-55	-57
25	-1	-4	-6	-13	-37	-54	-70	-82	-85
30	-1	-3	-4	-9	-27	-46	-78	-104	-109
35	0	-1	-1	-3	-9	-15	-31	-91	-112

Table 1: The longevity risk premium R_p in basis points, for different cohort sizes (N=10, 100, 1000 $\times 10^3$) and different risk aversion values ($CARA(\alpha) = 1, 3, 5$).

maturity	N = 10 CRRA = 3	CRRA = 5	CRRA = 8	N = 100 CRRA = 3	CRRA = 5	CRRA = 8	N = 1000 CRRA = 3	CRRA = 5	CRRA = 8
5	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	-1	-1	-2
15	0	0	0	0	0	0	-2	-3	-5
20	0	0	0	0	-1	-1	-3	-5	-8
25	0	0	0	0	-1	-1	-4	-6	-10
30	0	0	0	0	0	-1	-3	-4	-7
35	0	0	0	0	0	0	-1	-1	-2

Table 2: The longevity risk premium R_p in basis points, for different cohort sizes (N=10, 100, 1000 $\times 10^3$) and different risk aversion values ($CRRA(\gamma) = 3, 5, 8$). The initial wealth of the insurer is assumed to be $W_0 = 100$.

Common features of the results: 1) The required risk premium is negative, meaning that the insurer (or the bond issuer) must be compensated for bearing longevity risk. 2) The additional discount rate R_p (in absolute value) increases as the size of the risk increases. 3) The more risk averse the insurer is, the higher compensation is required. 4) Approximately, the longer the maturity, the higher compensation is required by this zero coupon bond.

2.3.5 Preference assumption

The above results also revealed some unsatisfied properties of CARA and CRRA preferences. For a CARA investor, he has the same worry about 1 additional euro loss no matter how rich or poor he is. For a CRRA investor, he cares much less when he is rich. In this case, the minimum required risk premium depends on his initial wealth. The CARA utility might overestimate the longevity risk premium. On the contrary, the CRRA utility might underestimate the risk premium. Therefore, we modify the CARA utility, to make the risk aversion depending on the initial insured capital or initial wealth of the company: $\alpha(W_0) = \overline{\alpha} W_0^{-b}$, where $b \in [0, 1]$. When $b = 1$, the modified utility approaches CRRA specification; when $b = 0$, the modified utility is back to the CARA specification.

$$u(S) = -\frac{1}{\alpha(W_0)} \exp(-\alpha(W_0) S) \tag{10}$$

Section 3 to 5 focus on seller's minimum required risk premium. Therefore the utility function (10) represents the preference of the shareholders of the seller. In the context of the EIB/BNP longevity bond, it is the preference of the shareholders of EIB/BNP. Section 6 discusses the buyer's maximum risk premium. Hence the utility function (10) represents the preference of the buyer, e.g. a pension fund.

The proposed preference (10) retains some convenient features of the negative exponential utility function. The expected utility is separable for independent risks x and y, i.e. $E[u(x+y)] = E[u(x)]\,E[\exp(-\alpha y)]$. For any given value of b, we have $u'(x) = \exp(-\alpha x)$, the inverse function of marginal utility $u'()$ is $I_v = -\frac{1}{\alpha}\ln(z)$, the inverse function of the utility function is $I_u = -\frac{1}{\alpha}\ln(-\alpha z)$.

3 Pricing of a coupon based longevity bond

This section derives the longevity risk premium of a coupon based longevity bond from seller's point of view. The shareholder of a financial company (like EIB/BNP) derives her utility from dividends and final wealth. We consider two alternative situations. In the first situation, the company does not insure longevity risk, hence not exposed to longevity risk. In the second situation, the company issues a longevity bond and hence bears longevity risk. Here we assume a large pool of population. There is only macro longevity risk in this model setup. The methodology used in this section combines the martingale approach with the equivalent utility pricing principle.

3.1 Setup

Problem 1 *without longevity risk*

Assume a complete financial market, with constant risk free rate, r. A company derives her utility from dividends and final wealth at the end of the horizon. The company maximize her utility by optimizing asset allocation (x_t) and dividend (D_t) decisions.

$$\max_{\{D_t,x_t\}_{t=0}^{T},W_T} V_0 = E\left[\int_0^T e^{-\delta t} u(D_t)\,dt + e^{-\delta T} u(W_T)\right] \tag{11}$$

$$s.t. \qquad E\left[\int_0^T M_t D_t dt + M_T W_T\right] = W_0 \tag{12}$$

where M_t is the pricing kernel for the complete financial market. M_t is defined by

$$dM_t/M_t = -rdt - \lambda dZ_t \tag{13}$$

where λ is the market price of equity risk.

Problem 2 *with longevity risk*

In the same financial market, this company issues 'coupon-based' longevity bond, in which the annual coupon is indexed to the 1939-born cohort survivor index. This cohort retires in 2004 at age 65. The company derives her utility from dividends and the residual claim $(E[S_t] - S_t)$ from the longevity risk. The longevity risk is not hedgeable from the financial market. The company maximizes her utility by optimizing asset allocation and dividend decisions.

$$\max_{\{D_t^\pi, x_t\}_{t=0}^T, W_T^\pi} V_0^\pi = E\left[\int_0^T e^{-\delta t} u\left(D_t^\pi + E[S_t] - S_t\right) dt + e^{-\delta T} u\left(W_T^\pi\right)\right] \tag{14}$$

$$s.t. \qquad E\left[\int_0^T M_t D_t^\pi dt + M_T W_T^\pi\right] = W_0 + \pi \tag{15}$$

Applying the equivalent utility pricing argument, we want to find the minimum risk compensation π such that the company is indifferent from bearing the longevity risk and without the longevity risk, that is,

$$V_0^\pi(\pi) = V_0. \tag{16}$$

3.2 Derivation

Problem 1 (continued)

Set up the Lagrange

$$L = E\left[\int_0^T e^{-\delta t} u\left(D_t\right) dt + e^{-\delta T} u\left(W_T\right)\right] + \phi\left(W_0 - E\left[\int_0^T M_t D_t dt + M_T W_T\right]\right)$$

$$\begin{aligned}
\frac{\partial L}{\partial D_t} &= 0 \quad \Rightarrow \quad e^{-\delta t} u'\left(D_t\right) = \phi M_t \\
\frac{\partial L}{\partial W_T} &= 0 \quad \Rightarrow \quad e^{-\delta T} u'\left(W_T\right) = \phi M_T
\end{aligned}$$

$$\begin{aligned}
D_t^* &= I_v\left(e^{\delta t}\phi M_t\right) = -\frac{1}{\alpha}\ln\left(e^{\delta t}\phi M_t\right) \\
&= -\frac{1}{\alpha}\delta t - \frac{1}{\alpha}\ln\phi - \frac{1}{\alpha}\ln M_t \\
W_T^* &= I_v\left(e^{\delta T}\phi M_T\right) = -\frac{1}{\alpha}\ln\left(e^{\delta T}\phi M_T\right)
\end{aligned}$$

Plug into the budget constraint and the indirect utility function, we have

$$
\begin{aligned}
W_0 &= E\left[\int_0^T M_t D_t^* dt + M_T W_T^*\right] &(17)\\
&= -\frac{1}{\alpha}E\left[\int_0^T M_t \ln\left(e^{\delta t}\phi M_t\right) dt + M_T \ln\left(e^{\delta T}\phi M_T\right)\right] &(18)
\end{aligned}
$$

$$
V_0 = E\left[\int_0^T e^{-\delta t}\left(-\frac{1}{\alpha}\exp\left(-\alpha D_t^*\right)\right) dt + e^{-\delta T} u\left(W_T^*\right)\right] = -\frac{1}{\alpha}\phi E\left[\int_0^T M_t dt + M_T\right] \qquad (19)
$$

Problem 2 (continued)

Set up the Lagrange

$$
L = E\left[\int_0^T e^{-\delta t} u\left(D_t^\pi + E[S_t] - S_t\right) dt + e^{-\delta T} u\left(W_T^\pi\right)\right] + \phi\left(W_0 + \pi - E\left[\int_0^T M_t D_t^\pi dt + M_T W_T^\pi\right]\right)
$$

Since longevity risk, $E[S_t]-S_t$, can not be hedged in the modelled financial market, the optimal strategy, D_t^π, is independent from $E[S_t]-S_t$. Under the assumed preference (10), the above Lagrange can be rewritten as

$$
\begin{aligned}
L &= -\frac{1}{\alpha}E\left[\int_0^T e^{-\delta t}\exp\left(-\alpha D_t^\pi\right) E\left[\exp\left(-\alpha\left(E[S_t]-S_t\right)\right)\right] dt + e^{-\delta T} u\left(W_T^\pi\right)\right]\\
&\quad +\phi\left(W_0 + \pi - E\left[\int_0^T M_t D_t^\pi dt + M_T W_T^\pi\right]\right)\\
&= E\left[\int_0^T e^{-\delta t} u\left(D_t^\pi\right) G_t dt + e^{-\delta T} u\left(W_T^\pi\right)\right] + \phi\left(W_0 + \pi - E\left[\int_0^T M_t D_t^\pi dt + M_T W_T\right]\right)
\end{aligned}
$$

where G_t denotes

$$
G_t \equiv E\left[\exp\left(-\alpha\left(E[S_t]-S_t\right)\right)\right]
$$

and α is the shorthand notation for $\alpha\left(W_0\right) = \overline{\alpha} W_0^{-b}$. G_t can be seen as a function of the certainty equivalent of $E[S_t]-S_t$.

The optimal dividend strategy can be found by

$$
\begin{aligned}
\frac{\partial L}{\partial D_t^\pi} &= 0 \quad\Rightarrow\quad e^{-\delta t} u'\left(D_t^{*\pi}\right) G_t = \phi^\pi M_t\\
\frac{\partial L}{\partial W_T^\pi} &= 0 \quad\Rightarrow\quad e^{-\delta T} u'\left(W_T^\pi\right) = \phi^\pi M_T
\end{aligned}
$$

$$D_t^{*\pi} = I_v\left(\frac{e^{\delta t}\phi^\pi M_t}{G_t}\right) = -\frac{1}{\alpha}\left[\ln\left(e^{\delta t}\phi^\pi M_t\right) - \ln G_t\right] \tag{20}$$

$$W_T^{*\pi} = I_v\left(e^{\delta T}\phi^\pi M_T\right) = -\frac{1}{\alpha}\ln\left(e^{\delta T}\phi^\pi M_T\right) \tag{21}$$

Plug into the budget constraint and the indirect utility function

$$\begin{aligned} W_0 + \pi &= E\left[\int_0^T M_t D_t^{*\pi} dt + M_T W_T^{*\pi}\right] \\ &= -\frac{1}{\alpha}E\left[\int_0^T M_t \ln\left(e^{\delta t}\phi^\pi M_t\right)dt + M_T \ln\left(e^{\delta T}\phi^\pi M_T\right)\right] + \frac{1}{\alpha}E\left[\int_0^T M_t \ln G_t dt\right] \end{aligned} \tag{22}$$

$$V_0^{*\pi} = -\frac{1}{\alpha}E\left[\int_0^T e^{-\delta t}\exp\left(-\alpha D_t^{*\pi}\right)G_t dt + e^{-\delta T}u\left(W_T^{*\pi}\right)\right] = -\frac{1}{\alpha}\phi^\pi E\left[\int_0^T M_t dt + M_T\right] \tag{23}$$

Equalizing the two indirect utilities (19) and (23), $V_0 = V_0^{*\pi}$, we find $\phi = \phi^\pi$.
Comparing the two budget constraints (17) and (22), with $\phi = \phi^\pi$, we have

$$W_0 = -\frac{1}{\alpha}E\left[\int_0^T M_t \ln\left(e^{\delta t}\phi^\pi M_t\right)dt + M_T \ln\left(e^{\delta T}\phi^\pi M_T\right)\right] \tag{24}$$

$$W_0 + \pi = -\frac{1}{\alpha}E\left[\int_0^T M_t \ln\left(e^{\delta t}\phi^\pi M_t\right)dt + M_T \ln\left(e^{\delta T}\phi^\pi M_T\right)\right] + \frac{1}{\alpha}E\left[\int_0^T M_t \ln G_t dt\right] \tag{25}$$

The difference between the two budget constraints gives the expression for longevity risk loading of the survival bond

$$\pi = \frac{1}{\alpha}E\left[\int_0^T M_t \ln G_t dt\right] = \frac{1}{\alpha}\int_0^T E[M_t]\ln G_t dt \tag{26}$$

$$= \frac{1}{\alpha}\int_0^T e^{-rt}\ln G_t dt. \tag{27}$$

The risk loading is a present value of the certainty equivalent compensations $\frac{1}{\alpha}\ln G_t$ for the risks $S_t - E[S_t]$. This compensation is paid out as part of dividend in each period, as shown in the optimal dividends policy (20).

3.3 Results

The value of the 'coupon-based' longevity bonds with maturity T can be decomposed into 'best estimated' price and longevity risk loading:

$$total\ price = best\ estimate\ +\ risk\ loading = \int_0^T e^{-rt} E[S_t] dt + \pi$$

where $best\ estimate := \int_0^T e^{-rt} E[S_t] dt$. Table 3 below shows the normalized risk loading, $\frac{\pi}{beste\ stimate}$, of the 'coupon-based' longevity bonds with maturity $T = 1, ..., 35$ years. The risk loading depends on the maturity of the bond, the size of initial capital and the risk aversion of the insurer. Recall that the risk aversion is inversely related to the size of the insured capital, $\alpha(W_0) = \overline{\alpha}(W_0)^{-b}$. As b changes from 1 to 0, the preference shifts from CRRA to CARA, which results in an increase in risk loading. CRRA investor requires virtually zero compensation. Whereas CARA investor requires a sizable compensation.

equity	w0 = 100000						w0 = 10000					
maturity	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0
5	0%	0%	0%	0%	0,0%	0,0%	0%	0%	0%	0%	0,0%	0,0%
10	0%	0%	0%	0%	0,0%	0,2%	0%	0%	0%	0%	0,1%	0,2%
15	0%	0%	0%	0%	0,1%	0,5%	0%	0%	0%	0%	0,2%	0,5%
20	0%	0%	0%	0%	0,2%	0,9%	0%	0%	0%	0%	0,3%	0,9%
25	0%	0%	0%	0%	0,3%	1,4%	0%	0%	0%	0%	0,4%	1,4%
30	0%	0%	0%	0%	0,4%	1,5%	0%	0%	0%	0%	0,5%	1,5%
35	0%	0%	0%	0%	0,4%	1,6%	0%	0%	0%	0%	0,5%	1,6%

equity	w0 = 1000						w0 = 100					
maturity	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0
5	0%	0%	0%	0,0%	0,0%	0,0%	0%	0%	0%	0,0%	0,0%	0,0%
10	0%	0%	0%	0,0%	0,1%	0,2%	0%	0%	0%	0,1%	0,1%	0,2%
15	0%	0%	0%	0,1%	0,2%	0,5%	0%	0%	0%	0,2%	0,3%	0,5%
20	0%	0%	0%	0,2%	0,4%	0,9%	0%	0%	0%	0,3%	0,6%	0,9%
25	0%	0%	0%	0,3%	0,6%	1,4%	0%	0%	0%	0,4%	0,8%	1,4%
30	0%	0%	0%	0,3%	0,7%	1,5%	0%	0%	0%	0,5%	0,9%	1,5%
35	0%	0%	0%	0,3%	0,7%	1,6%	0%	0%	0%	0,5%	0,9%	1,6%

Table 3: The relative risk loading per person, $\pi/best\ estimate$, of EIB/BNP type longevity bonds for different size of the initial equity capital of the firm, $W_0 = [100000, 10000, 1000, 100]$ million, and differet risk aversion specifications $\alpha(W_0) = \overline{\alpha}(W_0)^{-b}$, with $\overline{\alpha} = 3$, $b = [1, \frac{1}{2}, \frac{3}{8}, \frac{1}{4}, \frac{1}{8}, 0]$. The size of the insured pool is $N = 100$ million, $K = 1$.

We can also express the risk loading π in terms of risk premium, R_p, which is an additional discount rate above the risk free rate to adjust for the longevity risk.

$$
\begin{aligned}
total\ price &= \int_0^T e^{-(r+R_p)t} E[S_t] dt \\
i.e. \qquad \int_0^T e^{-rt} E[{}_t p_{65}] dt + \frac{\pi}{N} &= \int_0^T e^{-(r+R_p)t} E[{}_t p_{65}] dt
\end{aligned}
$$

equity	w0 = 100000						w0 = 10000					
maturity	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0
5	0	0	0	0	0	-1	0	0	0	0	0	-1
10	0	0	0	0	-1	-3	0	0	0	0	-1	-3
15	0	0	0	0	-2	-7	0	0	0	-1	-2	-7
20	0	0	0	-1	-3	-11	0	0	0	-1	-4	-11
25	0	0	0	-1	-4	-15	0	0	0	-2	-5	-15
30	0	0	0	-1	-4	-16	0	0	-1	-2	-5	-16
35	0	0	0	-1	-4	-16	0	0	-1	-2	-5	-16

equity	w0 = 1000						w0 = 100					
maturity	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0
5	0	0	0	0	0	-1	0	0	0	0	-1	-1
10	0	0	0	-1	-1	-3	0	0	-1	-1	-2	-3
15	0	0	-1	-1	-3	-7	0	-1	-1	-2	-4	-7
20	0	0	-1	-2	-5	-11	0	-1	-2	-4	-7	-11
25	0	0	-1	-3	-6	-15	0	-2	-3	-5	-8	-15
30	0	-1	-1	-3	-7	-16	0	-2	-3	-5	-9	-16
35	0	-1	-1	-3	-7	-16	0	-2	-3	-5	-9	-16

Table 4: The longevity risk premium R_p (in basis points) of EIB/BNP type longevity bonds for different size of the initial equity capital of the firm, $W_0 = [100000, 10000, 1000, 100]$ million, and differet risk aversion specifications $\alpha(W_0) = \overline{\alpha}(W_0)^{-b}$, with $\overline{\alpha} = 3$, $b = [1, \frac{1}{2}, \frac{3}{8}, \frac{1}{4}, \frac{1}{8}, 0]$. The size of the insured pool is $N = 100$ million, $K = 1$.

Table 4 above presents the risk premium R_p (in basis points) of the 'coupon-based' longevity bonds with maturity $T = 1, ..., 35$ year. The results show two observations. First, the risk premium increases as the maturity of the bond increases. The risk premium for short maturity ($T \leq 5$ years) is small, less than 1 basis point. Second, the risk premium depends on the financial position of the insurer. Except for the CARA case ($b = 0$), the larger initial equity a firm has, the lower risk premium the firm requires. The face value of the EIB/BNP bond issue was 540 million and the bond had a 25-year maturity. The initial coupon was set at 50 million, which is comparable with the initial payments $NK = 100$ million assumed here. By the end 2005, EIB's own fund amounts to nearly 30000 million, which is comparable with the initial equity $W_0 = 10000$ million assumed here. The upper right panel ($W_0 = 10000$) indicates a (minimum required) risk premium of 5 basis points with $b = 1/8$ for 25 years maturity.

3.4 Implications

The implication that we can get from the above results is that longevity risk premium depends on the financial position of the insurer. Large equity financial institutions may require a lower risk premium. Alternatively, smaller issues may require lower risk premium. In order to avoid too high risk premiums, it might be helpful to have many large institutions all issuing moderate amount of longevity bonds, linked to the same survivor index.

4 Pricing of other longevity linked securities

4.1 Vanilla longevity swaps

Vanilla longevity swaps have similar risk structure, $E[S_t] - S_t$, as the longevity bonds. The insurer or the investment bank pays the counterpart the difference between the expected and realized mortality. Analogize to interest rate swap, the fixed leg is $E[S_t]$, and the floating leg is S_t. The required risk premium of a longevity swap is the same as in a longevity bond with the same maturity and the same amount of notional issues (Table 4). The main advantages of swap lie in a much lower upfront capital requirement and lower credit risk, as compared with a longevity bond. The present value of the longevity swap is a small fraction of otherwise identical longevity bonds (Table 3).

4.2 Deferred longevity bond

An alternative design of a longevity bond is a deferred longevity bond. The issuance company starts paying the longevity linked coupons S years after the issuance, till the bond maturity in year T. An advantage of a deferred longevity bond is that it skips the ineffective hedge coupons in the first few years, and hence requires much less upfront capital than an immediate coupon paying bond, as pointed out by Blake, Cairns and Dowd (2006). Follow the same pricing principle as in Section 3, the risk loading of a deferred longevity bond is given by

$$\pi^{def} = \frac{1}{\alpha} \int_S^T e^{-rt} \ln G_t dt \tag{28}$$

Table 5 and 6 below show the relative risk loadings and the risk premiums of several deferred longevity bond. The following results assume that all deferred longevity bonds have a maturity of 35 years, but the coupon payments could start 5, 10, 15 or 20 years after the issuance. Notice the first row in the tables is an immediate starting bond for comparison. The initial capital is much smaller than the immediate starting bond, but the relative risk loading is much larger. As a consequence, the required risk premiums are also larger than the immediate starting bond.

# year	best	w0 = 100000						w0 = 10000					
defer	estim.	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0
0	13,1	0%	0%	0%	0%	0%	2%	0%	0%	0%	0%	1%	2%
5	8,6	0%	0%	0%	0%	1%	2%	0%	0%	0%	0%	1%	2%
10	5,2	0%	0%	0%	0%	1%	4%	0%	0%	0%	0%	1%	4%
15	2,7	0%	0%	0%	0%	1%	6%	0%	0%	0%	1%	2%	6%
20	1,2	0%	0%	0%	0%	2%	8%	0%	0%	0%	1%	2%	8%

# year	best	w0 = 1000						w0 = 100					
defer	estim.	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0
0	13,1	0%	0%	0%	0%	1%	2%	0%	0%	0%	1%	1%	2%
5	8,6	0%	0%	0%	0%	1%	2%	0%	0%	0%	1%	1%	2%
10	5,2	0%	0%	0%	1%	2%	4%	0%	0%	1%	1%	2%	4%
15	2,7	0%	0%	0%	1%	2%	6%	0%	1%	1%	2%	3%	6%
20	1,2	0%	0%	1%	1%	3%	8%	0%	1%	1%	2%	4%	8%

Table 5: The relative risk loading per person, $\pi/best\ estimate$, of the deferred starting longevity bonds, for different size of the initial equity capital, $W_0 = [100000, 10000, 1000, 100]$ million, and different risk aversion values $\overline{\alpha}\,(W_0)^{-b}$, with $\overline{\alpha} = 3$, $b = [1, \frac{1}{2}, \frac{3}{8}, \frac{1}{4}, \frac{1}{8}, 0]$. $N = 100$ million, $K = 1$.

# year	w0 = 100000						w0 = 10000					
defer	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0
0	0	0	0	-1	-4	-16	0	0	-1	-2	-5	-16
5	0	0	0	-1	-4	-17	0	0	-1	-2	-6	-17
10	0	0	0	-1	-5	-21	0	0	-1	-2	-7	-21
15	0	0	0	-2	-7	-26	0	0	-1	-3	-9	-26
20	0	0	0	-2	-7	-30	0	0	-1	-3	-10	-30

# year	w0 = 1000						w0 = 100					
defer	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0
0	0	-1	-1	-3	-7	-16	0	-2	-3	-5	-9	-16
5	0	-1	-1	-3	-8	-17	0	-2	-3	-6	-10	-17
10	0	-1	-2	-4	-9	-21	0	-2	-4	-7	-12	-21
15	0	-1	-2	-5	-12	-26	0	-3	-5	-9	-15	-26
20	0	-1	-2	-6	-13	-30	0	-3	-6	-10	-18	-30

Table 6: The longevity risk premium R_p (in basis points) of deferred longevity bonds, for different size of the initial equity capital of the firm, $W_0 = [100000, 10000, 1000, 100]$ million, and different risk aversion values $\overline{\alpha}\,(W_0)^{-b}$, with $\overline{\alpha} = 3$, $b = [1, \frac{1}{2}, \frac{3}{8}, \frac{1}{4}, \frac{1}{8}, 0]$. $N = 100$ million, $K = 1$.

4.3 Longevity floors and longevity caps

In case of a longevity floor, the payoff structure is $\min[E[S_t]-S_t,0]$. When the number of survivor is greater than the expected, the insurer faces a 'mortality' loss. In case of a longevity cap, the payoff structure is $max[E[S_t]-S_t,0]$. The insurer makes 'mortality' profit when the number of survivor is less than the expected. The risk premium of mortality floor and cap can be obtained similarly based on the equivalent utility approach. The company derives her utility from dividends and the residual claim $([E[S_t]-S_t]^-)$[9]. Following the martingale approach the risk loading is determined by

$$\pi^- = \frac{1}{\alpha}\int_0^T e^{-rt}\ln G_t^- dt \tag{29}$$

$$\pi^+ = \frac{1}{\alpha}\int_0^T e^{-rt}\ln G_t^+ dt \tag{30}$$

where $G_t^- \equiv E\left[\exp\left(-\alpha\left[E[S_t]-S_t\right]^-\right)\right]$, and $G_t^+ \equiv E\left[\exp\left(-\alpha\left[E[S_t]-S_t\right]^+\right)\right]$.

Table 7 below shows the longevity risk premium R_p (in basis points) of longevity bonds, longevity floors and longevity caps respectively.

	w 0 = 100000			w 0 = 10000			w 0 = 1000			w 0 = 100		
maturity	bond	floor	cap	bond	floor	cap	bond	floor	cap	bond	floor	cap
5	0	-3	3	0	-3	3	0	-3	3	-1	-3	3
10	-1	-5	4	-1	-5	4	-1	-5	4	-2	-5	4
15	-2	-7	6	-2	-7	5	-3	-7	5	-4	-8	5
20	-3	-9	7	-4	-9	7	-5	-10	6	-7	-11	6
25	-4	-10	8	-5	-11	8	-6	-12	7	-8	-14	7
30	-4	-11	9	-5	-12	8	-7	-13	8	-9	-15	8
35	-4	-11	9	-5	-12	9	-7	-13	8	-9	-15	8

Table 7: The longevity risk premium R_p (in basis points) of longevity bonds, longevity floors and longevity caps, for different size of the initial equity capital, $W_0 = [100000, 10000, 1000, 100]$ million, and different risk aversion values $\overline{\alpha}\,(W_0)^{-b}$, with $\overline{\alpha} = 3$, $b = \frac{1}{8}$. $N = 100$ million, $K = 1$.

We observe three things from this table. First, the risk premium of longevity floor is larger (in absolute terms) than that of the longevity bond, This is because that the payoff of longevity bond is symmetric, whereas the payoff of a longevity floor is highly asymmetric. Therefore higher risk premium is required for bearing losses only. Second, the risk premium of the long position in this longevity caps is positive, which means that the insurer pays for the call option. The more risk

[9] $[E[S_t]-S_t]^- \equiv \min\left(E[S_t]-S_t,0\right)$; $[E[S_t]-S_t]^+ \equiv max[E[S_t]-S_t,0]$

averse the insurer is, the less willingness to pay (read: compensate) the counterpart, for receiving the uncertain 'profit'. Thirdly, as the initial wealth decreases, the value of the cap decreases, while the value of the floor increases. The interpretation is that the 'profit' becomes more and more uncertain, therefore less willingness to pay for receiving the increasingly uncertain 'profit'.

5 The effect of natural hedgeing

It is known that term insurance policies provide a natural hedge for the immediate annuities (see e.g. Cox and Lin (2004)). The term insurance pays out certain amount of death benefit if the policy holder dies before the contract expires. Since longevity shocks affect all age cohorts in the same direction, the unexpected increase in annuity payments to the retirees can be partially offset by the unexpected reduction of death benefit payments linked to the younger cohorts. The availability of natural hedgeing clearly affects the risk premium of longevity bond issuance company. This section examines the magnitude of this effect on the pricing of longevity bonds.

Suppose that the longevity bond issuance company bears the risks from both the term insurance policies linked to a group of 35 year olds in 2004 and the longevity bonds linked to a group of 65 year olds in 2004. Furthermore, suppose that the estimated Lee-Carter 1992 model is the true process governing the future mortality dynamics. The number of death for the 35-year-old cohort in year t is $S_{t-1}^{35} - S_t^{35}$. The (unexpected) shocks from the term insurance policies are $B_t * \left(E\left[S_{t-1}^{35} - S_t^{35}\right] - \left(S_{t-1}^{35} - S_t^{35}\right)\right)$, where B_t denotes the ratio of death benefit relative to the annuity payout K (=1, which is the agreed annuity payments). The (unexpected) shocks from the longevity bonds are captured by $E[S_t^{65}] - S_t^{65}$.The combined shocks from the term insurance and the longevity bonds can be denoted by Z_t as

$$Z_t \equiv E[S_t^{65}] - S_t^{65} + B_t * \left(E\left[S_{t-1}^{35} - S_t^{35}\right] - \left(S_{t-1}^{35} - S_t^{35}\right)\right).$$

As explained in Section 3, the minimum longevity risk loading π^{hedge} required by the seller is determined by setting $V_0^{\pi,hedge} = V_0$, where $V_0^{\pi,hedge}$ is the indirect utility given by

$$\max_{\{D_t^{\pi,hd}, x_t\}_{t=0}^T} V_0^{\pi,hedge} = E\left[\int_0^T e^{-\delta t} u\left(D_t^{\pi,hedge} + Z_t\right) dt + e^{-\delta T} u\left(W_T\right)\right] \tag{31}$$

$$s.t. \quad E\left[\int_0^T M_t D_t^{\pi,hedge} dt + M_T W_T\right] = W_0 + \pi^{hedge} \tag{32}$$

Following a similar argument as in Section 3.2, we can find the corresponding risk premium as

$$\pi^{hedge} = \frac{1}{\alpha} \int_0^T e^{-rt} \ln G_t^{hedge} dt. \tag{33}$$

where G_t^{hedge} measures the certainty equivalent of the combined shocks Z_t, i.e.

$$G_t^{hedge} \equiv E\left[\exp\left(-\alpha\left(Z_t\right)\right)\right]$$

The following example illustrates the effective of natural hedgeing. In this example, the level of death benefit decreases over time, $B_t = T + 1 - t$, for $t = 1, 2, ..., T$.

Figure 3 compares the term insurance with the payout volatility of the longevity bonds with a term insurance with decreasing death benefits. The volatility of the combined shocks is much lower than that of the longevity bond alone. However, the hedgeing is not perfect.

Table 8 shows that the minimum required risk premiums, R_p, which is clearly reduced when natural hedgeing is available. The risk premiums are more than halved comparing with the case without natural hedgeing.

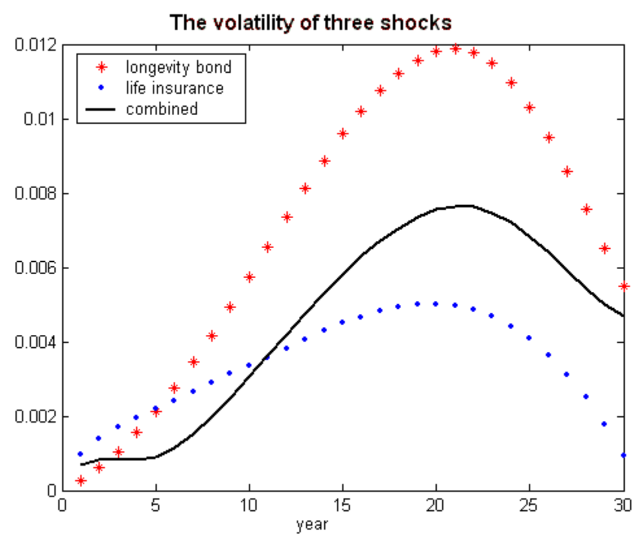

Figure 3: The volatility of payouts of the longevity bond and the term insurance separately and combined.

equity	w 0 = 100000						w 0 = 10000					
maturity	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0
5	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	-1	0	0	0	0	0	-1
15	0	0	0	0	-1	-2	0	0	0	0	-1	-2
20	0	0	0	0	-1	-5	0	0	0	0	-1	-5
25	0	0	0	0	-1	-6	0	0	0	-1	-2	-6
30	0	0	0	0	-2	-7	0	0	0	-1	-2	-7

equity	w 0 = 1000						w 0 = 100					
maturity	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0
5	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	-1	0	0	0	0	0	-1
15	0	0	0	0	-1	-2	0	0	0	-1	-1	-2
20	0	0	0	-1	-2	-5	0	0	-1	-1	-3	-5
25	0	0	0	-1	-3	-6	0	-1	-1	-2	-3	-6
30	0	0	0	-1	-3	-7	0	-1	-1	-2	-4	-7

Table 8: The longevity risk premium R_p (in basis points) of longevity bonds when natural hedgeing is available ($B_t = T + 1 - t$).

6 The demand side pricing and basis risk

6.1 Demand side pricing

The demand side pricing considers the maximum price π^{BUY} that the buyers (e.g. annuity providers) are willing to pay for the longevity bond or other securities in order to be fully insured against the longevity risk. From buyer's point of view, π^{BUY} can be derived in the same framework as in Section 3. Assume that an annuity provider sold annuities to a cohort retiring in 2004 at age 65. The shareholder of an annuity provider derives her utility from dividends and final wealth. In the first scenario, the annuity provider bought the ideal EIB/BNP survivor bonds for π^{BUY} so that the longevity risk from her annuity contracts are completely insured. In the second scenario, the annuity provider bears the longevity risk herself.

Problem 3 *without longevity risk*

Assume a complete financial market, with constant risk free rate, r. The annuity provider derives her utility from dividends and final wealth at the end of the horizon. The company bought the ideal EIB/BNP survivor bonds for π^{BUY}, such that the longevity risk is completely hedged. The company maximize her utility by optimizing asset allocation (x_t) and dividend (D_t) decisions.

$$\max_{\{D_t^\pi, x_t\}_{t=0}^T, W_T^\pi} V_0^\pi = E\left[\int_0^T e^{-\delta t} u\left(D_t^\pi\right) dt + e^{-\delta T} u\left(W_T^\pi\right)\right] \tag{34}$$

$$s.t. \qquad E\left[\int_0^T M_t D_t^\pi dt + M_T W_T^\pi\right] = W_0 - \pi^{BUY} \tag{35}$$

Problem 4 *with longevity risk*

In the same financial market, this annuity provider didn't buy any longevity bond, hence bears the longevity risk herself. The company derives her utility from dividends and the residual claim $(E[S_t] - S_t)$ from the longevity risk. The longevity risk is not hedgeable from the financial market. The company maximizes her utility by optimizing asset allocation and dividend decisions.

$$\max_{\{D_t, x_t\}_{t=0}^T, W_T} V_0 = E\left[\int_0^T e^{-\delta t} u\left(D_t + E[S_t] - S_t\right) dt + e^{-\delta T} u\left(W_T\right)\right] \tag{36}$$

$$s.t. \qquad E\left[\int_0^T M_t D_t dt + M_T W_T\right] = W_0 \tag{37}$$

Applying the equivalent utility pricing argument, we want to find the minimum risk compensation π^{BUY} such that the company is indifferent from bearing the longevity risk and without the longevity risk, that is,

$$V_0^\pi(\pi^{BUY}) = V_0. \tag{38}$$

Following the equivalent utility pricing argument,

$$\pi^{BUY} = \frac{1}{\alpha}\int_0^T e^{-rt} \ln G_t^{BUY} dt. \tag{39}$$

where G_t^{BUY} denotes

$$G_t^{BUY} \equiv E\left[\exp\left(-\alpha\left(E[S_t] - S_t\right)\right)\right] \tag{40}$$

The maximum premium that a buyer of the longevity bond is willing to pay has the same form as the minimum premium that the bond issuance company requires. It is common to assume that the longevity bond buyer is more risk averse than the bond issuance company, or the financial position of the buyer is weaker than the seller.

The buyer's maximum price is also influenced by whether or not natural hedgeing is possible. The availability of natural hedgeing could reduce the buyer's price significantly. Furthermore, the presence of basis risk and the risk sharing possibility will also affect the buyer's maximum price.

6.2 Basis risk

An ideal longevity bond which provides a perfect longevity hedge should be linked to the annuitant population of the annuity provider. However, often this is not the case. There is a discrepancy between the reference population from the bond seller and the annuitant population from the bond buyer. Although the survival probabilities of the two population might be (highly) correlated, the longevity bond buyer still exposes to the remaining unhedgeable part, the so-called basis risk. The EIB/BNP longevity bond, although linked to the British survivor index, was also marketed among Dutch pension funds. The idea is that the Dutch survivor index may be highly correlated with the British one. The question here is whether 20 basis points is a good deal or not for Dutch pension funds? This depends on the correlation between Dutch and British mortality rates. The correlation between $\triangle\gamma_t^{UK}$ and $\triangle\gamma_t^{NL}$ is about 0.8, based on 1880-2003 data from both countries, with $\triangle\gamma_t \equiv \gamma_t - \gamma_{t-1}$.

The basis risk between the British and the Dutch annuitant population can be captured by $Z_t^{BasisRisk}$ defined as

$$Z_t^{BasisRisk} \equiv E[S_t^{NL}] - S_t^{NL} - \left(E[S_t^{EN}] - S_t^{EN}\right).$$

Based on the expression for buyer's maximum acceptable price (39), we can show that the risk loading with basis risk is

$$\pi^{BasisRisk} = \frac{1}{\alpha}\int_0^T e^{-rt} \ln G_t^{BasisRisk} dt. \tag{41}$$

where $G_t^{BasisRisk}$ denotes

$$G_t^{BasisRisk} \equiv E\left[\exp\left(-\alpha\left(Z_t^{BasisRisk}\right)\right)\right] \tag{42}$$

If assuming that the pension fund has the same preference and the same level of equity as the longevity bond issuance company, and also assuming the Dutch pension fund has no natural hedgeing possibility, then we get the following maximum acceptable longevity risk premium R_p (Table 9). The risk premium in Table 9 is slightly higher than the minimum required risk premium in Table 4 is due to the fact that the estimated $\widehat{\sigma^{NL}}_{\varepsilon} = 0.2176$ is higher than the British counterpart (see Appendix A.1).

equity	w 0 = 100000						w 0 = 10000					
maturity	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0
5	0	0	0	0	0	-1	0	0	0	0	0	-1
10	0	0	0	0	-1	-2	0	0	0	0	-1	-2
15	0	0	0	0	-2	-6	0	0	0	-1	-2	-6
20	0	0	0	-1	-3	-11	0	0	0	-1	-4	-11
25	0	0	0	-1	-4	-16	0	0	-1	-2	-6	-16
30	0	0	0	-1	-5	-18	0	0	-1	-2	-7	-18
equity	w 0 = 1000						w 0 = 100					
maturity	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0	b=1	b=1/2	b=3/8	b=1/4	b=1/8	b=0
5	0	0	0	0	0	-1	0	0	0	0	0	-1
10	0	0	0	0	-1	-2	0	0	0	-1	-1	-2
15	0	0	-1	-1	-3	-6	0	-1	-1	-2	-4	-6
20	0	0	-1	-2	-5	-11	0	-1	-2	-4	-7	-11
25	0	-1	-1	-3	-8	-16	0	-2	-3	-6	-10	-16
30	0	-1	-2	-4	-9	-18	0	-2	-4	-7	-12	-18

Table 9: The buyer's maximum longevity risk premium R_p (in basis points) for pension fund with different initial equity capital level, and different risk aversion values $\overline{\alpha}\,(W_0)^{-b}$, with $\overline{\alpha} = 3$, $b = [1, \frac{1}{2}, \frac{3}{8}, \frac{1}{4}, \frac{1}{8}, 0]$. The size of the insured pool is $N = 100$ million, $K = 1$.

7 Conclusion

Longevity-linked securities are not traded in the financial markets yet, and pricing issue is one of the obstacles for the potential development of the longevity linked security market. Before the market opens up, this paper tries to quantify the potential sizes of the longevity risk premiums in various longevity-linked securities, including longevity bonds and derivatives. In this paper, we apply the equivalent utility pricing principle to tackle the longevity risk pricing problem. The range of the longevity risk premiums is captured by the seller's minimum price and the buyer's maximum price. In this paper, the longevity risk is modeled by Lee-Carter (1992) model, and estimated according to the Dutch mortality data. Given the plausible ranges of values of risk aversion, financial position and other assumptions, the resulting risk premiums found in this paper are consistent with the limited market observations. We also show that the impact of natural hedgeing is potentially significant. The results provide design implications for longevity-linked securities and longevity risk management.

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A The Lee-Carter 1992 model

Following the LC92 model, the time series property of the log mortality rate of the x-year-old, $\ln\left(\mu_{x,t}\right)$, is determined by a common latent factor γ_t with an age specific sensitivity β_x and an age specific level α_x

$$\ln\left(\mu_{x,t}\right) = \alpha_x + \beta_x\gamma_t + \delta_t \tag{43}$$

with the latent factor satisfies a random walk with drift process as

$$\gamma_t = c + \gamma_{t-1} + \varepsilon_t \tag{44}$$

where δ_t and ε_t are vectors of white noise, satisfying the distributional assumptions

$$\begin{pmatrix} \delta_t \\ \varepsilon_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_\delta & 0 \\ 0 & \sigma_\varepsilon^2 \end{pmatrix}\right) \tag{45}$$

The forecasted log mortality rate in s years' time of a then x-year-old is

$$\begin{aligned} \ln\left(\mu_{x,t+s}\right) &= \alpha_x + \beta_x\gamma_{t+s} + \delta_t \\ &= \ln\left(\mu_{x,t}\right) + \beta_x\left(\gamma_{t+s} - \gamma_t\right) + \left(\delta_{t+s} - \delta_t\right) \\ &= \ln\left(\mu_{x,t}\right) + \beta_x\left(sc + \sum_{i=1}^{s}\varepsilon_{t+i}\right) + \left(\delta_{t+s} - \delta_t\right) \end{aligned}$$

That is

$$\mu_{x,t+s} = \mu_{x,t}\exp\left(\beta_x\left(sc + \sum_{i=1}^{s}\varepsilon_{t+i}\right) + \left(\delta_{t+s} - \delta_t\right)\right) \tag{46}$$

Since about 95% of the variance in the long term forecasts is generated by the innovation of the latent factor γ_t, as reported by Lee and Carter (1992), one can simplify the forecast formula of $\mu_{x,t+s}$ as

$$\mu_{x,t+s} = \mu_{x,t}\exp\left(\beta_x\left(sc + \sum_{i=1}^{s}\varepsilon_{t+i}\right)\right) \tag{47}$$

The survival probability of the x-year-old over one year, assuming that the force of mortality is constant during the year $\mu_{x+u,t+u} = \mu_{x,t}$ $(0 \leq u < 1)$, is given by

$$p_{x,t} = p_{[x_0]+t,t} = \exp\left(-\mu_{x,t}\right) \tag{48}$$

The survival probability of the x-year-old over τ years is given by

$$_{\tau}p_{x,t} = \exp\left(-\sum_{i=1}^{\tau} \mu_{x+i,t+i}\right) \tag{49}$$

A.1 Estimation procedure of LC92 model

Let Y denote the matrix of log mortality rates, with each row for each age group $\ln \mu_x$ for N historical observations. We first construct a demeaned matrix of log mortalities, $X = Y - \alpha_x \iota$, where α_x is the mean value of $\ln \mu_x$, and ι is a row vector of ones. Then as proposed by Lee and Carter (1992), we can use Singular Value Decomposition (SVD) to estimate the latent factor γ_t and the age-specific sensitivity β_x. $X = USV'$. Since the first singular value is significantly larger than other singular values, one can use one factor to approximate the log of force of mortality, as proposed by Lee and Carter (1992). β_x is the first column of U (multiplied by -1 to keep γ_t a downward sloping trend), and γ_t is the first element of S times the first column of V (multiplied by -1 to keep γ_t a downward sloping trend). The straightforward estimations of the drift parameter c, the variance of the innovation of the latent factor, and the variance of the estimated c are given by:

$$\begin{aligned}
\widehat{c} &= \frac{1}{N-1}\sum_{n=2}^{N} \triangle\gamma_n = \frac{1}{N-1}(\gamma_N - \gamma_1) \\
\widehat{\sigma}_{\varepsilon}^2 &= \frac{1}{N-1}\sum_{n=2}^{N} \widehat{\varepsilon}_n^2 = \frac{1}{N-1}\sum_{n=2}^{N} (\triangle\gamma_n - \widehat{c})^2 \\
\sigma(\widehat{c}) &= \frac{\widehat{\sigma}_{\varepsilon}}{\sqrt{N-1}}
\end{aligned}$$

The estimated latent process (in U.K.) is the following, including two temporary shocks captured by a 'WWI' dummy and a 'WWII' dummy:

$$\gamma_t^{UK} = -0.0725 + \gamma_{t-1}^{UK} + 0.65 * WWI_t + 1.9 * WWII_t + \varepsilon_t \tag{50}$$

with $\widehat{\sigma^{UK}}_{\varepsilon} = 0.169$.

The estimated latent process (in the Netherlands) is the following, including two temporary shocks captured by a 'flu' dummy and a 'WWII' dummy:

$$\gamma_t^{NL} = -0.0748 + \gamma_{t-1}^{NL} + 1.85 * flu_t + 0.63 * WWII_t + \varepsilon_t \tag{51}$$

with $\widehat{\sigma^{NL}}_{\varepsilon} = 0.2176$. As pointed out in Lee-Carter (1992), the dummy variables only reduced the standard errors of the mortality forecast, but not the trend itself.

A.2 Simulation

The simulation steps:

1. simulate the latent factor for T periods according to $\gamma_{t+i} = \widehat{c} + \gamma_{t+i-1} + \varepsilon_i$, for $i = 1, ..., T$, where $\varepsilon_i \sim N\left(0, \widehat{\sigma}_{\varepsilon}^2\right)$, and $\gamma_t = \gamma_{2003}$ which is the last γ obtained from the estimation.

2. compute the force of mortality according to $\mu_{x+i,t+i} = \exp\left(\widehat{\alpha}_{x+i} + \widehat{\beta}_{x+i}\gamma_{t+i}\right)$, for $i = 1, ..., T$,

3. compute the survival probability of the x-year-old cohort according to ${}_{\tau}p_{x,t} = \exp\left(-\sum_{i=1}^{\tau} \mu_{x+i,t+i}\right)$, for $\tau = 1, ..., T$

4. compute the survival index $S_t = N_t p_x$, where N is the initial size of the cohort.

5. repeat 1-4 steps for M times. As a by-product, calculate the mean, variance, and confidence interval of the forecasted survival probabilities and the survival index.