HEDGING ENDOWMENT ASSURANCE PRODUCTS UNDER INTEREST RATE AND MORTALITY RISK

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Abstract. This paper analyzes how model misspecification associated with both interest rate and mortality risk influences hedging decisions of insurance companies. For this purpose, diverse risk management strategies which are risk–minimizing when model risk is ignored come into consideration. The effectiveness of these strategies is investigated by looking at the distribution of the resulting hedging errors under the combination of both sources of model risk. The analysis is based on endowment assurances which include an investment element together with a sum assured. Normally, the customer contributes periodic premiums. Compared to an upfront premium, this poses an additional risk to the insurance company. Since the premium payments stop in the case of an early death, it is not known today how many premium payments will be forthcoming. Theoretically, a loan corresponding to the present value of the expected delayed premium payments must be asked for by the insurance company in order to implement his hedging decisions. Therefore, we also consider how model risk affects this borrowing decision.

Keywords: *Model misspecification, mortality risk, interest rate risk, periodic premiums, asset liability management*

JEL–Codes: *G13, G22, G23*
Subject and Insurance Branch Codes: : *IM10, IE10, IE50, IB10*

1. Introduction

Endowment assurance products are the most popular policies among all insurance plans. For example, about 75% of the life insurance contracts sold in Germany belong to this category. The benefits of the contracts are given in terms of a life cover together with an investment element. In particular, the payoff is given by the maximum of a fixed amount (the sum assured) and an insurance account.[1] The maturity date where the payoff occurs is conditioned on the death time of the life insured. It is either given by a specified date or the nearest future reference date after an early death.[2] For contribution, the customer pays periodic premiums which are contingent on his death evolution, too. Obviously, periodic premiums make the insurer exposed

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[1] Normally, additional option features are included as well. One might think of an additional participation in the excess return of a benchmark index, c.f. Mahayni and Sandmann (2005). One can also or additionally think of an option to surrender the contract, c.f. for example Grosen and Jørgensen (2000).

[2] An early death is associated with a contract payoff which occurs prior to the specified date.

to more risk, because he has no idea whether future periodic premium payments will be forthcoming. Hence, the contracts contain both mortality and interest rate uncertainty.

Usually, the financial market and mortality risk are assumed to be independent, which allows a separate analysis of both uncertainties, in particular if the market is complete.[3] The mortality risk can be diversified by a continuum of contract policies. This is justified by the law of large numbers which states that the random maturity times can be replaced by deterministic numbers, i.e., the number of contracts which mature at each reference date is known with probability one. In addition, the financial market risk can be hedged perfectly by self–financing and duplicating trading strategies which are adjusted to the numbers of contracts which mature at each date.

In an incomplete market model, a separate analysis of financial market and mortality risk is no longer possible. Caused either by the financial market model and/or by a death distribution which changes over time stochastically, the market incompleteness makes it impossible to achieve a risk management strategy which exactly matches the liabilities. Therefore, it results in a non–zero hedging error with positive probabilities. In this paper, we consider market incompleteness which is caused by model misspecification associated with both the interest rate and mortality risk. In the analysis of pricing and hedging the risk exposure to the issued contracts, the insurance company makes model assumptions about the term structure of the interest rate and the death distribution. However, the contract fairness and the hedging effectiveness depend on the true interest rate dynamic and the true death distribution. Misspecification of the interest rate dynamic may lead to a hedging error associated with each strategy concerning the payout at one particular maturity date. Misspecification of the death distribution can be interpreted in the sense that the hedger assumes a wrong number of bonds concerning one particular maturity date.

It is worth emphasizing that independent of the concrete choice of term structure models, the true data generating interest rate process is reflected only partially. In addition to the problem of specifying the process class, there is an estimation problem concerning the process parameters. Furthermore, the insurance company faces the insurance typical risk. Death/survival probabilities are to be estimated from historical data. A particular estimation problem is implied by time–dependent probabilities.[4] In fact, it is realistic to assume that the mortality distribution even changes in a random way. Again, model risk is unavoidable, i.e. there is a deviation between true and assumed death/survival probabilities. The mortality misspecification can also be motivated by an intentional abuse of the insurer. For example, annuity providers often underestimate the survival probabilities deliberately. In this case, the (assumed) expected period of annuity payment is shortened. Consequently,

[3]The independence assumption is e.g. made in Aase and Persson (1994) and Nielsen and Sandmann (1995).

[4]Normally, it is assumed that there is a trend which reduces the death probabilities with respect to each age class. However, there are aspects which support the other way, too. For instance, a medical breakthrough or a catastrophe could increase/decrease life spans to a big extent. However, a (reasonable) factor which determines the trend of life expectancy is for example given in Wilmott (2006).

a higher annuity payment can be offered with the intention to acquire more customers.

Concerning the literature on model risk, there is an extensive analysis of financial market risk. Without postulating completeness, we refer to the papers of Avellaneda et al. (1995), Lyons (1995), Bergman et al. (1996), Dudenhausen et al. (1998), El Karoui et al. (1998), Hobson (1998), and Mahayni (2003). Certainly, there are also papers dealing with different scenarios of mortality risk and/or stochastic death distributions, for instance, Milevsky and Promislow (2001), Blake et al. (2004), Dahl (2004), Ballotta and Haberman (2006), and Gründl et al. (2006). A recent paper of Dahl and Møller (2006) considers the valuation and hedging problems of life insurance contracts when the mortality intensity is affected by some stochastic processes. However, to our knowledge, there are no papers which analyze the distribution of the hedging errors resulting from the combination of both. Therefore, the purpose of this paper is to analyze the effectiveness of risk management strategies stemming from the combination of diversification and hedging effects. In particular, it is interesting to look for a combination of diversification and hedging effects which is robust against model misspecification.

Neglecting model misspecification, the considered strategies are risk–minimizing. The concept of risk–minimizing is firstly introduced in Föllmer and Sondermann (1986) and applied to the context of insurance contracts in Møller (1998, 2001).[5] In contrast to Møller (1998, 2001), we do not assume a deterministic interest rate. Due to the long time to maturity of life insurance contracts, it is important that a meaningful risk management takes into account stochastic interest rates. However, the independence assumption of interest rate and mortality risk implies that Møller's (1998) results which concern the structure of the hedging strategies can be adopted here. Intuitively, the resulting risk–minimizing strategy can be explained as follows. Without the uncertainty about the random times of death, the cash flow of the benefits and contributions is deterministic. In particular, the benefits can be hedged perfectly by long–positions in bonds with matching maturities. Therefore, the most natural hedging instruments are given by the corresponding set of zero coupon bonds. Apparently, a strategy containing the entire term structure is an ideal case. Because of liquidity constraints in general or transaction costs in particular, it is not possible or convenient for the hedger to trade in all the bonds.[6] Therefore, we also consider the case that the set of hedging instruments is restricted, i.e. that it is only possible to hedge in a subset of bonds. In a complete financial market model, the unavailable bonds can be synthesized by using self–financing strategies such that the composition of risk–minimizing hedging strategies under mortality risk is straightforward. Thus, the main focus of the paper is not on the determination of the risk–minimizing hedging strategies for the endowment assurance with respect to one particular model, but on the implications of model risk to the effectiveness of these strategies.

[5]Møller (1998) applies this to the context of equity–linked life insurance and derives risk–minimizing hedging strategies for different equity–linked life insurance contracts. While Møller (2001) considers more general equity–linked life insurance contracts with payments incurring at random times within the term of the contract.

[6]For instance, there are trading constraints in the sense that not all zero coupons (maturities of zero coupon bonds) are traded at the financial market.

In order to initialize the above strategies, the insurer needs an amount corresponding to the initial contract value, while he only obtains the first periodic premium at the beginning. Therefore, a credit corresponding to the (assumed) expected discounted value of the delayed periodic premiums should be taken by the insurer, because the initial contract value equals the (assumed) present value of the entire periodic premiums. The insurance company trades with a simple selling strategy to pay back this loan. Apparently, the effectiveness of this strategy in the liability side depends on the model risk too.

It turns out that, independent of the model risk associated with the interest rates, an overestimation of the death probability yields a superhedge in the mean, i.e. the hedger is on the safe side on average. In the case that there is no misspecification with respect to the mortality risk, the model risk concerning the interest rate has no impact on the mean of the hedging error. In contrast, the effect of interest rate misspecification on the variance is crucial, in particular if the set of hedging instruments is restricted. In the case that there is no misspecification with respect to the interest rate dynamic, all strategies considered lead to the same variance level, independent of the mortality. Therefore, the interactivity of both sources of model risk is found to have a pronounced effect on the risk management of the insurer.

The remaining of the paper is organized as follows. Section 2 states the basic features of the insurance contract considered. In addition, some examples of commonly used model assumptions are given. Neglecting model risk, we also give a representation of fair contract specifications. Section 3 introduces the hedging problem and fixes some definitions which are needed for the analysis. In Section 4, we analyze hedging strategies consisting of a subset of zero coupon bonds and their cost processes under model risk. Mainly, we discuss the distribution of the hedging errors. Section 5 illustrates some numerical results for the cost distributions under different scenarios of model misspecification. Section 6 concludes the paper.

2. Product and Model Description

2.1. **Contract Specification.** We consider an endowment assurance product with periodic premiums A. In the following, $\underline{\underline{T}} = \{t_0, \ldots, t_{N-1}, t_N\}$ denotes a discrete set of equidistant reference dates where $\Delta t = t_{i+1} - t_i$ gives the distance between two reference dates. The insured pays, as long as he lives, a constant periodic premium A until the last reference date t_{N-1}.[7] In particular, if τ^x denotes the random time of death of a live aged x, then the last premium is due at the random time t_s where $s := \min\{N-1, n^*(\tau^x)\}$ and $n^*(t) := \max\{j \in \mathbb{N}_0 | t_j < t\}$. The insured receives his payoff at the next reference date after his last premium payment, i.e. he receives his payoff at random time $T := \min\left\{t_N, t_{n^*(\tau^x)+1}\right\}$. We denote the endowment part of the contract specification by h and assume that the insured receives at time T the higher amount of h and an insurance account G_T which depends on his paid premiums. Let $\bar{G}_T$ denote the payoff at T, then

$$\bar{G}_T := \max\{h, G_T\}.$$

[7]One can think of t_N as the customer's "retirement time" when his duty of premium payments terminates. In the simplest form, the accumulated funds are paid out as a lump sum.

Notice that the contract specification implies that the benefits and contributions depend on the time of death τ^x. In the case that $G_T = 0$, we have a simple endowment contract which always pays out h amount no matter how the death time of the customer evolves. In particular, the insurance knows exactly its amount of liability but does not know when it is due.[8] In contrast to the simple endowment contract, we consider contracts which also give a nominal capital guarantee, i.e., the insured gets back his paid premiums accrued with an interest rate g ($g \geq 0$), i.e. we use the following convention

$$\begin{aligned} \tilde{A}_{t_i} &:= \sum_{j=0}^{i} A e^{g(t_i - t_j)},\ i = 0, 1, \ldots, N-1 \\ \text{and } G_{t_i} &:= \tilde{A}_{t_{i-1}} e^{g(t_i - t_{i-1})},\ i = 1, \ldots, N. \end{aligned}$$

Concerning the above contract specification, several comments are helpful. Since the insurance account is monotonically increasing in the guaranteed rate, we only consider contracts where $G_{t_1} < h < G_{t_N}$.[9] It turns out that this condition $h < G_{t_N}$ restricts the set of fair parameter constellations (g^*, h^*), because under our contract specification, small guarantee values could lead to some h^*–values which are much higher than G_T. However, this problem is unlikely to appear when insurance products incorporate additional options, c.f. footnote 1. These additional options reduce the value of the resulting fair parameter h^* to a big extent.

To sum up the contract specification, it is convenient to notice that

$$\bar{G}_T = \sum_{i=0}^{N-1} \bar{G}_{t_{i+1}}\ 1_{\{t_i < \tau^x \leq t_{i+1}\}} + \bar{G}_{t_N}\ 1_{\{\tau^x > t_N\}}. \tag{1}$$

Equation (1) gives two basic death scenarios. One is given by an *early* death, i.e. a death during the interval $]t_{i-1}, t_i]$ $(i = 1, \ldots, N-1)$. The other refers to the surviving of the last premium date t_{N-1} where, in contrast to the first case, the insured pays all premiums. This implies that a death which occurs in the interval $]t_{N-1}, t_N]$ is not an early death in the technical sense of the insurance contract. A product example is given in Table 1. Obviously, the fair contract value (implicitly determined by the fair combination of G and h) crucially depends on the probabilities of death events and the term structure of the interest rate.

2.2. **Fair contract specification.** Now, we consider the question how to specify a *fair* contract, i.e., how to specify the *fair* contract parameters h^* and g^* for a given periodic premium A. The so–called equivalence principle states that a contract is fair if the present value of the contributions is equal to the present value of the benefits.

[8]Timing risk is also an interesting subject in a context different to the one given here, c.f. for example Korn (2006) and the literature given in this paper.

[9]Usually, the interest rate guarantee g is smaller than the instantaneous risk free rate of interest in the contract–issuing time, in particular, if the term structure is normal. Intuitively, as a compensation, a relative high h (at least higher than G_{t_1}) is provided as an endowment. The case $h \geq G_{t_N}$ implies $\max\{h, G_T\} = h$ such that the asymmetry which is introduced by the maximum operator vanishes.

Product Example

i	G_{t_i}	h	$\bar{G}_{t_i}$	$P\left(\tau^x \in \left]t_{i-1}, t_i\right]\right)$	$D(t_0, t_{i+1})$
1	525.6	20 673.6	20 673.6	0.00178031	0.949742
2	1078.2	20 673.6	20 673.6	0.00190781	0.899889
...					
...					
22	20 546.9	20 673.6	20 673.6	0.00947623	0.289887
23	22 126.0	20 673.6	22 126.0	0.01029050	0.274033
24	23 786.0	20 673.6	23 786.0	0.01116730	0.259051
...					
...					
≥ 30	35 694.6	20 673.6	35 694.6	0.789179	0.184932

TABLE 1. Insurance account G and death dependent payoff $\bar{G}$ for an insurance contract with maturity in $t_N = 30$ years, guaranteed rate $g = 0.05$ and $h = 20673.6$ and a life aged $x = 40$. In particular, the parameter constellation is summarized in Table 2.

The present value of the contributions of the customer is given by the discounted expected value, i.e.,

$$A \sum_{i=0}^{N-1} D(t_0, t_i)\ {}_{t_i}\tilde{p}_x$$

where $D(t_0, t_i)$ $(i = 1, \ldots, N)$ denotes the current (observable) market price of a zero coupon bond with maturity t_i and ${}_{t_i}\tilde{p}_x$ denotes the probability that a life aged x survives t_i (given that he has survived t_0).[10] Thus, if C_{t_0} denotes the contract value at the initialization date t_0, fair combinations of (g^*, h^*) result from the equality

$$C_{t_0} = A \sum_{i=0}^{N-1} D(t_0, t_i)\ {}_{t_i}\tilde{p}_x. \tag{2}$$

PROPOSITION 2.1 (Contract value). *Let h be a constant such that there exists a $k \in \{1, \ldots, N-1\}$ with $G_{t_k} < h \leq G_{t_{k+1}}$, then, in a complete arbitrage free market, the present value of the benefit is given by*

$$C_{t_0} = \sum_{i=0}^{k-1} h\ D(t_0, t_{i+1})\ {}_{t_i|t_{i+1}}\tilde{q}_x + \sum_{i=k}^{N-1} G_{t_{i+1}}\ D(t_0, t_{i+1})\ {}_{t_i|t_{i+1}}\tilde{q}_x + G_{t_N}\ D(t_0, t_N)\ {}_{t_N}\tilde{p}_x.$$

[10]We use conventional notations in life insurance mathematics, i.e.,

$${}_t\tilde{p}_x := P\left(\tau^x > t\right); \qquad {}_t\tilde{q}_x := P\left(\tau^x \leq t\right); \qquad {}_{u|t}\tilde{q}_x := P\left(u < \tau^x \leq t\right),$$

with ${}_t\tilde{p}_x$ denoting the probability of an x–aged life surviving time t, ${}_t\tilde{q}_x$ the probability of an x–aged life dying before time t and ${}_{u|t}\tilde{q}_x$ the probability that he dies between u and t. In addition, we use

$${}_t\tilde{p}_{x+v} := P\left(\tau^x > t | \tau^x > v\right); \qquad {}_{u|t}\tilde{q}_{x+v} := P\left(u < \tau^x \leq t | \tau^x > v\right),$$

to denote the corresponding conditional survival/death probabilities, i.e., given that he has survived time v. Obviously for $v \geq t$, it holds that ${}_t\tilde{p}_{x+v} = 1$ and ${}_{u|t}\tilde{q}_{x+v} = 0$.

Fair parameter combinations (g^*, h^*)

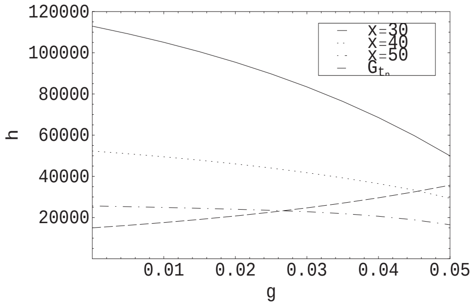

FIGURE 1. Fair parameter combinations for a contract as given in Table 2. In particular, the spot rate volatility is 0.02.

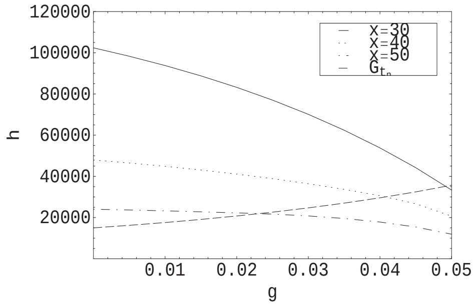

FIGURE 2. Fair parameter combinations for a contract as given in Table 2 and a spot rate volatility of 0.03 instead of 0.02.

PROOF: According to Equation (1), the present value in the sense of the expected discounted value of $\bar{G}_T$ is given by

$$E_P\left[e^{-\int_0^T r_u du}\bar{G}_T\right] = \sum_{i=0}^{N-1} \max\{G_{t_{i+1}}, h\}\ D(t_0, t_{i+1})\ {}_{t_i|t_{i+1}}\tilde{q}_x + G_{t_N}\ D(t_0, t_N)\ {}_{t_N}\tilde{p}_x.$$

□

The above proposition and its proof are based on the assumption that the mortality risk can be fully diversified among the insurance takers.[11]

COROLLARY 2.2.

$$h^*(g) = \frac{A\sum_{i=0}^{N-1} D(t_0, t_i)\ {}_{t_i}\tilde{p}_x - \sum_{i=k}^{N-1} G_{t_{i+1}}\ D(t_0, t_{i+1})\ {}_{t_i|t_{i+1}}\tilde{q}_x - G_{t_N}\ D(t_0, t_N)\ {}_{t_N}\tilde{p}_x}{\sum_{i=0}^{k-1} D(t_0, t_{i+1})\ {}_{t_i|t_{i+1}}\tilde{q}_x}.$$

PROOF: This corollary results from Proposition 2.1 and Equation (2) straightforwardly.

□

Notice that h is a decreasing function of g in view of fair contract analysis. As g goes up, G_T increases and so does $\bar{G}_T$. A rise in h leads to an increase in $\bar{G}_T$ as well. I.e., the customer of such a contract benefits from both a higher h and a higher g.

2.3. **Example.** Recall that it is not necessary to specify a term structure model if one assumes that the relevant bond prices are given by market data. However, to avoid the summary of all prices with respect to the long contract maturities, the

[11]An insurance risk is completely diversifiable (full diversification), if the law of large numbers can be applied. In this case, the random time of death can be replaced by deterministic numbers, the expected number of death or survival, i.e. the insurer can predict how many contracts become due at t_i, $i = 1, \cdots, N$. It's a usual and acceptable assumption in life insurance.

Benchmark parameter

contract parameter	interest rate parameter (Vasiček model)	mortality parameter (Makeham)
$g = 0.05$	initial spot rate = 0.05	
$h = 20673.6$ ($G_{t_N} = 35694.6$)	spot rate volatility = 0.03	H=0.0005075787
$t_N = 30$ (years)	speed of mean reversion = 0.18	K=0.000039342435
x=40, A=500	long run mean = 0.07	c=1.10291509

Table 2. Basic (assumed) model parameter.

following examples are given according to a term structure which fits to a Vasiček–model with a parameter constellation summarized in Table 2.[12] As an example for the death distribution, the insurer might use the death distribution according to Makeham where

$$
\begin{aligned}
{}_t\tilde{p}_x &= \exp\left\{-\int_0^t \mu_{x+s}\, ds\right\}, \\
\mu_{x+t} &:= H + Kc^{x+t}.
\end{aligned} \tag{3}
$$

As a benchmark case, we use a parameter constellation along the lines of Delbaen (1990) which is given in Table 2. Intuitively, it is clear that a very high h–value should be offered to the customer if the offered minimum rate of interest rate is much lower than the spot rate. This indicates that probably an h–value smaller than G_{t_N} would not give a fair contract. This phenomenon can be observed in Figures 1 and 2. For the small values of g, the fair values of h lie mostly above the G_{t_N} curve. However, as already mentioned, an additional bonus payment reduces the fair h–value substantially. Therefore, the interesting case here is that the issued contract offers a minimum interest rate guarantee (slightly) smaller than (or equal to) the instantaneous risk free rate of interest at the contract–issuing date, but as a compensation, that a minimum amount of money (h) will be guaranteed to the customer if an early death occurs. Finally, notice that an increase in the spot rate volatility leads to a rise in the price of zero coupon bonds. Consequently, this results in a lower fair value for h, i.e. a little more intersection areas between G_{t_N} and fair–h–curves are observed in the case of $\sigma = 0.03$ illustrated in Figure 2 than in the case of $\sigma = 0.02$, i.e. Figure 1.

3. Hedging

In an incomplete financial market, no perfect hedging can be achieved, i.e., no self–financing hedging strategies whose final values duplicate the contingent claim can be found. The deviation from the self–financing property can be described by a continuous–time rebalancing cost process. The deviation of the final portfolio value from the contingent claim is called duplication error. As a preparation for the next section, this section explains these terminologies resulting from the hedging in an

[12]The Vasiček–model implies that the volatility $\sigma_{\bar{t}}(t)$ of a zero coupon bond with maturity $\bar{t}$ is $\sigma_{\bar{t}}(t) = \frac{\bar{\sigma}}{\kappa}(1 - \exp\{-\kappa(\bar{t} - t)\})$ where κ and $\bar{\sigma}$ are non–negative parameters. $\bar{\sigma}$ is the volatility of the short rate and κ the speed factor of mean reversion.

incomplete market.

All the stochastic processes we consider are defined on an underlying stochastic basis $(\Omega, \mathcal{F}, \mathcal{F} = (\mathcal{F}_t)_{t\in[0,T^*]}, P)$ which satisfies the usual hypotheses. Trading terminates at time $T^* > 0$. We assume that the price processes of underlying assets are described by strictly positive, continuous semimartingales. By a contingent claim C with maturity $T \in [0, T^*]$, we simply mean a random payoff received at time T, which is described by the $\mathcal{F}_T$–measurable random variable C.

Definition 3.1 (Trading strategy, value process, duplication). Let $D^{(1)}, \ldots, D^{(N)}$ denote the price processes of underlying assets. A trading strategy ϕ in these assets is given by a $\mathbb{R}^N$–valued, predictable process which is integrable with respect to D. The value process $V(\phi)$ associated with ϕ is defined by

$$V_t(\phi) = \sum_{i=1}^{N} \phi_t^{(i)} D_t^{(i)}.$$

If C is a contingent claim with maturity T, then ϕ duplicates C iff

$$V_T(\phi) = C, \quad P\text{–a.s..}$$

The deviation of the terminal value of the strategy from the payoff is called duplication cost L^{dup}, i.e.,

$$L_T^{\text{dup}} := C - V_T(\phi).$$

Definition 3.2 (Gain process). If ϕ is a trading strategy in the assets $D^{(1)}, \ldots, D^{(N)}$, the gain process $(I_t(\phi))_{t\in[0,T]}$ associated with ϕ is defined as follows:

$$I_t(\phi) := \sum_{i=1}^{N} \int_0^t \phi_u^{(i)}\, dD_u^{(i)}.$$

Definition 3.3 (Rebalancing cost process). If ϕ is a trading strategy, the cost process $L^{reb}(\phi)$ associated with ϕ is defined as follows:

$$L_t^{reb}(\phi) := V_t(\phi) - V_0(\phi) - I_t(\phi).$$

By this definition, the rebalancing costs at two different trading dates are equally weighted when the costs are due. Notice that the above definition of the cost process neglects when the costs are due, i.e., rebalancing costs at two different trading dates are equally weighted. In order to take account of this, a numeraire is used, i.e., all the rebalancing costs are measured in terms of one reference date. Unless mentioned otherwise, we use the money account as numeraire and denote the discounted versions of D, V, L^{reb} and L^{dup} with a superscript *, e.g $D_t^* = e^{-\int_0^t r_u\, du}\, D_t$.

Definition 3.4 ((Discounted) Total Cost). The (discounted) total costs of a hedging strategy ϕ for a claim C are described as the sum of (discounted) rebalancing and duplication cost.

$$L_t^{tot}(\phi) \;=\; L_t^{reb}(\phi) + L_t^{dup}(\phi), \qquad L_t^{tot,*}(\phi) = L_t^{reb,*}(\phi) + L_t^{dup,*}(\phi).$$

Definition 3.5 (Super– and Subhedge). A hedging strategy ϕ for the claim C is called superhedge (subhedge) iff $L_t^{tot}(\phi) \leq 0$ ($L_t^{tot}(\phi) \geq 0$) for all $t \in [0,T]$. In particular, a strategy which is a superhedge and a subhedge at the same time is called perfect hedge.

It is noticed that super– and subhedge in the mean can be defined similarly, when the expectation of the total cost is considered. A strategy which is super– and subhedge in the mean at the same time is called mean–self–financing.

Lemma 3.6. *The total hedging costs* L_T^{tot} *and* $L_T^{tot,*}$ *are given by*

$$L_T^{tot}(\phi) = C_T - (V_0(\phi) + I_T(\phi)), \qquad L_T^{tot,*}(\phi) = C_T^* - (V_0^*(\phi) + I_T^*(\phi))$$

Proof: According to the above definitions we have

$$L_T^{tot} = L_T^{reb} + L_t^{dup} = V_T - (V_0 + I_T) + C_T - V_T = C_T - (V_0 + I_T)$$

□

4. Hedging with subsets of Bonds

In our context, there are two sources of market incompleteness. First, the insurance risk is a non–tradable risk. It cannot be hedged away by trading on the financial market and can only be reduced by diversification. Hence, with respect to one single contract, the relevant hedging strategy cannot be perfect. Second, it can be caused by model misspecification. Model misspecification includes the possibility of a wrong choice of the stochastic processes which describe the dynamic of the zero coupon bonds as well as the possibility that the hedger assumes a death distribution which deviates from the true one. Besides, the random death time can be reinterpreted as the real maturity of the insurance contract. This implies that even a hedge which is a perfect hedge under full diversification, i.e., when the random time of death can be replaced by deterministic numbers, gives a deviation between the value of the hedging strategy and the payoff of the insurance contract at the maturity.

The hedging possibility and effectiveness of a claim depend on the set of available hedging instruments. Hedging is easy if the hedging instrument coincides with the claim to be hedged, i.e. its payoff is given by a random variable which is indistinguishable from the one which represents the claim. However, this is not the case in our context. With respect to the insurance contract under consideration, the most natural hedging instruments are given by the set of zero coupon bonds with maturities $t_1, \ldots, t_N$, i.e., by the set $\{D(.,t_1), \ldots, D(.,t_N)\}$.[13] Thus, we consider the set Φ of hedging strategies which consist of these bonds, i.e.,

$$\Phi = \left\{ \phi = (\phi^{(1)}, \ldots, \phi^{(N)}) \,\middle|\, \phi \text{ is trading strategy with } V(\phi) = \sum_{j=1}^{N} \phi^{(i)} D(.,t_i) \right\}$$

However, due to liquidity constraints in general or transaction costs in particular, it is not possible or convenient to use all bonds for the hedging purpose. This is modelled in the following by restricting the class of strategies Φ. The relevant subset

[13] This is motivated by the contract value given in Proposition 2.1.

is denoted by $\Psi \subset \Phi$. Obviously, independent of the optimality criterion which is used to construct the hedging strategy, the effectiveness of the optimal strategy $\psi^* \in \Psi$ can be improved if there are additional hedging instruments available. To simplify the exposition, we propose that the assumed interest rate dynamic is given by a one–factor term structure model and set

$$\Psi = \left\{ \psi \in \Phi \,\middle|\, \psi = (0, \ldots, 0, \psi^{(N-1)}, \psi^{(N)}) \right\}.$$

Two comments are necessary. First, the assumption of a one–factor term structure model implies that two bonds are enough to synthesize any bond with maturity $\{t_1, \ldots, t_N\}$. However, the following discussion can easily be extended to a multi–factor term structure model. Second, as the bonds cease to exist as time goes by, it is simply convenient to use the two bonds with the longest time to maturity.[14]

Apparently, a certain criterion should be imposed on the hedging strategies. The first criterion we come up with is that the considered trading strategies should be mean–self–financing if no model risk exists. However, it is worth mentioning that the mean–self–financing feature is not enough to give a meaningful strategy. This is reasoned by the following proposition:

PROPOSITION 4.1. *For $\phi \in \Phi$ and a claim with payoff $C_T = \bar{G}_T$ at the random time $T = \min\left\{t_N, t_{n^*(\tau^x)+1}\right\}$, it holds*

$$E_{P^*}\left[L_T^{tot,*}(\phi)\right] = C_{t_0} - V_{t_0}(\phi)$$

where C_{t_0} is given as in Proposition 2.1.

PROOF: Due to the fact that T is bounded above by t_N and that C^* and I^* are P^*–martingales, Lemma 3.6 combined with optional stopping theorem leads to [15]

$$E_{P^*}\left[L_T^{tot,*}(\phi)\right] = E_{P^*}[C_T^*] - (V_0^*(\phi) + E^{P^*}[I_T^*(\phi)]) = C_{t_0}^* - V_{t_0}^*(\phi).$$

□

The above proposition states that any strategy where the initial investment coincides with the price of the claim to be hedged is self–financing in the mean. Therefore, it is necessary to use an additional hedging criterion. In the following, we consider a conventional hedging criterion used in the incomplete market, i.e., the considered hedging strategies are risk–minimizing if model risk is neglected. First of all, if a strategy is risk–minimizing, it is mean–self–financing. Therefore, risk–minimizing feature contains mean–self–financing feature. In the analysis of risk–minimizing hedging, we look for an admissible strategy which minimizes the variance of the future costs at any time $t \in [0, T]$. Along the lines of Møller (1998), we derive the risk–minimizing hedging strategy for both case: when the entire term structure or when only the last two zero bonds are used. They are simply denoted by ϕ and ψ respectively. The motivation and derivation of the hedging strategies is based on the value process of the claim to be hedged.

[14]It might be more practical to use two hedging instruments which differ much from each other, e.g., two bonds whose maturities are not very close, like t_1 and t_N–bond. However, which two bonds to choose will not be discussed here. Those who are interested in this topic please refer to Dudenhausen and Schlögl (2002).

[15]In the above context, the martingale measure coincides with the real world measure P.

Proposition 4.2 (Value Process). *In our arbitrage–free model setup, the contract value at time $t \in [0, \tau^x]$ is given by*

$$C_t = \left[\sum_{j=n^*(t)+1}^{N-1} \bar{G}_{t_j}\ D(t,t_j)\ {}_{t_{j-1}|t_j}\tilde{q}_{x+t} + \bar{G}_{t_N}\ D(t,t_N)\ \underbrace{\left({}_{t_{N-1}|t_N}\tilde{q}_{x+t} + {}_{t_N}\tilde{p}_{x+t}\right)}_{={}_{t_{N-1}}\tilde{p}_{x+t}} \right].$$

Proof: Using standard theory of pricing by no arbitrage implies that the contract value at t ($0 \leq t < T$) is given by the expected discounted payoff under the martingale measure P^*, i.e.,

$$C_t = E_{P^*}[e^{-\int_t^T r_u du}\bar{G}_T|\mathcal{F}_t].$$

In particular, the above proposition is a straightforward generalization of Proposition 2.1.

□

The above proposition immediately motivates a duplication strategy on the set $\{t \leq \tau^x\}$. Prior to the death time τ^x, the contract value (at time t) can be synthesized by a trading strategy which consists of bonds with maturities t_i ($i = n^*(t)+1, \ldots, N$). Assuming that the insurance company will not learn the death of the customer until no further premiums are paid by the insured implies that the strategy proceeds on the set $t \in]\tau^x, T]$ in the same way as on the set $t \in [0, \tau^x]$. Notice that the number of available instruments, i.e. the number of bonds, decreases as time goes by. At time t, only bonds with maturities later than $n^*(t)$ are traded, i.e., the hedger buys $\bar{G}_{t_i} \cdot {}_{t_{i-1}|t_i}\tilde{q}_x$ units of $D(t,t_i)$ and $\bar{G}_{t_N}\,{}_{t_{N-1}}\tilde{p}_{x+t}$ units of $D(t,t_N)$. The advantage of using this strategy is that the strategy itself is not dependent of the model assumptions of the interest rate.

Proposition 4.3. *Let $\phi \in \Phi$ denote a risk– (variance–) minimizing trading strategy with respect to the set of trading strategies Φ. Assume that the insurance company notices the death of the customer only when no further premium is paid by the insured. If one additionally restricts the set of admissible strategies to the ones which are independent of the term structure, then it holds: ϕ is uniquely determined and for $t \in [0,T]$*

$$\begin{aligned} \phi_t^{(i)} &= 1_{\{t \leq t_i\}}\,\bar{G}_{t_i}\ {}_{t_{i-1}|t_i}\tilde{q}_{x+t} \qquad i = 1, \cdots, N-1 \\ \phi_t^{(N)} &= \bar{G}_{t_N}\ {}_{t_{N-1}}\tilde{p}_{x+t}. \end{aligned}$$

Proof: Without the introduction of model risk it is easily seen that V_{t_0} and the contract value C_{t_0} according to Proposition 4.2 coincide. Thus, with Proposition 4.2 it follows that ϕ is self–financing in the mean. Since the stochastic interest rate risk can be eliminated by trading in all "natural" zero coupon bonds, Møller's (1998) results concerning the independence of mortality and market risk can be adopted here. Since an endowment insurance is a mixture of pure endowment and term insurance, the results immediately follow from Theorem 4.4 and Theorem 4.9 of Møller (1998).

□

A one–factor short rate model is complete in two bonds, i.e. the availability of two bonds with different maturities is enough to synthesize any further bond. Therefore, without postulating the independence from the interest rate model, the variance–minimizing strategy is not defined uniquely.

PROPOSITION 4.4. *Let ψ denote the risk– (variance–) minimizing trading strategy with respect to the set of trading strategies $\Psi \subset \Phi$. Assuming that the insurance company notices the death of the customer only when no further premiums are paid by the insured implies that for $t \in [0,T]$*

$$\begin{aligned}
\psi_t^{(N-1)} &= 1_{\{\tau^x \geq t\}} \left(1_{\{t \leq t_{N-2}\}} \sum_{i=n^*(t)+1}^{N-2} \bar{G}_{t_i\ t_{i-1}|t_i}\tilde{q}_{x+t} \frac{D(t,t_i)}{D(t,t_{N-1})} \lambda_1^{(i)}(t) \right. \\
&\quad \left. + 1_{\{t \leq t_{N-1}\}} \bar{G}_{t_{N-1}\ t_{N-2}|t_{N-1}}\tilde{q}_{x+t} \right) \\
\psi_t^{(N)} &= 1_{\{\tau^x \geq t\}} \left(1_{\{t \leq t_{N-2}\}} \sum_{i=n^*(t)+1}^{N-2} \bar{G}_{t_i\ t_{i-1}|t_i}\tilde{q}_{x+t} \frac{D(t,t_i)}{D(t,t_N)} \lambda_2^{(i)}(t) \right. \\
&\quad \left. + \bar{G}_{t_N} \left({}_{t_{N-1}|t_N}\tilde{q}_{x+t} + {}_{t_N}\tilde{p}_{x+t} \right) \right) \\
\textit{where } \lambda_1^{(i)}(t) &:= \frac{\tilde{\sigma}_{t_i}(t) - \tilde{\sigma}_{t_N}(t)}{\tilde{\sigma}_{t_{N-1}}(t) - \tilde{\sigma}_{t_N}(t)} \ \textit{and } \lambda_2^{(i)}(t) = \frac{\tilde{\sigma}_{t_{N-1}}(t) - \tilde{\sigma}_{t_i}(t)}{\tilde{\sigma}_{t_{N-1}}(t) - \tilde{\sigma}_{t_N}(t)}
\end{aligned}$$

with $\tilde{\sigma}_{\bar{t}}(t)$ denoting the assumed volatility of a zero coupon bond with maturity date $\bar{t}$ at time t.

PROOF: Notice that, in the setup of a one-factor short rate model, there is a self–financing strategy $\tilde{\phi}^{(i)} = \left(\alpha^{(i)}, \beta^{(i)}\right)$ with value process $V_t\left(\tilde{\phi}^{(i)}\right) = \alpha_t^{(i)} D(t,t_{N-1}) + \beta^{(i)} D(t,t_N) = D(t,t_i)$ for $i = 1,\ldots,N$. With Proposition A.1 of Appendix A, one immediately can write down the strategy for $D(.,t_i)$, i.e.,

$$\alpha_t^{(i)} = \frac{D(t,t_i)}{D(t,t_{N-1})}\lambda_1^{(i)}(t), \quad \beta_t^{(i)} = \frac{D(t,t_i)}{D(t,t_N)}\lambda_2^{(i)}(t)$$

where $\lambda_1^{(i)}(t)$ and $\lambda_2^{(i)}(t)$ are given as above. Notice that $V_t\left(\tilde{\phi}^{(i)}\right) = D(t,t_i)$ P–almost surely implies $Var[L_T^*(\psi)] = Var[L_T^*(\phi)]$ (alternatively, this can be deducted from Proposition 4.6). This together with $\Psi \subset \Phi$ ends the proof.

The above proposition states that ψ corresponds to the strategy which is defined along the lines of Proposition 4.3 where the hedging instruments $D(.,t_1),\ldots,D(.,t_{N-2})$ are synthesized by the traded zero bonds $D(.,t_{N-1})$ and $D(.,t_N)$. Obviously, the strategy depends on the term structure model. Basically, by using a one–factor interest model, the risk–minimizing strategy for the insurance contract can be implemented in any subset of bonds with at least two elements. A generalization is straightforward if a hedging instrument is added for every dimension of risk factor which is introduced to the short rate model.

Taking into account a high degree of model risk, it is in particular necessary to distinguish between true and assumed death and survival probabilities. Throughout this paper, we put a tilde to denote expressions which are only assumed by the insurance company and do not necessarily correspond to the true parameters which are denoted without a tilde.

Just because of the existence of model risk, an extra cost from the liability side is not negligible in addition to the total cost (under model misspecification) from the asset side. It is noticed that the implementation of the above strategies is based on taking a credit at t_0. Since the initial value of the hedging strategies is given by the expected value of the premium inflows, the insurer must in fact borrow the amount $\sum_{i=1}^{N-1} A\, {}_{t_i}\tilde{p}_x\, D(t_0,t_i)$. The underpinning strategy for this is to sell $A\, {}_{t_i}\tilde{p}_x$ bonds with maturity t_i $(i=1,\ldots,t_{N-1})$. Under mortality risk, it is not necessarily the case that the insurer achieves exactly the number of periodic premiums which are necessary to pay back the credit. These discrepancies lead to extra costs. In particular, these costs can be understood as a sequence of cash flows, i.e., the insurer has to pay back $A\, {}_{t_i}\tilde{p}_x$ at each time t_i $(i=1,\ldots,t_{N-1})$, i.e. independent of whether the insured survives. Therefore, the additional discounted costs $L_T^{add,*}$ associated with the above borrowing strategy are given by

$$L_T^{add,*} = \sum_{i=1}^{N-1} e^{-\int_0^{t_i} r_u\,du} A\left({}_{t_i}\tilde{p}_x - 1_{\{\tau^x > t_i\}}\right). \tag{4}$$

Proposition 4.5 (Expected total discounted hedging costs). *Let L_T^* denote the discounted total costs from both, the asset and the liability side, i.e. $L_T^* = L_T^{tot,*} + L_T^{add,*}$. ϕ (ψ) denotes the strategy given in Proposition 4.3 (4.4). For $w \in \{\phi,\psi\}$ it holds (under model risk)*

$$E_{P^*}[L_T^*(w)] = E_{P^*}[L_T^{tot,*}(w)] + E_{P^*}[L_T^{add,*}(w)]$$

where

$$E_{P^*}[L_T^{tot,*}(w)] = D(t_0,t_N)\bar{G}_{t_N}({}_{t_N}p_x - {}_{t_N}\tilde{p}_x) + \sum_{j=1}^{N-1}({}_{t_{j-1}|t_j}q_x - {}_{t_{j-1}|t_j}\tilde{q}_x)D(t_0,t_j)\bar{G}_{t_j}$$

and $$E_{P^*}[L_T^{add,*}(w)] = \sum_{i=1}^{N-1} D(t_0,t_i)A\left({}_{t_i}\tilde{p}_x - {}_{t_i}p_x\right).$$

Proof: This proposition is an immediate consequence of Propositions 4.1 and 4.2, in addition to taking the expectation of the addition cost term given in Equation (4). □

Notice that, independent of the set of bonds, the expected costs are the same. Furthermore, independent of the model risk related to the interest rate, mortality misspecification determines the sign of the expected value, i.e., that decides when a superhedge in the mean can be achieved. When no mortality misspecification is available, the model risk related to the interest rate has no impact on the expected value. When there exists mortality misspecification, the model risk related to the interest rate will influence the size of the expected value. Therefore, the effect of model risk associated with the interest rate depends on the mortality misspecification. However, when it comes to the analysis of the variance, model risk associated with the interest rate has a more pronounced effect than mortality misspecification.

PROPOSITION 4.6 (Additional variance). *It holds*

$$
\begin{array}{lrcl}
(i) & Var_{P^*}[L_T^{tot,*}(\psi)] & = & Var_{P^*}[L_T^{tot,*}(\phi)] + AV_T \\
(ii) & Var_{P^*}[L_T^*(\psi)] & = & Var_{P^*}[L_T^*(\phi)] + AV_T
\end{array}
$$

with $AV_T = 0$ when there exists no model risk related to the interest rate, otherwise

$$AV_T = {}_{t_N}p_x\, E_{P^*}\left[(I^*_{t_N}(\psi) - I^*_{t_N}(\phi))^2\right] + \sum_{j=0}^{N-1} {}_{t_j|t_{j+1}}q_x\, E_{P^*}\left[(I^*_{t_{j+1}}(\psi) - I^*_{t_{j+1}}(\phi))^2\right] > 0.$$

PROOF: The proof is given in Appendix B.

□

It should be emphasized that the effect of mortality misspecification depends on the model risk related to the interest rate. If there exists no interest rate misspecification, mortality misspecification plays no role in the additional variance. However, if there exists model risk related to the interest rate, an additional variance part results always when the restricted subset of zero coupon bonds are used as hedging instruments.

As stated in the introduction, mortality misspecification can be caused by a deliberate use of the insurance company for certain purposes, e.g. safety reasons. I.e., a deviation of the assumed mortality from the true one is generated by a shift in the parameter x which leads to a shift in the life expectancy. For this purpose, we let ${}_tp_{\tilde{x}}$ and ${}_tq_{\tilde{x}}$ denote the assumed probabilities ${}_t\tilde{p}_x$ and ${}_t\tilde{q}_x$.[16]

PROPOSITION 4.7. *For any realistic death/survival probability which satisfies*

$$\frac{\partial\, {}_tp_x}{\partial x} < 0, \text{ and } \frac{\partial_{u|t}q_{x+v}}{\partial x} > 0, \; v \leq u < t,$$

we obtain that

(i) $\frac{\partial E_{P^*}[L_T^*]}{\partial \tilde{x}} < 0$. *Furthermore, an overestimation of the death probability (an underestimation of the survival probability) leads to a superhedge in the mean, i.e.,* $E_{P^*}[L_T^*] \leq 0$.
(ii) The additional variance given in Proposition 4.6 is increasing in $\tilde{x}$.

PROOF: (i) It holds

$$
\begin{aligned}
\frac{\partial E_{P^*}[L_T^*]}{\partial \tilde{x}} &= \frac{\partial E_{P^*}[L_T^{tot,*}]}{\partial \tilde{x}} + \frac{\partial E_{P^*}[L_T^{add,*}]}{\partial \tilde{x}}; \\
\frac{\partial E_{P^*}[L_T^{add,*}]}{\partial \tilde{x}} &= \sum_{i=0}^{N-1} D(t_0, t_i) A \frac{\partial\, {}_{t_i}p_{\tilde{x}}}{\partial \tilde{x}} < 0.
\end{aligned}
$$

[16] Since we want to obtain some general results, we make the sensitivity analysis with respect to $\tilde{x}$. If a specific death/survival distribution is used, similar sensitivity analyses can be made. For instance, concerning the illustrative death/survival distribution according to Makeham, naturally a sensitivity analysis can be made with respect to the parameter $\tilde{c}$. However, it should be emphasized that the same consequence will result, because only the effect of these parameters on the death/survival probabilities is of importance.

In addition, Proposition 4.1 states

$$\begin{aligned} E_{P^*}[L_T^{*,tot}] &= f(x) - f(\tilde{x}) \\ \text{where } f(\tilde{x}) &:= \bar{G}_{t_N}\ D(t_0,t_N)\ {}_{t_N}p_{\tilde{x}} + \sum_{i=0}^{N-1} \bar{G}_{t_{i+1}}\ D(t_0,t_{i+1})\ {}_{t_i|t_{i+1}}q_{\tilde{x}}. \end{aligned}$$

Consequently, we obtain

$$\frac{\partial E_{P^*}[L_T^{*,tot}]}{\partial \tilde{x}} = -\frac{\partial f(\tilde{x})}{\partial \tilde{x}} = -\sum_{i=0}^{N-1} \underbrace{\left(\bar{G}_{t_{i+1}}\ D(t_0,t_{i+1}) - \bar{G}_{t_N}\ D(t_0,t_N)\right)}_{>0} \underbrace{\frac{\partial_{t_i|t_{i+1}}q_{\tilde{x}}}{\partial \tilde{x}}}_{>0} < 0.$$

Since under this condition $E_{P^*}[L_T^*]$ is a decreasing monotonic function of $\tilde{x}$ and $E_{P^*}[L_T^*]|_{x=\tilde{x}} = 0$, for the region $\{\tilde{x} > x\}$ (overestimation of the death probability), a superhedge in the mean results.
(ii) The derivative of the additional variance with respect to $\tilde{x}$ is given by

$${}_{t_N}p_x \underbrace{\frac{\partial\left(E_{P^*}\left[(I_{t_N}^*(\psi) - I_{t_N}^*(\phi))^2\right]\right)}{\partial \tilde{x}}}_{>0} + \sum_{j=0}^{N-1} {}_{t_j|t_{j+1}}q_x \underbrace{\frac{\partial\left(E_{P^*}\left[(I_{t_{j+1}}^*(\psi) - I_{t_{j+1}}^*(\phi))^2\right]\right)}{\partial \tilde{x}}}_{>0} > 0.$$

A detailed derivation is given in Appendix C.

□

Independent of the choice of the hedging instruments, an overestimation of the death probability ($\tilde{x} > x$) makes the insurance company achieve a superhedge in the mean. However, as the assumed $\tilde{x}$ goes up, the additional variance increases. I.e., a traditional tradeoff between the expected hedging costs and the additional variance is observed here. Furthermore, the impact of restricting the set of hedging instruments is highlighted only when the variance is taken into consideration and when the model risk related to the interest rate is available.

5. Illustration of results

To illustrate the results of the last sections, we use a one–factor Vasiček–type model framework to describe the financial market risk and a death distribution according to Makeham. The benchmark parameter constellation is given in Table 2.

5.1. **Expected total costs.** Figures 3 and 4 demonstrate how the death and survival probability, i.e., ${}_{t_{j-1}|t_j}q_x$ and ${}_tp_x$ change with the age x. With the change of x, the death and survival probability demonstrate a parallel shift. If the true age of the customer is 40, then an assumed age of 50 leads to an overestimation of the death probability and an assumed age of 30 results in an underestimation of the death probability. Of course the survival probability has exactly a reversed trend.

How the expected discounted total costs from both asset and liability side change with the assumed age $\tilde{x}$ is depicted by Figures 5 and 6. It is noticed that, for the given parameters, the expected discounted total cost exhibits a negative relation in $\tilde{x}$. It is a monotonically decreasing concave function of $\tilde{x}$. Especially, for a given t_N value in Figure 6, the higher $\tilde{x}$, the lower the expected total costs. From both figures,

Death and Survival Probabilities for Varying x Values

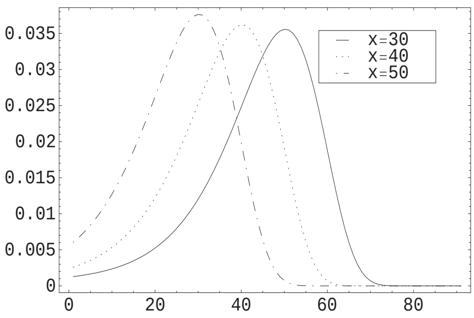

FIGURE 3. ${}_{t_{j-1}|t_j}q_x$ for $x = 30, 40, 50$. The other parameters are given in Table 2

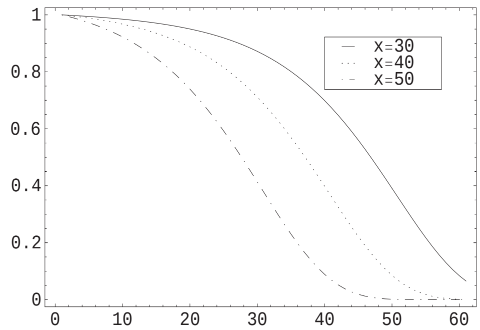

FIGURE 4. ${}_tp_x$ for $x = 30, 40, 50$. The other parameters are given in Table 2

it is observed that, independent of the set of hedging instruments (bonds), the hedger achieves profits in mean (negative expected discounted cost) if he overestimates the death probabilities. [17] Hence, negative expected discounted costs result when true x is smaller than the assumed one. Converse effects are observed when the insurer underestimates the death probability. Here, a real age of 40 is taken and it is observed that for $\tilde{x} = 45$, 50, the expected costs have negative values (blue curves), and for $\tilde{x} = 30$, 35, the expected costs exhibit positive values. When the true age coincides with the assumed one, the considered strategy is mean–self–financing because the expected discounted cost equals zero. These observations coincide with the result stated in Proposition 4.7.

5.2. **Variance of total costs/ distribution of total costs.** In contrast to the expected total costs, the distribution of the costs depends on the set of hedging instruments. This subsection attempts to illustrate how the variance difference depends on the model risk, i.e., some illustrations are exhibited to support Proposition 4.7. The model risk associated with the interest rate influences the variance difference through the functions $|g^{(i)}|$, $i = 1, \cdots, N-2$ [18], which is given by

$$|g_u^{(i)}| = \left| \frac{\tilde{\sigma}_{t_i}(u) - \tilde{\sigma}_{t_N}(u)}{\tilde{\sigma}_{t_{N-1}}(u) - \tilde{\sigma}_{t_N}(u)} \sigma_{t_{N-1}}(u) + \frac{\tilde{\sigma}_{t_{N-1}}(u) - \tilde{\sigma}_{t_i}(u)}{\tilde{\sigma}_{t_{N-1}}(u) - \tilde{\sigma}_{t_N}(u)} \sigma_{t_N}(u) - \sigma_{t_i}(u) \right|.$$

Only if it holds that

$$\sigma_{t_i}(u) \quad = \quad \frac{\tilde{\sigma}_{t_i}(u) - \tilde{\sigma}_{t_N}(u)}{\tilde{\sigma}_{t_{N-1}}(u) - \tilde{\sigma}_{t_N}(u)} \sigma_{t_{N-1}}(u) + \frac{\tilde{\sigma}_{t_{N-1}}(u) - \tilde{\sigma}_{t_i}(u)}{\tilde{\sigma}_{t_{N-1}}(u) - \tilde{\sigma}_{t_N}(u)} \sigma_{t_N}(u), \tag{5}$$

i.e., only if it is possible to write the volatility of the t_i–bond as a linear combination of the hedge instruments' volatilities, it is possible to find a self–financing replicating strategy for the bond with maturity t_i, and consequently, it is possible that no

[17]This result is opposite to the result in pure endowment insurance contracts, where a negative expected discounted cost is achieved when an overestimation of the survival probability exists.
[18]C.f. Appendix B

Expected Discounted Cost for Varying $\tilde{x}$

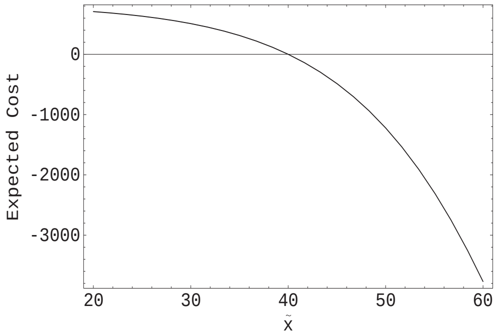

FIGURE 5. Expected cost as a function of $\tilde{x}$ with $x = 40$. The other parameters are given in Table 2

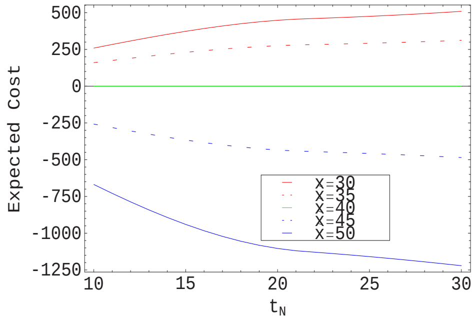

FIGURE 6. Expected cost for $\tilde{x}$ = 30, 35, 40, 4550 with the real $x = 40$. The other parameters are given in Table 2

variance difference results, independent of mortality misspecification. This indicates, if there is no model misspecification associated with the interest rate, the choice of the hedging instruments has no impact on the variance of the total cost. However, condition (5) is a very demanding condition, i.e., there always exists model misspecification related to the interest rate.

Assuming that the short rate is driven by a one–factor Vasiček model, model risk associated with the interest rate can be characterized either by the mismatch of the volatility ($\bar{\sigma}$) or the speed factor (κ), which are determining factors in the volatility function of the zero coupon bonds. Due to the Vasiček modelling, the misspecification of $\bar{\sigma}$ has no impact on $g_u^{(i)}$ functions, hence, no impact on the variance difference. Therefore, in the following, we concentrate on the interest rate misspecification characterized by the deviation of the assumed $\tilde{\kappa}$ from the true κ.

The volatility of the zero coupon bond (with any maturities) is a decreasing function of κ. I.e., a $\tilde{\kappa} < \kappa$ leads to an overestimation of the bond volatility. Under this condition, $|g^{(i)}|$ is a decreasing function of $\tilde{\kappa}$. On the contrary, in the case of $\tilde{\kappa} > \kappa$ (underestimation of the bond volatility), $|g^{(i)}|$ is a increasing function of $\tilde{\kappa}$. Therefore, we obtain some values for the variance difference as exhibited in Table 3. Firstly, there exists a deviation of $\tilde{\kappa}$ from κ, the variances of these two strategies differ, even when there is no mortality misspecification. Secondly, mortality misspecification does not have impact on the variance difference, if there are no interest rate misspecification available. I.e., these two strategies make no difference to the variance of the total cost if no model risk associated with the interest rate appears. Therefore, for $\tilde{\kappa} = \kappa = 0.18$, overall the variance difference exhibits a value of 0. These two observations validate the argument that the model misspecification resulting from the term structure of the interest rate has a substantial effect when the variance is taken into account. The effect of mortality risk is partly contingent

	Expected total cost			Variance Difference			The Ratio		
$\tilde{\kappa}$	$\tilde{x}=35$	$\tilde{x}=40$	$\tilde{x}=45$	$\tilde{x}=35$	$\tilde{x}=40$	$\tilde{x}=45$	$\tilde{x}=35$	$\tilde{x}=40$	$\tilde{x}=45$
0.150	287.795	0	-449.842	774.983	1876.71	4660.65	0.0967	-	-0.1518
0.155	293.229	0	-457.831	568.654	1375.73	3414.78	0.0813	-	-0.1276
0.160	297.998	0	-464.843	385.079	930.677	2308.87	0.0659	-	-0.1034
0.165	302.192	0	-471.014	229.521	554.135	1373.97	0.0501	-	-0.0787
0.170	305.891	0	-476.458	108.254	261.073	646.952	0.0340	-	-0.0534
0.175	309.159	0	-481.270	28.7651	69.2933	171.608	0.0173	-	-0.0272
0.180	312.054	0	-485.532	0	0	0	0	-	0
0.185	314.621	0	-489.315	32.6569	78.4791	194.106	0.0182	-	-0.0285
0.190	316.903	0	-492.677	139.546	334.922	827.806	0.0373	-	-0.0584
0.195	318.934	0	-495.670	336.029	805.425	1989.29	0.0575	-	-0.0890
0.200	320.745	0	-498.338	640.547	1533.21	3783.96	0.0789	-	-0.1234
0.205	322.361	0	-500.719	1075.28	2570.10	6338.04	0.1017	-	-0.1590
0.210	323.805	0	-502.847	1666.92	3978.35	9802.85	0.1261	-	-0.1969

Table 3. Expected total cost, variance differences and the ratio of the standard deviation of the variance difference and the expected total cost for varying $\tilde{\kappa}$ with $x = 40$ and the other parameters are given in Table 2.

on the model risk associated with the interest rate. Thirdly, only the absolute distance of $\tilde{\kappa}$ from κ counts. The bigger this absolute distance is, the higher variance differences these two strategies result in. Therefore, overall you observe parabolic curves for the variance difference. In addition, the variance difference increases in $\tilde{x}$, as stated in Proposition 4.7. This positive effect can be observed in Figures 7 and 8.

To sum up, if the hedger substantially overestimates ($\tilde{\kappa} << \kappa$) or underestimates ($\tilde{\kappa} >> \kappa$) the bond volatilities, and if at the same time he highly overestimates the death probability ($\tilde{x} >> x$), the diverse choice of the hedging instruments leads to a huge difference in the variance. On the contrary, a $\tilde{\kappa}$ value close to κ combined with a big overestimation of the survival probability ($\tilde{x} << x$) almost leads to very small variance difference. I.e., very close variances result. The choice of the hedging instrument does not have a significant effect under this circumstance. These result leads to a very interesting phenomenon, with an overestimation of the death probability ($\tilde{x} > x$), the insurance company is always on the safe side in mean, i.e., it achieves a superhedge in the mean. However, if the set of hedging instruments is restricted, an overestimation of the death probability does not necessarily decrease the shortfall probability under a huge misspecification associated with the interest rate (characterized by a big deviation of $\tilde{\kappa}$ from κ). This is due to the observation that a quite high variance difference is reached under this parameter constellation.

In addition, due to the tradeoff between the expected value and the variance difference[19], it is interesting to have a look at the relative size, like the ratio of the

[19]An overestimation of the death probability leads to a superhedge in the mean but at the same time a higher variance difference.

Variance Difference

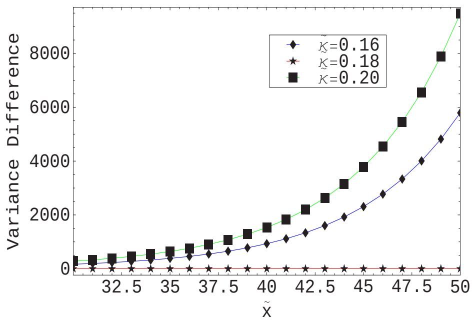

FIGURE 7. Variance difference as function of $\tilde{x}$ with the real $x = 40$ for $\tilde{\kappa} = 0.16$, $\tilde{\kappa} = 0.18$ and $\tilde{\kappa} = 0.20$. The other parameters are given in Table 2.

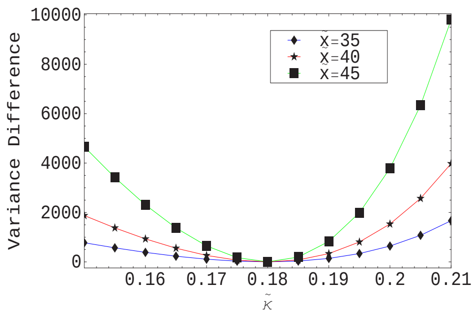

FIGURE 8. Variance difference as function of $\tilde{\kappa}$ with the real $\kappa = 0.18$ for $\tilde{x} = 35$, $\tilde{x} = 40$ and $\tilde{x} = 45$. The other parameters are given in Table 2.

standard deviation of the variance difference and the expected value of the total cost from both asset and liability side. First of all, this ratio is not defined when the assumed and real age coincide. Second of all, here for the given parameters, an overestimation of the death probability ($\tilde{x} = 45$) has a higher effect than an underestimation ($\tilde{x} = 35$), i.e. the absolute value of this ratio is larger for the case of $\tilde{x} = 45$. Finally, this ratio can give a hint to the safety loading factor. Assume, the insurer uses standard–deviation premium principle. The ratio given in Table 3 suggests him how much safety loading to take when he uses the last two bonds instead of the entire term structure.

6. Conclusion

The risk management of an insurance company must take into account model risk. The uncertainty about the true model concerns the insurance typical risk and the market risk. Due to the long maturities which are observed in life insurance products, it is even not enough to consider stochastic interest rates but to take into account that the true data generating process might deviate from the assumed one. In comparison with the market risk, the uncertainty about the life expectancy is low. However, we show that even a small difference between assumed and realized death scenarios may have a great impact on the hedging performance because of the existence of interest rate risk. In practice, this is particularly important because a deviation of true and assumed mortality/survival probabilities is unavoidable and sometimes even caused intentionally by the insurance company itself.

The problem which is associated with the interdependence of model risk concerning the interest rate dynamic and the mortality distribution is even more severe if there is a restriction on the set of hedging instruments. In practice, it is not necessarily the case that zero bonds with every (possible) maturity of insurance contracts are available. Besides, there are different sources of liquidity constraints to consider. Thus, it is safe to assume that, even if it is possible to trade in all bond maturities, the insurance company would restrict the number of hedging instruments. We measure the risk implied by the restriction of hedging instruments by calculating the additional variance of the hedging costs, i.e. the variance which is to be added to the variance term without the restriction.

We also stress an important problem which arises if, as it is normally done, the contributions of the insured are given in terms of periodic premiums instead of an up-front premium. If the contributions of the insured are delayed to a future, uncertain time, model risk influences the liability side in addition to the asset side. Theoretically, a credit must be taken by the insurer in order to implement the considered hedging strategies in the asset side. The insurer achieves not necessarily the number of periodic premiums which is needed to pay back his credit. Therefore, one further focus is on the extra costs stemming from periodic premiums.

To sum up, neither the model risk which is related to the death distribution nor the one associated with the financial market model is negligible for a meaningful risk management.

Appendix A. Synthesising the pseudo asset X

We study the case where the hedge instrument X is not liquidly traded in the market and a potential hedger must use other assets $Y^1, ..., Y^n$ to synthesize X. We place ourselves in a diffusion setting, i.e. the prices X, $Y^1, \ldots, Y^n$ are given by Itô processes which are driven by a d-dimensional Brownian motion W defined on $\big(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, P\big)$:

$$\begin{aligned} dX_t &= X_t\{\mu_t^X dt + \sigma_t^X dW_t\} \\ dY_t^i &= Y_t^i\{\mu_t^i dt + \sigma_t^i dW_t\} \end{aligned}$$

where μ^X, σ^X and μ^i, σ^i are suitably integrable stochastic processes. We assume the prices X, $Y^1, \ldots, Y^n$ are arbitrage-free. This implies that there is a "market price of risk" process φ such that for any $i \in \{1, \ldots, n\}$:

$$\mu^X - \sigma^X \varphi = \mu^i - \sigma^i \varphi.$$

Synthesizing X out of $Y^1, \ldots, Y^n$ involves finding a self-financing strategy ϕ with a position of ϕ^i in asset Y^i for each $i \in \{1, \ldots, n\}$ such that $X = \sum_{i=1}^n \phi^i Y^i$. The following proposition characterizes these strategies ϕ.

Proposition A.1. *Suppose that $\lambda^1, \ldots, \lambda^n$ are predictable processes satisfying the following two conditions:*

$$(1) \ \sum_{i=1}^n \lambda_t^i = 1 \ \textit{and} \ (2) \ \sum_{i=1}^n \lambda_t^i \sigma_t^i = \sigma_t^X.$$

For each $i \in \{1, \ldots, n\}$, we set $\phi^i := \frac{X}{Y^i}\lambda^i$. Then ϕ is a self-financing strategy which identically duplicates X. In particular, any such strategy is of the form above.

Proof: Suppose that weights $\lambda^1, \ldots, \lambda^n$ are given which satisfy conditions (1) and (2) and that ϕ is the corresponding strategy. By condition (1), it is clear that $\sum_{i=1}^n \phi^i Y^i = X$. By the no-arbitrage condition and because of (2) we have

$$\sum_{i=1}^n \lambda^i \mu^i = \sum_{i=1}^n \lambda^i \{\mu^X + \varphi(\sigma^i - \sigma^X)\} = \mu^X.$$

From this we see that ϕ is also self-financing because

$$\sum_{i=1}^n \phi_t^i dY_t^i = X_t \sum_{i=1}^n \lambda_t^i \{\mu_t^i dt + \sigma_t^i dW_t\} = X_t\{\mu^X dt + \sigma_t^X dW_t\} = dX_t.$$

Conversely, if ϕ is a self-financing strategy which identically duplicates X, then the weights $\lambda^1, \ldots, \lambda^n$ determined by $\lambda^i := \frac{Y^i}{X}\phi^i$ will satisfy the two conditions.

□

The weights $\lambda^1, \ldots, \lambda^n$ are to be interpreted as portfolio weights, i.e. λ^i is the proportion of total capital to be invested in asset Y^i.

Appendix B. Proof of Proposition 4.6

(i) Lemma 3.6 immediately gives $\mathrm{Var}[L_T^{tot,*}(\phi)] = \mathrm{Var}[C_T^* - I_T^*(\phi)]$ such that

$$\mathrm{Var}[L_T^{tot,*}(\phi)] - \mathrm{Var}[L_T^{tot,*}(\psi)] \quad = \quad \mathrm{Var}[I_T^*(\phi)] - \mathrm{Var}[I_T^*(\psi)] + 2\mathrm{Cov}[C_T^*, I_T^*(\psi) - I_T^*(\phi)].$$

The relation between ϕ and ψ can be represented as follows:

$$\begin{array}{rl} & \psi^{(N-1)} = \sum_{i=1}^{N-2} \alpha^i \phi^{(i)} + \phi^{(N-1)}, \quad \psi^{(N)} = \sum_{i=1}^{N-2} \beta^i \phi^{(i)} + \phi^{(N)} \\ \text{where} & \alpha_u^{(i)} = \lambda_1^{(i)}(u)\frac{D(t,t_i)}{D(t,t_{N-1})}, \beta_u^{(i)} = \lambda_2^{(i)}(u)\frac{D(t,t_i)}{D(t,t_N)}, \quad \lambda_1^{(i)} + \lambda_2^{(i)} = 1. \end{array}$$

Notice that α_i is function of $\frac{D(t,t_i)}{D(t,t_{N-1})}$ and β_i function of $\frac{D(t,t_i)}{D(t,t_N)}$. With respect to the gain process of ψ this implies:

$$\begin{aligned} I_t^*(\psi) &= \int_0^t \psi_u^{(N-1)} dD^*(u, t_{N-1}) + \int_0^t \psi_u^{(N)} dD^*(u, t_N) \\ &= \int_0^t \left(\sum_{i=1}^{N-2} \alpha_u^i \phi_u^{(i)} + \phi_u^{(N-1)}\right) dD^*(u, t_{N-1}) + \int_0^t \left(\sum_{i=1}^{N-2} \beta_u^i \phi_u^{(i)} + \phi_u^{(N)}\right) dD^*(u, t_N) \\ &= \sum_{i=1}^{N-2} \int_0^t \alpha_u^i \phi_u^{(i)} dD^*(u, t_{N-1}) + \int_0^t \phi_u^{(N-1)} dD^*(u, t_{N-1}) \\ &\quad + \sum_{i=1}^{N-2} \int_0^t \beta_u^i \phi_u^{(i)} dD^*(u, t_N) + \int_0^t \phi_u^{(N)} dD^*(u, t_N). \end{aligned}$$

With respect to the difference in the gains of ψ and ϕ this leads to

$$
\begin{aligned}
& I_t^*(\psi) - I_t^*(\phi) \\
= & \sum_{i=1}^{N-2} \int_0^t \alpha_u^i \phi_u^{(i)} dD^*(u, t_{N-1}) + \sum_{i=1}^{N-2} \int_0^t \beta_u^i \phi_u^{(i)} dD^*(u, t_N) - \sum_{i=1}^{N-2} \int_0^t \phi_u^{(i)} dD^*(u, t_i) \\
= & \sum_{i=1}^{N-2} \int_0^t \left(\alpha_u^i \phi_u^{(i)} dD^*(u, t_{N-1}) + \beta_u^i \phi_u^{(i)} dD^*(u, t_N) - \phi_u^{(i)} dD^*(u, t_i) \right) \\
= & \sum_{i=1}^{N-2} \int_0^t \phi_u^{(i)} \Bigg(\lambda_1^{(i)}(u) \frac{D(u, t_i)}{D(u, t_{N-1})} dD^*(u, t_{N-1}) + \lambda_2^{(i)}(u) \frac{D(u, t_i)}{D(u, t_N)} dD^*(u, t_N) \\
& -dD^*(u, t_i) \Bigg).
\end{aligned}
$$

The assumed model associated with the interest rate leads to

$$
\begin{aligned}
D^*(u, t_i) = e^{-\int_0^u r_s ds} D(u, t_i) &= D(t_0, t_i) \exp\left\{ -\frac{1}{2} \int_0^u (\sigma_{t_i}(s))^2 ds + \int_0^u \sigma_{t_i}(s) dW_s^* \right\} \\
dD^*(u, t_i) &= D^*(u, t_i) \sigma_{t_i}(u) dW_u^*.
\end{aligned}
$$

This results in

$$
\begin{aligned}
I_t^*(\psi) - I_t^*(\phi) &= \sum_{i=1}^{N-2} \int_0^t \phi_u^{(i)} \Bigg(\lambda_1^{(i)}(u) D^*(u, t_i) \sigma_{t_{N-1}}(u) dW_u^* + \lambda_2^{(i)} D^*(u, t_i) \sigma_{t_N}(u) dW_u^* \\
& \quad -D^*(u, t_i) \sigma_{t_i}(u) dW_u^* \Bigg) \\
&= \sum_{i=1}^{N-2} \int_0^t \phi_u^{(i)} D^*(u, t_i) \left(\lambda_1^{(i)}(u) \sigma_{t_{N-1}}(u) + \lambda_2^{(i)}(u) \sigma_{t_N}(u) - \sigma_{t_i}(u) \right) dW_u^* \\
&:= \sum_{i=1}^{N-2} \int_0^t \phi_u^{(i)} D^*(u, t_i)\, g_u^{(i)}\, dW_u^*.
\end{aligned}
$$

If there exists **no** model risk concerning the interest rate, i.e., $\sigma_{t_i}(u) = \tilde{\sigma}_{t_i}(u)$, $u \leq T$, the gain process of ϕ coincides with that of ψ. Therefore, under this circumstance, it holds $\mathrm{Var}[I_t^*(\psi)] = \mathrm{Var}[I_t^*(\phi)]$ Consequently, it leads to $\mathrm{Var}[L_T^{tot,*}(\phi)] = \mathrm{Var}[L_T^{tot,*}(\psi)]$. The following transformation enlightens this argument.

$$
\begin{aligned}
& \mathrm{Var}[L_T^{tot,*}(\phi)] - \mathrm{Var}[L_T^{tot,*}(\psi)] \\
= & \mathrm{Var}[I_T^*(\phi)] - \mathrm{Var}[I_T^*(\psi)] + 2\mathrm{Cov}[C_T^*, I_T^*(\psi) - I_T^*(\phi)] \\
= & \mathrm{Var}[I_T^*(\phi)] - \mathrm{Var}[I_T^*(\phi) + I_T^*(\psi) - I_T^*(\phi)] + 2\mathrm{Cov}[C_T^*, I_T^*(\psi) - I_T^*(\phi)] \\
= & \mathrm{Var}[I_T^*(\phi)] - \mathrm{Var}[I_T^*(\phi)] - \mathrm{Var}[I_T^*(\psi) - I_T^*(\phi)] - 2\mathrm{Cov}[I_T^*(\phi), I_T^*(\psi) - I_T^*(\phi)] \\
& +2\mathrm{Cov}[C_T^*, I_T^*(\psi) - I_T^*(\phi)] \\
= & -\mathrm{Var}[I_T^*(\psi) - I_T^*(\phi)] + 2\mathrm{Cov}[C_T^* - I_T^*(\phi), I_T^*(\psi) - I_T^*(\phi)].
\end{aligned}
$$

Now let us have a look at the variance difference if there does exist model misspecification related to the interest rate. If T is a deterministic time point,

$$\mathrm{Var}[I_T^*(\psi) - I_T^*(\phi)] = \mathrm{Var}\left[\sum_{i=1}^{N-2} \int_0^T \phi_u^{(i)} D^*(u,t_i)\, g_u^{(i)}\, dW_u^*\right]$$

$$= \sum_{i=1}^{N-2} \mathrm{Var}\left[\int_0^T \phi_u^{(i)} D^*(u,t_i)\, g_u^{(i)}\, dW_u^*\right] = \sum_{i=1}^{N-2} E^*\left[\left(\int_0^T \phi_u^{(i)} D^*(u,t_i)\, g_u^{(i)}\, dW_u^*\right)^2\right]$$

$$= \sum_{i=1}^{N-2} E^*\left[\left(\int_0^T \phi_u^{(i)} D^*(u,t_i)\, g_u^{(i)}\right)^2 du\right]$$

Since

$$\begin{aligned}
\phi_u^{(i)} &= 1_{u\le t_i} \bar{G}_{t_i t_{i-1}|t_i} \tilde{q}_{x+u} \\
E^*[(D^*(u,t_i))^2] &= (D(t_0,t_i))^2 \exp\left\{\int_0^u (\sigma_{t_i}(s))^2 ds\right\},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}[I_T^*(\psi) - I_T^*(\phi)] &= \sum_{i=1}^{N-2} \int_0^{t_i} \left(\bar{G}_{t_i t_{i-1}|t_i} \tilde{q}_{x+u}\right)^2 \left(g_u^{(i)}\right)^2 E^*[(D^*(u,t_i))^2] du \\
&= \sum_{i=1}^{N-2} (D(t_0,t_i))^2 \int_0^{t_i} \left(\bar{G}_{t_i t_{i-1}|t_i} \tilde{q}_{x+u}\right)^2 \left(g_u^{(i)}\right)^2 \exp\left\{\int_0^u (\sigma_{t_i}(s))^2 ds\right\} du
\end{aligned}$$

And if T is a stopping time as specified in our contract, we obtain

$$\begin{aligned}
&\mathrm{Var}[I_T^*(\psi) - I_T^*(\phi)] \\
=\;& E^*\left[(I_T^*(\psi) - I_T^*(\phi))^2 1_{\{\tau^x > t_N\}}\right] + \sum_{i=0}^{N-1} E^*\left[(I_T^*(\psi) - I_T^*(\phi))^2 1_{\{t_j < \tau^x \le t_{j+1}\}}\right] \\
=\;& E^*\left[(I_{t_N}^*(\psi) - I_{t_N}^*(\phi))^2 1_{\{\tau^x > t_N\}}\right] + \sum_{j=0}^{N-1} E^*\left[(I_{t_{j+1}}^*(\psi) - I_{t_{j+1}}^*(\phi))^2 1_{\{t_j < \tau^x \le t_{j+1}\}}\right] \\
=\;& {}_{t_N}p_x\, E^*\left[(I_{t_N}^*(\psi) - I_{t_N}^*(\phi))^2\right] + \sum_{j=0}^{N-1} {}_{t_j|t_{j+1}}q_x\, E^*\left[(I_{t_{j+1}}^*(\psi) - I_{t_{j+1}}^*(\phi))^2\right]
\end{aligned}$$

where

$$\begin{aligned}
E^*[(I_{t_j}^*(\psi) - I_{t_j}^*(\phi))^2] =& \sum_{i=1}^{\min\{j,N-2\}} (D(t_0,t_i))^2 \int_0^{t_i} \left(\bar{G}_{t_i t_{i-1}|t_i} \tilde{q}_{x+u}\right)^2 \left(g_u^{(i)}\right)^2 \\
& \exp\left\{\int_0^u (\sigma_{t_i}(s))^2 ds\right\} du, \qquad j = 1, \cdots, N
\end{aligned}$$

In addition,

$$\mathrm{Cov}[C_T^* - I_T^*(\phi), I_T^*(\psi) - I_T^*(\phi)] = \mathrm{Cov}[L_T^{tot,*}(\phi), I_T^*(\psi) - I_T^*(\phi)]$$

$$= {}_{t_N}p_x\, \mathrm{Cov}[L_{t_N}^{tot,*}(\phi), I_{t_N}^*(\psi) - I_{t_N}^*(\phi)] + \sum_{j=0}^{N-1} {}_{t_j|t_{j+1}}q_x\, \mathrm{Cov}[L_{t_j+1}^{tot,*}(\phi), I_{t_{j+1}}^*(\psi) - I_{t_{j+1}}^*(\phi)]$$

Due to the fact that $I_t^*(\psi) - I_t^*(\phi)$ is not of bounded variation, but $L_T^{tot,*}(\phi)$ is, the above covariance equals zero. To sum up, after taking account of the mortality risk, the variance difference is given by

$$
\begin{aligned}
& \mathrm{Var}[L_T^{tot,*}(\phi)] - \mathrm{Var}[L_T^{tot,*}(\psi)] \\
= & -\mathrm{Var}[I_T^*(\psi) - I_T^*(\phi)] + 2\mathrm{Cov}[C_T^* - I_T^*(\phi), I_T^*(\psi) - I_T^*(\phi)] \\
= & -\left({}_{t_N}p_x\, E^*\left[(I_{t_N}^*(\psi) - I_{t_N}^*(\phi))^2\right] + \sum_{j=0}^{N-1} {}_{t_j|t_{j+1}}q_x\, E^*\left[(I_{t_{j+1}}^*(\psi) - I_{t_{j+1}}^*(\phi))^2\right]\right) < 0.
\end{aligned}
$$

(ii) Concerning the second part it follows with $\mathrm{Var}[L_T^*(\phi)] = \mathrm{Var}[L_T^{tot,*}(\phi) + L_T^{add,*}]$

$$
\mathrm{Var}[L_T^*(\psi)] - \mathrm{Var}[L_T^*(\phi)] = \mathrm{Var}[L_T^{tot,*}(\psi)] - \mathrm{Var}[L_T^{tot,*}(\phi)] + 2\mathrm{Cov}[L_T^{tot,*}(\psi) - L_T^{tot,*}(\phi), L_T^{add,*}]
$$

Since it holds that

$$
\begin{aligned}
L_T^{tot,*}(\psi) - L_T^{tot,*}(\phi) &= I_T^*(\phi) - I_T^*(\psi) \\
&= -\sum_{i=1}^{N-2} \int_0^t 1_{\{u \le t_i\}} \bar{G}_{t_i\ t_{i-1}|t_i}\tilde{q}_{x+u} D^*(u,t_i)\, g_u^{(i)}\, dW_u^* \\
L_T^{add,*} &= \sum_{i=0}^{N-1} e^{-\int_0^{t_i} r_u\, du} A\left({}_{t_i}\tilde{p}_x - 1_{\{\tau^x > t_i\}}\right),
\end{aligned}
$$

the covariance part is given by

$$
\mathrm{Cov}\left[\sum_{i=1}^{N-2} \int_0^t 1_{\{u \le t_i\}} \bar{G}_{t_i\ t_{i-1}|t_i}\tilde{q}_{x+u} D^*(u,t_i)\, g_u^{(i)}\, dW_u^*, \sum_{i=0}^{N-1} e^{-\int_0^{t_i} r_u\, du} A \cdot 1_{\{\tau^x > t_i\}}\right].
$$

Now we claim it equals zero because of the independence assumption between the financial and mortality risk. It is observed that the first part depends only on the financial risk, while the second only on the mortality risk.

Appendix C. Proof of Proposition 4.7

In this part of appendix, we will demonstrate you some sensitivity analysis, in particular, how the mortality misspecification affects the expected value and variance of the total hedging costs. Mortality misspecification will be characterized by the deviation of the assumed age $\tilde{x}$ from the true age x–value. Recall that

$$
\begin{aligned}
{}_tp_x &= e^{-\int_0^t \mu_{x+s} ds} \\
{}_{u|t}q_x &= {}_up_x - {}_tp_x = e^{-\int_0^t \mu_{x+s} ds} - e^{-\int_0^u \mu_{x+s} ds}, \quad t > u
\end{aligned}
$$

μ is the so called hazard rate of mortality. Furthermore, concerning the death/survival probabilities, we make the following assumptions:

$$
\begin{aligned}
(a) \qquad \frac{\partial\, {}_tp_x}{\partial x} &= {}_tp_x\left(-\int_0^t \frac{\partial \mu_{x+s}}{\partial x} ds\right) < 0 \Leftrightarrow \qquad \frac{\partial \mu_{x+s}}{\partial x} > 0 \\
(b) \qquad \frac{\partial\, {}_tp_x}{\partial t} &= {}_tp_x\left(-\int_0^t \frac{\partial \mu_{x+s}}{\partial t} ds\right) = -{}_tp_x\, \mu_{x+t} < 0
\end{aligned}
$$

$$(c) \qquad \frac{\partial\, {}_{u|t}q_x}{\partial x} = \frac{\partial_u p_x}{\partial x} - \frac{\partial_t p_x}{\partial x} > 0$$

$$\Leftrightarrow \quad \frac{\frac{\partial_s p_x}{\partial x}}{\partial s} < 0 \quad \Leftrightarrow \quad {}_s p_x \left(\mu_{x+s} \int_0^s \frac{\partial \mu_{x+v}}{\partial x}\, dv - \frac{\partial \mu_{x+s}}{\partial x} \right) < 0$$

$$(d) \qquad \frac{\partial\, {}_{t_{i-1}|t_i}q_{x+u}}{\partial x} = \frac{\partial \left({}_{t_{i-1}}p_{x+u} - {}_{t_i}p_{x+u} \right)}{\partial x} = \frac{\partial \left(\frac{{}_{t_{i-1}}p_x}{{}_u p_x} - \frac{{}_{t_i}p_x}{{}_u p_x} \right)}{\partial x} > 0$$

$$\Leftrightarrow \quad \frac{\partial \left(\partial \frac{{}_s p_x}{{}_u p_x} / \partial x \right)}{\partial s} < 0 \quad \Leftrightarrow \quad \frac{{}_s p_x}{{}_u p_x} \left(\mu_{x+s} \int_u^s \frac{\partial \mu_{x+v}}{\partial x}\, dv - \frac{\partial \mu_{x+s}}{\partial x} \right) < 0,\ s > u$$

These assumptions are indeed quite realistic. Assumption (a) says that the survival probability decreases in the age. Assumptions (c) and (d) tell that the (conditional) death probability increases in the age. Condition (b) holds always. Technically, it should hold

$$\frac{\partial \mu_{x+s}}{\partial x} > 0, \qquad \mu_{x+s} \int_u^s \frac{\partial \mu_{x+v}}{\partial x}\, dv - \frac{\partial \mu_{x+s}}{\partial x} < 0,\ u < s.$$

These conditions hold e.g. for De Moivre hazard rate, where $\mu_{x+t} = \frac{1}{w-x-t}$ with w the highest attainable age, and Makeham hazard rate, where $\mu_{x+t} = H + Ke^{x+t}$.
(i) It is known that the expected discounted total hedging cost is the difference between the initial price of the contract conditional on the true death distribution and that conditional on the true one.

$$\begin{aligned} E_{P^*}[L_T^{tot,*}(\phi)] &= D(t_0,t_N)\bar{G}_{t_N}({}_{t_N}p_x - {}_{t_N}\tilde{p}_x) + \sum_{j=1}^{N-1} ({}_{t_{j-1}|t_j}q_x - {}_{t_{j-1}|t_j}\tilde{q}_x) D(t_0,t_j)\bar{G}_{t_j} \\ &= f(x) - f(\tilde{x}) \end{aligned}$$

Since the true x is always considered given, we are interested in how exactly this expected cost depends on the assumed age $\tilde{x}$, i.e.,

$$\frac{\partial E_{P^*}[L_T^{tot,*}(\phi)]}{\partial \tilde{x}} = -\frac{\partial f(\tilde{x})}{\partial \tilde{x}}$$

Since the initial value can be reformulated as follows:

$$\begin{aligned} f(\tilde{x}) &= \bar{G}_{t_N}\ D(t_0,t_N)\ {}_{t_N}p_{\tilde{x}} + \sum_{i=0}^{N-1} \bar{G}_{t_{i+1}}\ D(t_0,t_{i+1})\ {}_{t_i|t_{i+1}}q_{\tilde{x}} \\ &= \bar{G}_{t_N}\ D(t_0,t_N)(1 - {}_{t_N}q_{\tilde{x}}) + \sum_{i=0}^{N-1} \bar{G}_{t_{i+1}}\ D(t_0,t_{i+1})\ {}_{t_i|t_{i+1}}q_{\tilde{x}} \\ &= \bar{G}_{t_N}\ D(t_0,t_N) - \bar{G}_{t_N}\ D(t_0,t_N) \sum_{i=0}^{N-1} {}_{t_i|t_{i+1}}q_{\tilde{x}} + \sum_{i=0}^{N-1} \bar{G}_{t_{i+1}}\ D(t_0,t_{i+1})\ {}_{t_i|t_{i+1}}q_{\tilde{x}} \\ &= \bar{G}_{t_N}\ D(t_0,t_N) + \sum_{i=0}^{N-1} \left(\bar{G}_{t_{i+1}}\ D(t_0,t_{i+1}) - \bar{G}_{t_N}\ D(t_0,t_N) \right) {}_{t_i|t_{i+1}}q_{\tilde{x}} \end{aligned}$$

And

$$\frac{\partial E_{P^*}[C_T^{tot,*}(\phi)]}{\partial \tilde{x}} = -\frac{\partial f(\tilde{x})}{\partial \tilde{x}} = -\sum_{i=0}^{N-1} \underbrace{\left(\bar{G}_{t_{i+1}}\ D(t_0,t_{i+1}) - \bar{G}_{t_N}\ D(t_0,t_N)\right)}_{>0} \underbrace{\frac{\partial_{t_i|t_{i+1}} q_{\tilde{x}}}{\partial \tilde{x}}}_{>0} < 0$$

Since under this condition $E_{P^*}[L_T^{*,tot}]$ is a decreasing monotonic function of $\tilde{x}$ and $E_{P^*}[L_T^{*,tot}]|_{x=\tilde{x}} = 0$, for the region $\{\tilde{x} > x\}$ (overestimation of the death probability), a superhedge in the mean results.

(ii) The derivative of the variance difference with respect to $\tilde{x}$.

$$\frac{\partial\left(\mathrm{Var}[L_T^{tot,*}(\psi)] - \mathrm{Var}[L_T^{tot,*}(\phi)]\right)}{\partial \tilde{x}}$$

$$= {}_{t_N}p_x \frac{\partial\left(E_{P^*}\left[(I^*_{t_N}(\psi) - I^*_{t_N}(\phi))^2\right]\right)}{\partial \tilde{x}} + \sum_{j=0}^{N-1} {}_{t_j|t_{j+1}}q_x\ \frac{\partial\ \ E_{P^*}\left[(I^*_{t_{j+1}}(\psi) - I^*_{t_{j+1}}(\phi))^2\right]}{\partial \tilde{x}} > 0$$

because

$$\frac{\partial E^*\left[(I^*_{t_N}(\psi) - I^*_{t_N}(\phi))^2\right]}{\partial x}$$

$$= \sum_{i=1}^{N-2} (D(t_0,t_i))^2 \int_0^{t_i} (\bar{G}_{t_i})^2 2_{t_{i-1}|t_i}\tilde{q}_{x+u} \underbrace{\frac{\partial_{t_{i-1}|t_i}\tilde{q}_{x+u}}{\partial x}}_{>0} \left(g_u^{(i)}\right)^2 \exp\left\{\int_0^u (\sigma_{t_i}(s))^2 ds\right\} du > 0.$$

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