Pricing and capital requirements for with profit contracts: modelling considerations

Laura Ballotta

Faculty of Actuarial Science and Insurance, Cass Business School,

106 Bunhill Row, London, EC1Y 8TZ, United Kingdom

E-mail: L.Ballotta@city.ac.uk

This version: January 2006

Abstract

The aim of this paper is to provide an assessment of alternative frameworks for the fair valuation of life insurance contracts with a predominant financial component, in terms of impact on the market consistent price of the contracts, the options embedded therein, and the capital requirements for the insurer. In particular, we model the dynamics of the log-returns of the reference fund using the so-called Merton process (Merton, 1976), which is given by the sum of an arithmetic Brownian motion and a compound Poisson process, and the Variance Gamma (VG) process introduced by Madan and Seneta (1990), and further refined by Madan and Milne (1991) and Madan et al. (1998). We conclude that, although the choice of the market model does not affect significantly the market consistent price of the overall benefit due at maturity, the consequences of a model misspecification on the capital requirements are quite severe.

Keywords: fair value, incomplete markets, Lévy processes, Monte Carlo simulation, participating contracts, solvency requirements

1 Introduction

Our problem is motivated by the recent move towards market consistent valuation of insurance companies' assets and liabilities for accounting and solvency purposes. Although asset prices can be observed directly in the financial market, in general insurance liabilities are not fully traded, which implies the lack of proper market prices. Consequently, according to the regulators' directives, insurance companies need to develop suitable (internal) models which incorporate both market risk and insurance risk, and are market consistent, i.e. are based on the up-to-date information available at the time of valuation. These models will be used to generate market consistent distributions for the future cash flows originated by the relevant liabilities, from which a proxy for the market price can be extracted. In terms of how this is implemented in practice, we note that two approaches are currently being debated (see, for example, FSA, 2006): on the one hand, the market value of the liability can be calculated on the basis of the fair value principle, which, using the terminology of contingent claim pricing theory, is equivalent to risk neutral valuation. This approach should be adopted when hedges are readily available, like in the case of financial risks. In each case where risks are not hedgeable (like, for example, in the case of some insurance risk), the market value can be calculated as the sum of the expected present value of the liability itself (the so-called best estimate), and an arbitrary but quantified risk margin. Regulators agree on the use of the risk free rate of interest as discount factor to reflect the time value of money (CEIOPS, 2006).

In terms of solvency, instead, the market values of assets and liabilities need to be used to calculate the target capital or Solvency Capital Requirement (SCR); thus, the SCR should reflect the amount of capital required to meet all obligations over a specified time horizon to a defined confidence level. Hence, the calculation of the target capital should be based on suitable risk measures, like VaR and TVaR, over a 1-year time horizon (CEIOPS, 2006).

Given the regulatory framework described above, one of the key factors the insurance companies need to deal with carefully is the development of a suitable valuation model incorporating both the market risk and the insurance risk. The features and the complexity of this model will depend on the nature of the liability to be priced; for example, very common policy types in the insurers' portfolio of products are the so-called participating contracts with minimum guarantee, which are essentially path-dependent contingent claims and, consequently, particularly sensitive to the underlying dynamics of the asset returns.

In the light of the discussion above, the aim of this paper is to analyze the impact on the market consistent price and the target capital of financially sound models for the market risk; to this purpose, we consider the example of a participating contract with minimum guarantee. In recent years, a series of studies have applied classical contingent claim theory to different types of participating contracts, building on the pioneering work of Brennan and Schwartz (1976) on unit-linked policies; thus, amongst some of the most recent works, we would cite Bacinello (2001, 2003), Ballotta et al. (2006.a,.c), Grosen and Jørgensen (2000, 2002), Guillén et al. (2004), and Tanskanen and Lukkarinen (2003). As all these contributions use a Black-Scholes (1973) framework, based on the assumption of a geometric Brownian motion model for the dynamics of the asset fund backing the insurance policy, Ballotta (2005), Kassberger et al. (2006) and Siu et al. (2006) extend the pricing framework to the case of a market specification based on different Lévy processes.

In his respect, this contribution aims at extending these more recent works in two directions. Firstly, we want to assess the relevance of the (financial) model error by calculating the impact on the contract fair value of neglecting or not correctly capturing market shocks. Hence, we compare the performance of three different assumptions for the dynamic of the log-returns of the reference portfolio backing the insurance policy; specifically, we use the traditional Brownian motion, which provides the "standard" model, and two Lévy processes which allow us to depart from the assumption of normal distributed log-returns, and incorporate market shocks. The first alternative is the so-called Merton process (Merton, 1976), given by the sum of an arithmetic Brownian motion and a compound Poisson process; the second alternative is the Variance Gamma (VG) process introduced by Madan and Seneta (1990) and further refined by Madan and Milne (1991) and Madan et al. (1998). Secondly, we assess the mispricing generated by the above mentioned models not only with respect to the fair value of the insurance policy, but also in terms of the target capital. The numerical experiments carried out show that, although the choice of the driving process does not affect significantly the market consistent value of the contract, the impact of the model misspecification becomes relevant when the target capital is involved, since this quantity is computed using the tails of the reference fund's distribution.

The paper is hence organized as follows. In the next section, we present the features of the insurance contract considered for this analysis, and we also introduce the framework for the fair valuation and the calculation of the target capital. We then provide in section 3 the market setup and the resulting market consistent price. In section 4, we describe and test a number of numerical algorithms available to perform the required computations, the results of which are discussed in section 5. The last section presents our concluding remarks on few issues related to the pricing procedure and the model setup.

2 The participating contract: fair valuation and capital requirements

In order to assess the impact of the choice for the market model on the balance sheet of the insurance company and the corresponding capital requirements, we make use of an example based on a participating contract with minimum guarantee. More specifically, for ease of exposition, we adopt the same contract considered in Ballotta (2005); however, we consider the full specification of the policy, allowing for both leverage and terminal bonus rate (like in Ballotta et al., 2006.c, for example). Thus, at maturity the policy pays a guaranteed benefit which incorporates the minimum guarantee and a scheme for the distribution of the annual returns earned by the company's assets (called the reversionary bonus in the UK insurance industry), together with a discretionary benefit which depends on the insurer's final surplus, i.e. the terminal bonus. The accumulation scheme for the guaranteed benefit is based on the smoothed asset share approach. This policy is representative of a typical UK accumulating (unitized) with profit contract. Further, since our focus is on the market model, in this analysis we ignore lapses and mortality.

The features of the participating contract design under analysis will be described in the next section. Based on these features, we proceed to identify the options embedded in the insurance policy, for which we develop a general framework for the calculation of the fair value, and a possible approach for the calculation of the target capital.

2.1 Contract design

As mentioned above, the policy is initiated at time $t = 0$ by the payment of a single premium, P_0, from the policyholder to the insurance company. The premium is invested in the company's assets, A, together with the contribution from the shareholders, E_0; hence, $P_0 = \vartheta A_0$ and $E_0 = (1 - \vartheta) A_0$, where $\vartheta \in (0, 1]$ represents the policyholder contribution or leverage coefficient (Ballotta et al., 2006.c), and A_0 is the value of the insurer's assets at time $t = 0$. The contract entitles the policyholder to receive at maturity, T, an overall benefit given by the guaranteed component, $P(T)$, which includes the minimum guarantee and a scheme for the distribution of the annual returns generated by the reference fund A, and a discretionary component representing the terminal bonus

$$\gamma R(T) = \gamma (\vartheta A(T) - P(T))^+ , \tag{1}$$

where $\gamma \in [0, 1]$ is the terminal bonus rate. Hence, the terminal bonus redistributes part of the final surplus generated by the policyholder share in the insurance company.

As to the accumulation scheme governing the guaranteed benefit, $P(T)$, as mentioned above we follow Ballotta (2005) and adopt the smoothed asset share scheme, so that every year after inception the guaranteed benefit is calculated as

$$\begin{aligned} P(t) &= \alpha P^1(t) + (1 - \alpha) P(t - 1), \qquad \alpha \in (0, 1), \quad t > 0, \\ P(0) &= P_{0,} \end{aligned}$$

where $P^1(t)$ is the unsmoothed asset share defined by

$$\begin{aligned} P^1(0) &= P_0, \\ P^1(t) &= P^1(t - 1)(1 + r_P(t)), \\ r_P(t) &= \max\left\{ r_G, \beta \frac{A(t) - A(t - 1)}{A(t - 1)} \right\}, \end{aligned} \tag{2}$$

and $r_G \in \mathbb{R}^{++}$ and $\beta \in (0, 1)$ are the guaranteed rate and the participation rate respectively. Therefore, at maturity, T, the value of the policy reserve is

$$P(T) = P_0 \left[\alpha \sum_{k=0}^{T-1} (1 - \alpha)^k \prod_{t=1}^{T-k} (1 + r_P(t)) + (1 - \alpha)^T \right]. \tag{3}$$

If, at the claim date, the insurance company is not capable of paying the liability due, then the policyholder sizes the available assets, whilst the shareholders "walk away" empty handed. This implies that the payoff at expiration of the participating contract is

$$\Pi(T) = P(T) + \gamma R(T) - D(T), \tag{4}$$

where $D(T) = (P(T) - A(T))^+$ is the payoff of the so-called default option.

2.2 Fair valuation

If the insurance company aims at setting an initial premium, P_0, which is fair, in the sense that it does not originate arbitrage opportunities (and therefore is market consistent), then

$$P_0 = \hat{\mathbb{E}}\left[\tilde{\Pi}(T)\right],$$

where $\hat{\mathbb{E}}$ denotes the expectation taken under some risk neutral probability measure $\hat{\mathbb{P}}$, and $\tilde{\Pi}$ represents the payoff at maturity discounted at the current risk free rate of interest. Let us define

$$V^P(0) = \hat{\mathbb{E}}\left[\tilde{P}(T)\right]; \quad V^R(0) = \hat{\mathbb{E}}\left[\tilde{R}(T)\right]; \quad V^D(0) = \hat{\mathbb{E}}\left[\tilde{D}(T)\right];$$

then, it follows from equation (4) that the fair value condition returns

$$P_0 + V^D(0) = V^P(0) + \gamma V^R(0). \tag{5}$$

Equation (5) shows that the price of the default option represents an additional premium that the policyholder has to pay in order to gain an "insurance" against a possible default of the company; in this sense, the default option premium can be then interpreted as a safety loading (see Ballotta et al,. 2006.a, .c, and Bernard et al., 2006, for a more detailed discussion of this point).

Further, equation (5) also implies that the fair terminal bonus rate is given by

$$\gamma = \frac{P_0 + V^D(0) - V^P(0)}{V^R(0)}.$$

Hence, if the policyholder's contribution is 100% of the reference fund (i.e. $\vartheta = 1$), then $\gamma = 1$. This is consistent with intuition, since in this case the policyholders would be the only group contributing to the financing of the reference portfolio, and as such they would have the right to receive the entire surplus of the company; consequently they would fix the terminal bonus rate at its maximum value.

2.3 Target capital

As previously mentioned, the market consistent values of assets and liabilities related to insurance contracts are the key ingredients not only for the preparation of the company's balance sheet, but also for the calculation of the capital requirements.

For ease of exposition, in this paper the approach for the calculation of the target capital is based on the comparison between the so-called Risk Bearing Capital (RBC) and the target capital (FOPI, 2004). The RBC is defined as the difference between the total value of the assets and the market consistent price of the liabilities. Thus, we notice that, according to equation (5), the total value of the assets of the insurance company is given by the reference portfolio and the safety loading, i.e. the default option premium, which, in the following, we assume to be invested in the same fund backing the participating contract (in this respect, in this study we assume that the insurance company is passive in terms of risk management). Therefore, the RBC at time $t \in [0, T]$ is given by

$$RBC(t) = A_{tot}(t) - V^P(t) - \gamma V^R(t),$$

where A_{tot} is the total value of the insurer's assets, such that $A_{tot}(0) = A_0 + V^D(0)$. The fair value condition (5) implies that $RBC(0) = A_0 (1 - \vartheta)$. We note that, based on our model, the RBC is a stochastic process evolving under the real probability measure, which depends on the fair value process of the liabilities defined as a conditional expectation under the risk neutral martingale measure.

The target capital is, instead, based on the calculation of a downside risk measure relative to the change in the RBC over a 1 year time horizon. In order to take into account the time value of money, discounting at the risk free rate is applied. This reflects the implicit assumption that the target capital should represent the amount that, once invested in the money market account, guarantees enough capital strength to maintain appropriate policyholder protection and market stability with a certain confidence level. Therefore, the target capital at year t is based on the variation

$$\widetilde{RBC(t+1)} - RBC(t).$$

For ease of exposition of the results, we prefer to construct a solvency index expressing the change in the RBC as a percentage of the value of the total assets of the insurance company at the valuation time, i.e.

$$s_t = \frac{\widetilde{RBC(t+1)} - RBC(t)}{A_{tot}(t)}.$$

In this study, we focus our attention on the TVaR (or Tail Conditional Expectation) with confidence level $1 - x$, i.e.

$$TVaR\,(x; t, t+1) := -\mathbb{E}\left(s_t | \, s_t \leq c_{s_t}\,(x; t, t+1)\right),$$

where c_{s_t} is the VaR of the solvency index s_t with confidence level $1 - x$.

In order to proceed to the actual calculation of the contract market consistent value, and the related distributions of the assets and the liability which are needed to obtain the target capital, we need to specify the relevant market model and the stochastic process driving the reference portfolio. This is covered in the next section, in which we also derive the valuation framework.

3 Market consistent pricing of the embedded options

In order to price the components of the participating contract shown in section 2, we need to define a possible dynamic for the evolution of the price of the fund A. We note at this point that there is no specific recommendation from the regulators as to which model should be adopted; however, a common benchmark seems to be the RiskMetrics model (Mina and Xiao, 2001) with a given number of factors capturing the several sources of risk in the market. The Swiss Solvency Test (FOPI, 2004), for example, recommends a RiskMetrics-based standard asset model with 75 risk factors, including interest rates, FX rates, implied volatilities, credit spreads and hedge funds amongst the others (Keller, 2005). However, such a complex model creates a significant challenge in terms of intuition and understanding (which, consequently, makes transparency of information more difficult to achieve). For this reason, in this paper we prefer to adopt a simpler, parsimonious approach to the modelling of the reference portfolio evolution. More specifically, we rely on the recent advances in the area of financial mathematics, and choose as a standard model the traditional Black-Scholes paradigm. Consequently, assets' log-returns follow a normal distribution which, in spirit, is very similar to the main assumption of the RiskMetrics model. However, since this assumption has been proven not to hold in real markets, we also consider two alternative asset models that depart from the assumption of Gaussian log-returns in order to incorporate market shocks. These two alternatives make use of the Merton process (Merton, 1976) and the Variance Gamma (VG) model (Madan et al., 1990, 1991, 1998).

The idea is to assess the impact of the model error when shocks are either neglected, or not correctly captured by the driving process. In the following sections, we introduce the three asset models and analyze their most relevant features; we then proceed to discuss the issue of market incompleteness originated by the inclusion of shocks in the model, and hence we show how the participating contract introduced in section 2 can be evaluated.

3.1 Market modelling

Consider as given a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ under the real probability measure $\mathbb{P}$, and assume a frictionless market with continuous trading, in which a risk free security $B(t) = e^{rt}, r \in \mathbb{R}^{++}$, is traded. The insurance company's reference portfolio is then assumed to be given by

$$\begin{aligned} A(t) &= A(0)\, e^{L(t)}, \\ A(0) &= A_0, \end{aligned}$$

where L is the process governing the log-returns.

The standard model As mentioned above, the standard asset model proposed in this note agrees with the Black-Scholes paradigm, so that

$$L(t) = \mu t + \sigma_A W(t),$$

where W is a one-dimensional standard Brownian motion under the real probability measure $\mathbb{P}$, $\mu \in \mathbb{R}$ is the mean log-return and $\sigma_A \in \mathbb{R}^{++}$ is the instantaneous volatility. It follows that the expected rate of growth on the fund is $\mu + \sigma_A^2/2$.

It is, however, a well known fact that asset log-returns exhibit fatter tails than those accommodated by the normal distribution, implying a misunderestimation of the likelihood of extreme events.

The Merton process-based model In order to take into account the occurrence of market shocks, the first alternative we propose is based on the so-called Merton process (Merton, 1976), which is given by the sum of a Brownian motion with drift and an independent compound Poisson process. Thus

$$L(t) = (n - \lambda \mu_X) t + \sigma W(t) + \sum_{k=1}^{N(t)} X(k), \quad n, \mu_X \in \mathbb{R}, \sigma \in \mathbb{R}^{++}, \tag{6}$$

where W is a one-dimensional standard $\mathbb{P}$-Brownian motion capturing the "marginal" price changes; $X \sim N(\mu_X, \sigma_X^2)$ models the size of the jumps, i.e. the "abnormal" changes in the prices due to the arrival of important new information, whose flow is regulated by a Poisson process, N, of rate $\lambda \in \mathbb{R}^{++}$. Note that W, N and X are assumed to be independent one of the other, which implies that L is a Lévy process. In particular, the characteristic function of the Merton process is

$$\begin{aligned} \phi_L(u;t) &= e^{t\left(iu(n-\lambda\mu_X) - u^2 \frac{\sigma^2}{2} + \lambda(\phi_X(u)-1)\right)}, \\ \phi_X(u) &= e^{iu\mu_X - u^2 \frac{\sigma_X^2}{2}}; \end{aligned} \tag{7}$$

consequently, the characteristic triplet of the process L is $(n - \lambda\mu_X, \sigma, \upsilon_M(dx))$, with

$$\upsilon_M(dx) = \frac{\lambda}{\sigma_X \sqrt{2\pi}} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} dx.$$

Hence, the Merton process is a finite activity process. It follows that the mean log-return is $n \in \mathbb{R}$, whilst the instantaneous variance is $\sigma^2 + \lambda(\mu_X^2 + \sigma_X^2)$ and the expected rate of growth on the fund A is $(n - \lambda\mu_X) + \sigma^2/2 + \lambda(\phi_X(1) - 1)$. The Merton process exhibits skewness and kurtosis as described by the Pearson index of asymmetry

$$\gamma_1 = \frac{\lambda \mu_X (\mu_X^2 + 3\sigma_X^2)}{(\sigma^2 + \lambda(\mu_X^2 + \sigma_X^2))^{3/2}},$$

and the excess of kurtosis index

$$\gamma_2 = \frac{\lambda(\mu_X^4 + 6\mu_X^2\sigma_X^2 + 3\sigma_X^4)}{(\sigma^2 + \lambda(\mu_X^2 + \sigma_X^2))^2}.$$

Finally, we observe that

$$\begin{aligned} sign(\gamma_1) &= sign(\mu_X); \\ \gamma_2 &> 0. \end{aligned}$$

Therefore, the distribution of the Merton process is positively or negatively skewed according to the sign of the expected jump' size; further, it is leptokurtic.

The Variance Gamma process-based model A recent analysis offered by Carr et al. (2002) shows that, in general, market prices lack of a diffusion component, as if it was diversified away; consequently, they conclude that there is an argument for using pure jump processes, particularly of infinite activity and finite variation given their ability to capture both frequent small changes and rare large jumps. A process of this kind used in finance due to its analytical and numerical tractability is the Variance Gamma (VG) process, which is a normal tempered stable process obtained by time changing an arithmetic Brownian motion by a gamma subordinator. We follow this approach and define the second alternative asset model by

$$L(t) = (m - \theta)\, t + Z(t), \quad m \in \mathbb{R}, \tag{8}$$

where

$$Z(t) = \theta \tau_t + \xi W(\tau_t), \qquad \theta \in \mathbb{R}, \xi \in \mathbb{R}^{++},$$

is the VG process, $W = (W_t : t \geq 0)$ is a standard Brownian motion and $\tau = (\tau_t : t \geq 0)$ is a gamma process, with parameters a, $b > 0$, and independent of W. The parameter a represents the time scale of the process, i.e. it alters the intensity of the jumps of all sizes simultaneously, whilst the parameter b captures the decay rate of big jumps. It is easy to show that the characteristic function of the process L is

$$\phi_L(u,t) = e^{iu(m-\theta)t} \left[\frac{b}{b - iu\theta + u^2 \frac{\xi^2}{2}} \right]^{at}. \tag{9}$$

The VG process has been introduced by Madan and Seneta (1990), and has been further refined by Madan and Milne (1991) and Madan et al. (1998); in particular, these authors consider as subordinator a gamma process with unit mean rate, i.e. with parameters $a = b = 1/k$, where $k \in \mathbb{R}^{++}$ is the variance rate. For this parametrization, Z is $VG(\theta, \xi, 1/k, 1/k)$, and therefore

$$\phi_L(u,t) = e^{iu(m-\theta)t} \left[\frac{1}{1 - iu\theta k + u^2 \frac{\xi^2}{2} k} \right]^{\frac{t}{k}}. \tag{10}$$

Note that equation (10) implies that the VG process is well defined for

$$\frac{-\theta - \sqrt{\theta^2 + \frac{2\xi^2}{k}}}{\xi^2} < \Re(z) < \frac{-\theta + \sqrt{\theta^2 + \frac{2\xi^2}{k}}}{\xi^2} \qquad z \in \mathbb{C}.$$

The VG process Z can also be represented as the difference between two independent gamma processes, which follows from the fact that

$$\phi_Z(u,t) = \left[\frac{1}{1 - iu\theta k + u^2 \frac{\xi^2}{2} k} \right]^{\frac{t}{k}} = \left[\frac{b_+}{b_+ - iu} \right]^{\frac{t}{k}} \left[\frac{b_-}{b_- + iu} \right]^{\frac{t}{k}}, \tag{11}$$

where

$$\begin{aligned} b_+ &= \frac{2}{k\left(\sqrt{\theta^2+\frac{2\xi^2}{k}}+\theta\right)}, \\ b_- &= \frac{2}{k\left(\sqrt{\theta^2+\frac{2\xi^2}{k}}-\theta\right)}. \end{aligned}$$

Consequently, the Lévy measure of the process $Z(t)$ is given by

$$\upsilon_Z(dx) = \frac{1}{k}|x|^{-1}\left(e^{-b_+x}1_{(x>0)} + e^{b_-x}1_{(x<0)}\right), \tag{12}$$

and the characteristic triplet of the process L is $(m-\theta, 0, \upsilon_Z(dx))$.

In this model, the mean log-return is given by $m \in \mathbb{R}$; the instantaneous variance is $(\xi^2+\theta^2 k)$; the expected rate of growth on A is $m-\theta-\frac{1}{k}\ln\left(1-\theta k-\frac{\xi^2}{2}k\right)$. As far as skewness and kurtosis are concerned, the Pearson index of asymmetry and the excess of kurtosis index are respectively

$$\begin{aligned} \gamma_1 &= \frac{(3\xi^2\theta k + 2\theta^3 k^2)}{(\xi^2+\theta^2 k)^{3/2}} \\ \gamma_2 &= \frac{(3\xi^4 k + 12\xi^2\theta^2 k^2 + 6\theta^4 k^3)}{(\xi^2+\theta^2 k)^2}. \end{aligned}$$

Therefore, the VG distribution is positively or negatively skewed according to whether $\theta > 0$ or $\theta < 0$, since $sign(\gamma_1) = sign(\theta)$; further, we observe that $\gamma_2 > 0$ and, consequently, the distribution is leptokurtic as well.

3.2 Pricing the embedded options

The models proposed in the previous section have as common feature the fact that the driving process is a Lévy process, i.e. a process with independent and stationary increments. This actually allows us to reduce the problem of obtaining the price of the guaranteed benefit $P(T)$ to the pricing of a European call option.

The payoff equation (3), in fact, implies that

$$P(T) = \alpha \sum_{k=0}^{T-t-1} (1-\alpha)^k P^1(t) \prod_{i=1}^{T-t-k} (1+r_P(t+i)) + (1-\alpha)^{T-t} P(t);$$

therefore

$$\begin{aligned} V^P(t) &= \hat{\mathbb{E}}\left[\tilde{P}(T)\middle|\mathbb{F}_t\right] \\ &= \alpha P^1(t) \sum_{k=0}^{T-t-1} e^{-rk}(1-\alpha)^k \prod_{i=1}^{T-t-k} \hat{\mathbb{E}}\left[e^{-r}(1+r_P(t+i))\middle|\mathbb{F}_t\right] \\ &\quad + e^{-r(T-t)}(1-\alpha)^{T-t} P(t). \end{aligned} \tag{13}$$

Because of equation (2), it follows that

$$\hat{\mathbb{E}}\left[e^{-r}\left(1+r_P\left(t+i\right)\right)\middle|\,\mathbb{F}_t\right]=e^{-r}\left(1+r_G\right)+\hat{\mathbb{E}}\left[e^{-r}\left(\beta e^{L'(1)}-\left(\beta+r_G\right)\right)^+\right],\tag{14}$$

where L' denotes an independent copy of the Lévy process L. Analytical formulae are hence available for this part of the participating contract (see, for example, Ballotta, 2005); we note, though, that it is not possible to derive analytical formulae for both the terminal bonus and the default option given the complex recursive nature of $P(T)$, and their high dependency on the path of the reference fund A; hence we resort to Monte Carlo simulation in order to approximate the price of these two components of the insurance contract. We need however to specify the risk neutral martingale measure $\hat{\mathbb{P}}$: except for the Brownian motion case, in fact, the market is incomplete due to the presence of market shocks and, therefore, there are infinitely many pricing measures.

For the purpose of this analysis, we select the Esscher risk neutral martingale measure; however, such a probability measure imposes a specific form of the investors' preferences. A more general approach would be to extract the market's pricing measure from option prices via calibration. However, the portfolio backing insurance policies is in general a mixture of securities such as equities, bonds and properties, that rarely is tradeable in the market. Consequently, the "market-implied martingale measure" cannot be extracted due to the lack of suitable option prices. In the remaining of this section, we show how the risk neutral Esscher measure can be determined for the models considered in this paper, and how it is used to price the guaranteed benefit $P(T)$.

3.2.1 The Esscher transform and the risk neutral martingale Esscher measure

Let $L(t)$ be a Lévy motion; then the process

$$\eta(t)=\left\{e^{hL(t)}\phi_L\left(\frac{h}{i},t\right)^{-1}:t\geq 0\right\},\tag{15}$$

is a positive $\mathbb{P}$-martingale that can be used to define a change of probability measure, i.e. the Radon-Nikodým derivative of a new equivalent probability measure $\hat{\mathbb{P}}^h$, called the Esscher measure of parameter h. The process $\eta(t)$ is called the Esscher transform of parameter h. If we use the Esscher transform to determine a risk neutral martingale measure, i.e. a measure under which discounted asset prices behave like martingales, the Esscher parameter h needs to satisfy the following condition (see, for example, Gerber and Shiu, 1994):

$$r=\ln\phi_L\left(\frac{h+1}{i},1\right)-\ln\phi_L\left(\frac{h}{i},1\right).\tag{16}$$

In virtue of equation (7), the $\hat{\mathbb{P}}^h$-characteristic function of the log-returns driven by the Merton process is

$$\begin{aligned}
\hat{\phi}_L^h(u,t) &= \frac{\phi_L\left(\frac{iu+h}{i},t\right)}{\phi_L\left(\frac{h}{i},t\right)} = e^{t\left(iu\left(n+h\sigma^2-\lambda\mu_X\right)-u^2\frac{\sigma^2}{2}+\lambda^h\left(\hat{\phi}_X^h(u)-1\right)\right)} \\
\lambda^h &= \lambda e^{h\mu_X+h^2\frac{\sigma_X^2}{2}}, \\
\hat{\phi}_X^h(u) &= e^{iu\left(\mu_X+h\sigma_X^2\right)-u^2\frac{\sigma_X^2}{2}}
\end{aligned}$$

This implies that the $\hat{\mathbb{P}}^h$-triplet is $(n+h\sigma^2-\lambda\mu_X,\sigma,\hat{\upsilon}_M(dx))$, with

$$\hat{\upsilon}_M(dx) = \frac{\lambda^h}{\sigma_X\sqrt{2\pi}}e^{-\frac{\left(x-\left(\mu_X+h\sigma_X^2\right)\right)^2}{2\sigma_X^2}}dx.$$

From the risk neutral condition (16) it follows that

$$r = n+h\sigma^2-\lambda\mu_X+\frac{\sigma^2}{2}+\lambda^h\left(\hat{\phi}_X^h(1)-1\right);$$

therefore, the reference fund under $\hat{\mathbb{P}}^h$ is given by

$$A(t) = A(0)\,e^{\left(r-\frac{\sigma^2}{2}-\lambda^h\left(\hat{\phi}_X^h\left(\frac{1}{i}\right)-1\right)\right)t+\sigma\hat{W}^h(t)+\sum_{k=1}^{\hat{N}^h(t)}\hat{X}^h(k)}, \tag{17}$$

where $\hat{W}^h$ is a $\hat{\mathbb{P}}^h$-Brownian motion, $\hat{N}^h$ is a $\hat{\mathbb{P}}^h$-Poisson process with rate λ^h, and $\hat{X}^h \sim N\left(\mu_X+h\sigma_X^2,\sigma_X^2\right)$ under $\hat{\mathbb{P}}^h$.

Similarly, using equation (10), it follows that the characteristic function of the VG-based log-returns under $\hat{\mathbb{P}}^h$ is

$$\hat{\phi}_L^h(u,t) = e^{iu(m-\theta)t}\left[\frac{1}{1-iu\theta^h k^h+u^2\frac{\xi^2}{2}k^h}\right]^{\frac{t}{k}}, \tag{18}$$

where

$$\begin{aligned}
\theta^h &= \theta+h\xi^2, \\
k^h &= \frac{k}{1-h\theta k-h^2\frac{\xi^2}{2}k}.
\end{aligned}$$

Moreover, the Esscher measure exists if and only if $\phi_Z\left(\frac{h}{i},t\right)$ exists (similar results but for a simpler setting have been obtained also by Hubalek and Sgarra, 2005), i.e. if and only if

$$\frac{-\theta-\sqrt{\theta^2+\frac{2\xi^2}{k}}}{\xi^2} < h < \frac{-\theta+\sqrt{\theta^2+\frac{2\xi^2}{k}}}{\xi^2}.$$

The $\hat{\mathbb{P}}^h$- Lévy measure can be easily recovered either from equation (18), or via exponential tilting of the $\mathbb{P}$- Lévy measure (12), so that

$$\hat{\upsilon}^h_Z(dx) = e^{hx}\upsilon_Z(dx) = \frac{1}{k}|x|^{-1}\left(e^{-b^h_+ x}1_{(x>0)} + e^{b^h_- x}1_{(x<0)}\right), \tag{19}$$

$$b^h_+ = \frac{2 - hk\left(\sqrt{\theta^2 + \frac{2\xi^2}{k}} + \theta\right)}{k\left(\sqrt{\theta^2 + \frac{2\xi^2}{k}} + \theta\right)}, \tag{20}$$

$$b^h_- = \frac{2 + hk\left(\sqrt{\theta^2 + \frac{2\xi^2}{k}} + \theta\right)}{k\left(\sqrt{\theta^2 + \frac{2\xi^2}{k}} - \theta\right)}. \tag{21}$$

The Esscher parameter h is the solution to

$$f(h) = r - (m - \theta) - \frac{1}{k}\ln\frac{1 - h\theta k - h^2\frac{\xi^2}{2}k}{1 - (h+1)\theta k - (h+1)^2\frac{\xi^2}{2}k} = 0. \tag{22}$$

Note that the function $f(h)$ exists only for $h \in \left(\frac{-\theta - \sqrt{\theta^2 + \frac{2\xi^2}{k}}}{\xi^2}, \frac{-\theta + \sqrt{\theta^2 + \frac{2\xi^2}{k}}}{\xi^2} - 1\right)$, provided that $\theta^2 + \frac{2\xi^2}{k} > \frac{\xi^4}{4}$. Further, let

$$h_1 = \frac{-\theta - \sqrt{\theta^2 + \frac{2\xi^2}{k}}}{\xi^2}; \quad h_2 = \frac{-\theta + \sqrt{\theta^2 + \frac{2\xi^2}{k}}}{\xi^2} - 1,$$

then

$$\lim_{h\to h_1} f(h) = \infty; \qquad \lim_{h\to h_2} f(h) = -\infty;$$

hence, equation (22) admits at least one solution on the support of f. Solving equation (22) directly, we obtain that

$$h^* = -\frac{\theta}{\varepsilon} - \frac{1}{\varepsilon} \pm \frac{1}{\xi^2\varepsilon}\sqrt{\xi^4 + \theta^2\varepsilon^2 - \xi^4\varepsilon + \frac{2\xi^2}{k}\varepsilon^2}$$
$$\varepsilon = 1 - e^{k(m-\theta-r)}.$$

This solution to equation (22) together with equation (18) fully characterizes the risk neutral dynamic of the stock price process under $\hat{\mathbb{P}}^h$, which, in virtue of the Girsanov theorem, is given by

$$A(t) = A(0)\,e^{\left(r - \ln\hat{\phi}^h_Z\left(\frac{1}{i},1\right)\right)t + \hat{Z}^h(t)}, \tag{23}$$

and $\hat{Z}^h$ is $VG\left(\theta^h, \xi, 1/k, 1/k^h\right)$.

3.2.2 Pricing of the guaranteed benefit P

Based on the results presented in the previous section, it is possible to solve analytically equation (14), and therefore determine closed formulae for the price of the guaranteed benefit, expressed in equation (13), under the three market paradigms introduced in section 3.1. The result is summarized in the following

Proposition 1 *The market consistent value of the guaranteed benefit $P(T)$, when the reference fund is driven by a Lévy process, is (for the risk neutral Esscher measure $\hat{\mathbb{P}}^h$)*

$$V^P(t) = \alpha P^1(t) \sum_{k=0}^{T-t-1} e^{-rk}(1-\alpha)^k \left[e^{-r}(1+r_G) + \beta \hat{\mathbb{E}}^h\left(e^{-r} e^{L'(1)} 1_{\left(L'(1) > \frac{\beta+r_G}{\beta}\right)} \right) - e^{-r}(\beta+r_G)\hat{\mathbb{P}}^h\left(L'(1) > \frac{\beta+r_G}{\beta} \right) \right]^{T-t-k} + e^{-r(T-t)}(1-\alpha)^{T-t} P(t). \quad (24)$$

In particular, let Φ denote the distribution of the standard normal random variable. Then

i) under the standard model, $V^P(t)$ is

$$\alpha P^1(t) \sum_{k=0}^{T-t-1} e^{-rk}(1-\alpha)^k \left[e^{-r}(1+r_G) + \beta\Phi(d_1) - e^{-r}(\beta+r_G)\Phi(d_2) \right]^{T-t-k} + e^{-r(T-t)}(1-\alpha)^{T-t} P(t), \quad (25)$$

where

$$d_1 = \frac{\ln\frac{\beta}{\beta+r_G} + \left(r + \frac{\sigma_A^2}{2}\right)}{\sigma_A}; \quad d_2 = d_1 - \sigma_A.$$

ii) Under the Merton model, $V^P(t)$ is given by

$$\alpha P^1(t) \sum_{k=0}^{T-t-1} e^{-rk}(1-\alpha)^k \left[e^{-r}(1+r_G) + \beta \sum_{n=0}^{\infty} \frac{e^{-\lambda^h \hat{\phi}_X^h\left(\frac{1}{i}\right)} \left(\lambda^h \hat{\phi}_X^h\left(\frac{1}{i}\right)\right)^n}{n!} \Phi(d_{n;h}) - e^{-r}(\beta+r_G) \sum_{n=0}^{\infty} \frac{e^{-\lambda^h}\left(\lambda^h\right)^n}{n!} \Phi\left(d'_{n;h}\right) \right]^{T-t-k} + e^{-r(T-t)}(1-\alpha)^{T-t} P(t), \quad (26)$$

where

$$d_{n;h} = \frac{\ln\frac{\beta}{\beta+r_G} + \left(r_{n;h} + \frac{v_n^2}{2}\right)}{v_n}, \quad d'_{n;h} = d_{n;h} - v_n,$$

and

$$r_{n;h} = r - \lambda^h\left(\hat{\phi}_X^h\left(\frac{1}{i}\right) - 1\right) + n \ln \hat{\phi}_X^h\left(\frac{1}{i}\right),$$
$$v_n^2 = \sigma^2 + n\sigma_X^2.$$

iii) Under the VG model, $V^P(t)$ is

$$\alpha P^1(t)\sum_{k=0}^{T-t-1}e^{-rk}(1-\alpha)^k\left[e^{-r}(1+r_G)+\beta\Psi\left(d\sqrt{1-s},\frac{\theta^h+\xi^2}{\xi\sqrt{1-s}},\frac{1}{k}\right)\right.$$
$$\left.-e^{-r}(\beta+r_G)\Psi\left(d,\frac{\theta^h}{\xi},\frac{1}{k}\right)\right]^{T-t-k}+e^{-r(T-t)}(1-\alpha)^{T-t}P(t), \qquad (27)$$

where

$$\begin{aligned}\Psi(a,b,c) &= \int_0^\infty \Phi\left(\frac{a}{\sqrt{\tau}}+b\sqrt{\tau}\right)\frac{\tau^{c-1}e^{-\frac{\tau}{k^h}}}{(k^h)^c\Gamma(c)}d\tau;\\ d &= \frac{\ln\frac{\beta}{\beta+r_G}+r-\ln\hat{\phi}_Z^h\left(\frac{1}{i},1\right)}{\xi}, \quad s=k^h\left(\theta^h+\frac{\xi^2}{2}\right).\end{aligned}$$

Proof. We only focus on the calculation the expectation in equation (14), bearing in mind that the pricing measure is the risk neutral Esscher measure $\hat{\mathbb{P}}^h$ defined in the previous section. Hence, we note that

$$\hat{\mathbb{E}}^h\left[e^{-r}\left(\beta e^{L'(1)}-(\beta+r_G)\right)^+\right]=\hat{\mathbb{E}}^h\left(e^{-r}e^{L'(1)}1_{\left(L'(1)>\frac{\beta+r_G}{\beta}\right)}\right)-\hat{\mathbb{P}}^h\left(L'(1)>\frac{\beta+r_G}{\beta}\right).$$

Now, we calculate the expectation and the probability in the previous expression under the three market model introduced in this paper.

i) The result follows from the application of the Black-Scholes formula (see Ballotta, 2005).

ii) Note that, conditioning on the number of jumps occurring in 1 year, the process $L'(1)$ follows a normal distribution with mean $r_{n;h}-\frac{v_n^2}{2}$ and variance v_n^2. Therefore,

$$\hat{\mathbb{E}}^h\left(e^{L'(1)}1_{\left(L'(1)>\frac{\beta+r_G}{\beta}\right)}\middle|\hat{N}^h(1)=n\right)=e^{r_{n;h}}\Phi(d_{n;h});$$

consequently

$$\begin{aligned}\hat{\mathbb{E}}^h\left(e^{-r}e^{L'(1)}1_{\left(L'(1)>\frac{\beta+r_G}{\beta}\right)}\right) &= \sum_{n=0}^\infty\frac{e^{-\lambda^h}\left(\lambda^h\right)^n}{n!}e^{r_{n;h}}\Phi(d_{n;h})\\ &= \sum_{n=0}^\infty\frac{e^{-\lambda^h\hat{\phi}_X^h\left(\frac{1}{i}\right)}\left(\lambda^h\hat{\phi}_X^h\left(\frac{1}{i}\right)\right)^n}{n!}\Phi(d_{n;h}).\end{aligned}$$

Moreover,

$$\begin{aligned}\hat{\mathbb{P}}^h\left(L'(1)>\frac{\beta+r_G}{\beta}\right) &= \sum_{n=0}^\infty\frac{e^{-\lambda^h}\left(\lambda^h\right)^n}{n!}\hat{\mathbb{P}}^h\left(L'(1)>\frac{\beta+r_G}{\beta}\middle|\hat{N}^h(1)=n\right)\\ &= \sum_{n=0}^\infty\frac{e^{-\lambda^h}\left(\lambda^h\right)^n}{n!}\Phi\left(d'_{n;h}\right),\end{aligned}$$

as required.

iii) In this case, we condition on the random time $\hat{\tau}^h(1)$, which implies that the process $L'(1)$ follows a normal distribution with mean $r_{\tau;h} - \frac{\xi^2}{2}$ and variance ξ^2, where $r_{\tau;h} = \frac{r - \ln \hat{\phi}_Z^h\left(\frac{1}{i},1\right)}{\tau} + \theta^h + \frac{\xi^2}{2}$. Therefore,

$$\hat{\mathbb{E}}^h\left(e^{L'(1)} 1_{\left(L'(1) > \frac{\beta + r_G}{\beta}\right)} \middle| \hat{\tau}^h(1) = \tau \right) = e^{r_{\tau;h}} \Phi\left(d_{\tau;h}\right);$$

consequently

$$\hat{\mathbb{E}}^h\left(e^{-r} e^{L'(1)} 1_{\left(L'(1) > \frac{\beta + r_G}{\beta}\right)} \right) = e^r \int_0^\infty \hat{\phi}_Z^h\left(\frac{1}{i},1\right)^{-1} e^{\left(\theta^h + \frac{\xi^2}{2}\right)\tau} \Phi\left(d_{\tau;h}\right) \frac{\tau^{\frac{1}{k}-1} e^{-\frac{\tau}{k^h}}}{\left(k^h\right)^{\frac{1}{k}} \Gamma\left(\frac{1}{k}\right)} d\tau,$$

with

$$d_{\tau;h} = \frac{\ln \frac{\beta}{\beta + r_G} + \left(r_{\tau;h} + \frac{\xi^2}{2}\right)}{\xi\sqrt{\tau}}.$$

Further, set $s = k^h\left(\theta^h + \frac{\xi^2}{2}\right)$ and $u = (1-s)\tau$. Then

$$\hat{\mathbb{E}}^h\left(e^{-r} e^{L'(1)} 1_{\left(L'(1) > \frac{\beta + r_G}{\beta}\right)} \right) = e^r \int_0^\infty \Phi\left(d_{u;h}\right) \frac{u^{\frac{1}{k}-1} e^{-\frac{u}{k^h}}}{\left(k^h\right)^{\frac{1}{k}} \Gamma\left(\frac{1}{k}\right)} du$$

where

$$d_{u;h} = \frac{d\sqrt{1-s}}{\sqrt{u}} + \frac{\theta^h + \xi^2}{\xi\sqrt{1-s}}\sqrt{u}.$$

Moreover,

$$\begin{aligned}
\hat{\mathbb{P}}^h\left(L'(1) > \frac{\beta + r_G}{\beta} \right) &= \int_0^\infty \frac{\tau^{\frac{1}{k}-1} e^{-\frac{\tau}{k^h}}}{\left(k^h\right)^{\frac{1}{k}} \Gamma\left(\frac{1}{k}\right)} \hat{\mathbb{P}}^h\left(L'(1) > \frac{\beta + r_G}{\beta} \middle| \hat{\tau}^h(1) = \tau \right) d\tau \\
&= \int_0^\infty \frac{\tau^{\frac{1}{k}-1} e^{-\frac{\tau}{k^h}}}{\left(k^h\right)^{\frac{1}{k}} \Gamma\left(\frac{1}{k}\right)} \Phi\left(d'_{\tau;h}\right) d\tau, \\
d'_{\tau;h} &= \frac{d}{\sqrt{\tau}} + \frac{\theta^h}{\xi}\sqrt{\tau},
\end{aligned}$$

which concludes the proof.

■

For the case of the model based on the VG process, a closed form for the option price can be obtained by using the closed formula for the function $\Psi(a,b,c)$ developed by Madan et al. (1998) (Theorem 2, equation $(A.11)$), which involves the modified Bessel function of the second kind and the degenerate hypergeometric function. However, the evaluation of these special functions is computationally time-consuming; for this reason, we resort to numerical methods for the computation of the price of the guaranteed benefit. Further, in order to obtain the fair value of the participating contract we also need to evaluate the premium of the terminal bonus option and the default option, which can only be done using a Monte Carlo procedure as discussed before. The price $V^P(t)$ can be then easily obtained as a by-product of these computations.

	Sequential Monte Carlo (with variance reduction)			Stratified Monte Carlo		
	GBM	**Merton**	**VG**	**GBM**	**Merton**	**VG**
$\mathbf{V}^P(0)$	190.7750	191.7985	191.4060	190.773	191.812	191.8260
	(0.00003%)	(0.00003%)	(0.00002%)	(0.00003%)	(0.00004%)	(0.00003%)
$\mathbf{V}^R(0)$	8.72811	9.02418	9.3426	8.7261	9.0228	9.3099
	(0.00084%)	(0.00083%)	(0.46%)	(0.61%)	(0.59%)	(0.59%)
$\mathbf{V}^D(0)$	99.5084	100.759	100.8730	99.6916	100.756	100.805
	(0.00006%)	(0.00006%)	(0.04%)	(0.11%)	(0.11%)	(0.11%)
$\mathbf{V}^P(0)$ (closed analytical formula)						
GBM	190.7739					
Merton	191.8112					

Table 1: Fair values of the embedded option for the benchmark set of parameters specified in section 5, and $\vartheta = 1$. Values based on 1,000,000 Monte Carlo simulations. The numbers in parenthesis represent the error of the estimate expressed as percentage of the corresponding option price

	n. of runs	**10,000**	**100,000**	**1,000,000**
GBM model	$\mathbf{V}^R(0)$	0.030	0.006	0.0020
	$\mathbf{V}^D(0)$	0.006	0.002	0.0006
Merton model	$\mathbf{V}^R(0)$	0.049	0.010	0.0032
	$\mathbf{V}^D(0)$	0.017	0.003	0.0011
	$\mathbf{V}^P(0)$	1.07	6.34	12.01
VG model	$\mathbf{V}^R(0)$	1.86	2.10	2.54
	$\mathbf{V}^D(0)$	1.07	1.08	1.19

Table 2: Stratified sampling vs Sequential sampling with variance reduction. The table reports the efficiency gain, E_{AB} of method A (stratified Monte Carlo) to method B (sequential Monte Carlo). If $E_{AB} > 1$, then method A is more efficient than method B; in fact, E_{AB} is a multiple of the time method B takes to achieve a particular standard deviation compared to method A.

4 Numerical algorithms

As mentioned in the previous section, we need to develop numerical algorithms for the pricing of the participating contract analyzed, and for the calculation of the corresponding risk margin and capital requirements. Since this represents a critical issue for insurance companies which need to develop suitable software architectures, in this section we review the available alternative algorithms, and test their efficiency for the case of the contract considered in this note.

In order to price the guaranteed benefit, we use the closed analytical formulae developed in the previous section for both the cases of the standard asset model and the Merton-based model; for the case of the VG-based model, instead, we resort to Monte Carlo simulation.

In more details, the infinite series determining the Poisson distribution for the calculation of V^P under the Merton model (see equation (26)) is computed with relative tolerance $\epsilon = 10^{-15}$; in other words, we neglect all terms smaller than ϵ times the current sum.

The price of both the terminal bonus option and the default option are obtained by Monte Carlo method, irrespective of the model, due to their path-dependent payoff design.

All Monte Carlo codes generate random deviate by inversion of the distribution function using the Newton algorithm, except for the case of beta deviates (see below). Further, variance reduction techniques are used in order to speed up the convergence and improve the accuracy of the estimates. Specifically, for all models considered in this paper, we develop both sequential and stratified algorithms. The sequential algorithms use, for variance reduction purposes, the antithetic variate method in addition with, wherever possible, the control variate technique. The stratified algorithms make use of the Latin Hypercube Sampling technique to generate uniform deviates; further, the Brownian bridge and the Brownian-Gamma bridge (Ribeiro and Webber, 2004) are used to produce respectively the paths of the Brownian motion and the Variance Gamma process. As far as the Merton process is concerned, we use the standard property that, given the (stratified) total number of jumps in $[0, T]$, $N(T) = k$, the arrival times of the jumps have the the joint distribution of the order statistics of k independent random variables uniformly distributed over $[0, T]$ (see Glasserman, 2004). The gamma function and the lower incomplete gamma function are computed using the algorithms recommended in the Numerical Recipes in C++ (Press et al., 2002). Finally, the beta deviates from a Beta distribution, $B(a, b)$, required for the gamma bridge are generated using the algorithms of Atkinson and Whittaker for the cases in which $a, b < 1$ with $a+b \geq 1$, and $a \leq 1, b \geq 1$; the Johnk's algorithm is used instead for the case in which $a, b < 1$ with $a+b < 1$ (see Devroye, 1986).

For benchmarking purposes, we compute V^P by Monte Carlo simulation as well using equation (24), with numerical approximation restricted to the embedded 1-year European call option. Results are shown in Table 1.

In order to test the efficiency of the algorithms developed, we consider the efficiency gain index, E_{AB}, of the stratified sampling procedure (method A with standard error σ_A and execution time t_A) versus the sequential sampling algorithm (method B with standard error σ_B and execution time t_B), which is defined as

$$E_{AB} = \frac{\sigma_B^2 t_B}{\sigma_A^2 t_A}$$

Results are shown in Table 2.

The results reported in Tables 1 and 2 show that for both the cases of the standard asset model and the Merton process model, Monte Carlo with variance reduction provides the best prices in terms of both accuracy, as measured by the standard error, and efficiency, as quantified by the efficiency index E. The reason for this high performance has to be sought for in the combination of the antithetic variate technique with the control variate method, which is made possible by the closed analytical formulae developed for the guaranteed benefit in these two market frameworks. Stratified Monte Carlo is, instead, the most competitive method for the VG based model, above all when it is used to calculate the price of the guaranteed benefit.

In virtue of these results, in the remaining of the analysis, we will refer to the estimates generated by the sequential algorithm for the standard model and the Merton model; whilst we will use the estimates generated by the stratified algorithm for the VG model.

5 Results

In this section, we use the results presented in section 3 and the numerical algorithms discussed in section 4 to analyze the impact of a model misspecification on the market consistent price of the participating contract, and the corresponding target capital.

For the analysis to be consistent, the parameters of the models considered in this note are chosen so that the first four moments of the underlying distributions of the asset log-returns are matched as closely as possible under the real probability measure. The base set of parameters is presented in Table 3; the resulting moments are instead reported in Table 4.

The mispricing of the benefits' prices are analysed in Table 5 for $\vartheta = 1$ only, since the leverage coefficient ϑ is a mere rescaling factor of the value of the benefits; equations (1) and (3) in fact imply

$$\begin{aligned} P(T) &= \vartheta A(0)\left[\alpha \sum_{k=0}^{T-1} (1-\alpha)^k \prod_{t=1}^{T-k} (1 + r_P(t)) + (1-\alpha)^T\right] = \vartheta P^U(T), \\ R(T) &= \vartheta \left(A(T) - P^U(T)\right) = \vartheta R^U(T). \end{aligned}$$

Therefore

$$V^P(0) = \vartheta V_U^P(0); \quad V^R(0) = \vartheta V_U^R(0);$$

where $V_U^{\cdot}$ denotes the "unlevered" option's value. The same argument though does not apply to the default option, as its value depends on the leverage coefficient as shown in Table 6; the mispricing generated by the model error is reported in Table 7.

The results show that the standard asset model underprices each single component of the insurance contract, although the mispricing is particularly significant for the case of the terminal bonus (call) option and the default (put) option. A more detailed examination shows that both the Merton process and the VG process originate leptokurtic and negatively skewed distributions, although the kurtosis is higher in the case of the VG process. This is reflected in the mispricing of both the terminal bonus and the default option, in particular when the default option is deep out-of-the-money, i.e. for low values of ϑ. The mispricing is less severe when shocks are somehow incorporated in the model: although the VG process overprices out-of-the-money default options when compared to the Merton model, the price difference reduce sensibly as ϑ approaches 1. The observed mispricing is also reflected in the terminal bonus rate γ; for the benchmark set of parameters we obtain $\gamma = 14.17\%$ under the standard asset model, $\gamma = 18.19\%$ under the Merton process model, and $\gamma = 26.26\%$ under the VG model.

Market models parameters	
Standard model (GBM)	$\mu = 10\%$ p.a.; $\sigma_A = 20\%$ p.a.
Merton model	$n = 10\%$ p.a.; $\sigma = 18.82\%$ p.a.; .
	$\lambda = 0.59$; $\mu_X = -5.37\%$ p.a.; $\sigma_X = 7\%$ p.a.
VG model	$m = 10\%$ p.a.; $\theta = -3.04\%$ p.a.; $k = 0.15$; $\xi = 19.56\%$ p.a.
	$r = 3.5\%$ p.a.
Contract parameters	
$P_0 = 100$; $T = 20$ years; $\alpha = 60\%$; $\beta = 50\%$; $\vartheta = 90\%$; $r_G = 4\%$ p.a.	

Table 3: Base parameter set. The parameters are taken from Ballotta (2005).

	GBM model	**Merton model**	**VG model**
Expected rate of growth	0.12	0.1199	0.1199
$\mathbb{E}(L_1)$	0.1	0.1	0.1
	(0.1)	(0.1)	(0.1)
$\mathbb{V}ar(L_1)$	0.04	0.04	0.04
	(0.04)	(0.04)	(0.04)
γ_1	0	-0.6964	-0.06836
	(0)	(-0.0693)	(-0.0655)
γ_2	0	0.0609	0.45312
	(0)	(0.0679)	(0.0459)

Table 4: Moments of the asset log-returns at time $t = 1$, based on the models considered in section 3 and the base set of parameters given in Table 3. The numbers in parenthesis represent the estimated moments based on 1,000,000 Monte Carlo runs.

	GBM vs Merton	**GBM vs VG**	**VG vs Merton**
$\mathbf{V}^P(0)$	-0.54%	-0.55%	0.01%
$\mathbf{V}^R(0)$	-3.27%	-6.25%	3.18%

Table 5: Model error: impact on the fair value of the guaranteed benefit and the terminal bonus for the benchmark set of parameters (unlevered contract). Mispricing calculated using the prices reported in Table 1.

ϑ	**GBM**	**Merton**	**VG**
0.1	0.1325	0.1879	0.2194
0.2	1.9670	2.1509	2.3368
0.3	6.7038	7.1569	7.4603
0.4	14.5295	15.1898	15.5587
0.5	25.0377	25.7831	26.1078
0.6	37.2608	38.3416	38.5989
0.7	51.4867	52.5789	52.6699
0.8	66.9076	67.7502	67.9518
0.9	82.8095	84.1079	84.8437
1.0	99.5084	100.7590	100.8052

Table 6: Fair value of the default option for different levels of the leverage coefficient ϑ, and the benchmark set of parameters. Values based on 1,000,000 runs.

ϑ	**GBM vs Merton**	**GBM vs VG**	**VG vs Merton**
0.1	-39.60%	-29.46%	16.79%
0.2	-15.82%	-8.55%	8.64%
0.3	-10.14%	-6.33%	4.24%
0.4	-6.62%	-4.35%	2.43%
0.5	-4.10%	-2.89%	1.26%
0.6	-3.47%	-2.82%	0.67%
0.7	-2.25%	-2.08%	0.17%
0.8	-1.54%	-1.24%	0.30%
0.9	-2.40%	-1.54%	0.87%
1.0	-1.24%	-1.29%	0.05%

Table 7: Model error: impact on the default option price for different levels of the leverage coefficient ϑ. Mispricing calculated using the prices reported in Table 6.

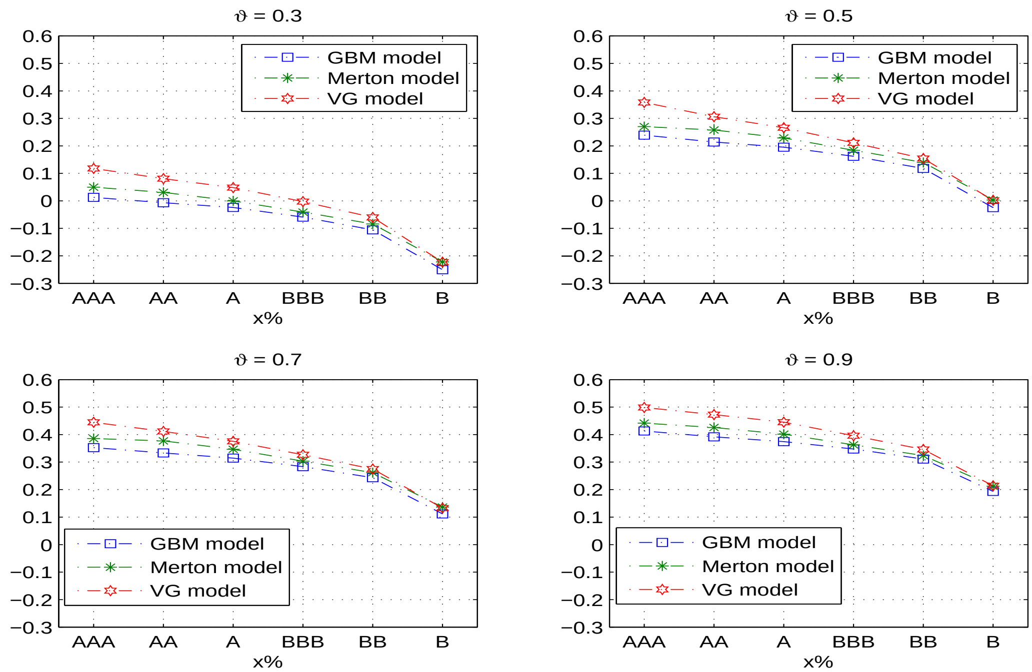

Figure 1: Risk Bearing Capital: $TVaR(s_1)$ for different levels of the leverage coefficient ϑ. For the confidence levels, we consider the percentiles provided by the Standard & Poor's classification (AAA = 99.99%; AA = 99.97%; A = 99.93%; BBB = 99.77%; BB = 99.31%; B = 93.46%). Estimation based on 100,000 simulations for the benchmark set of parameters.

The RBC, as measured by the TVaR of the solvency index s_t defined in section 2.3, is presented in Figure 1 for different values of the leverage coefficient ϑ. In particular, for this numerical example, we consider the change in the RBC over 1 year after the inception of the contract, i.e.

$$TVaR\,(x;0,1) = -\mathbb{E}\left(s_1 \middle| \, s_1 \leq c_{s_1}\,(x;0,1) \right),$$

with

$$s_1 = \frac{RB\tilde{C}(1) - A_0\,(1-\vartheta)}{A_0 + V^D(0)}.$$

The analysis can be easily extended to any two points in time over the lifetime of the contract (although the results would depend on the trajectory of the underlying fund). As the RBC depends on the price of the default option, the capital requirements change with the policyholder's contribution level to the reference fund of the insurance company. In particular, we note that the RBC increases with ϑ, due to the fact that the probability of default increases (as quantified by the premium for the default option), and therefore a higher safety loading is required. Similarly to what has been observed in the case of the default option, the mispricing generated by the model error reduces as ϑ increases; further, the mispricing becomes more significant in correspondence of higher confidence levels, x, which is a reflection of the different probability mass assigned by the three models to the tails of the distribution. Consistently with the findings related to the fair value, also in this case we observe that the standard model underestimates quite significantly the capital requirements; the inclusion of shocks in the model though reduces the magnitude of the error.

6 Conclusions

In this note we have developed a general framework for the market consistent pricing of insurance liabilities based on the fair value principle and the calculation of the corresponding capital requirements. In particular, we have used this framework to analyze the impact on these quantities of the inclusion in the model of market shocks. We conclude from the numerical results presented that the standard Black-Scholes economy significantly underestimates the amount of the total liabilities and, more importantly, the value of the default option. This fact has therefore repercussions on the safety loadings and consequently on the size of the target capital. However, the mispricing is less severe when shocks are incorporated in the model, even if this might be done in a sub-optimal way.

The analyses presented above are based on the assumption of constant interest rates; the techniques presented in this paper can be extended to incorporate a stochastic term structure, although this would prevent the derivation of closed form solutions for the price of the guaranteed benefit. This consideration highlights the importance of computationally efficient and accurate algorithms for the pricing of these contracts. In this note, we compare sequential and stratified Monte Carlo methods; the numerical procedures described in section 4 can be further improved by using, for example, QMC or RQMC techniques. Further, Avramidis and L'Ecuyer (2006) have introduced a competitive alternative procedure for pricing options under the VG model, exploiting dyadic partition, the VG representation as difference of gamma processes, and a suitably tailored beta generator. The generation of VG trajectories on the basis of a Dirichlet bridge and a fast gamma generator is currently being explored (see Ballotta et al., 2006.b).

An important open issue related to the implementation of fair valuation schemes in incomplete markets is the selection of the pricing measure. The analyses presented rely on the risk neutral Esscher measure; however, as discussed above, this might prove a quite restrictive approach. The lack of a market of derivative securities written on the contract reference portfolio, though, prevents the adoption of a more suitable market measure. Furthermore, the incompleteness of the market also means that the valuation framework has to take into account some non hedgeable financial risk. In this sense, perhaps it would be more appropriate to use the "best estimate plus risk margin" approach for the market consistent valuation of the liabilities; however, the full definition of the best estimate is still under discussion at regulatory level. Based on financial theory, a possible suggestion which we think might be appropriate to the task, could be interpreting the best estimate as the market price of a hedging strategy, whilst the risk margin would represent a protection against the inevitable hedging error. Investigation of this approach is left for future research.

Acknowledgements

The author would like to thank Enrico Biffis for many useful discussions about many steps of the development of this project, Steven Haberman, Vladimir Kaishev and Jiwook Jang for reading previous drafts of this paper and making very helpful suggestions and comments. Earlier versions of this work have been presented at the 12^{th} International Conference on Computing in Economics and Finance, the EURO XXI conference and the 10^{th} International Congress in Insurance: Mathematics and Economics. The author would like to thank the participants of these conferences, in particular Farid AitSahlia and Mogens Steffensen. Usual caveats apply. This research was partially supported by the UK Actuarial Profession.

References

[1] Avramidis, A. and P. L'Ecuyer (2006). Efficient Monte Carlo and Quasi-Monte Carlo option pricing under the variance-gamma model. *Management Science*, forthcoming.

[2] Bacinello, A. R. (2001). Fair pricing of life insurance participating contracts with a minimum interest rate guaranteed. *Astin Bulletin*, 31, 275-97.

[3] Bacinello, A. R. (2003). Pricing guaranteed life insurance participating policies with annual premiums and surrender option. *North American Actuarial Journal*, 7 (3), 1-17.

[4] Ballotta, L. (2005). Lévy process-based framework for the fair valuation of participating life insurance contracts. *Insurance: Mathematics and Economics*, 37, 2, 173-196.

[5] Ballotta, L., G. Esposito and S. Haberman (2006.a). The IASB Insurance Project for life insurance contracts: impact on reserving methods and solvency requirements. *Insurance: Mathematics and Economics*, 39, 3, 356-375.

[6] Ballotta, L., D. Dimitrova and V. Kaishev (2006.b). A Dirichlet bridge sampling of the Variance Gamma process: pricing path-dependent options. Quantitative Methods in Finance Conference, Sydney, December 2006.

[7] Ballotta, L., S. Haberman and N. Wang (2006.c). Guarantees in with-profit and unitised with profit life insurance contracts: fair valuation problem in presence of the default option. *Journal of Risk and Insurance*, 73, 1, 97-121.

[8] Bernard, C., O. Le Courtois and F. Quittard-Pinon (2006). Assessing the market value of safety loadings. Working Paper.

[9] Black, F. and M. Scholes (1973). The pricing of options on corporate liabilities. *Journal of Political Economy*, 81, 637-59.

[10] Brennan, M. J. and E. S. Schwartz (1976). The pricing of equity-linked life insurance policies with an asset value guarantee. *Journal of Financial Economics*, 3, 195-213.

[11] Carr, P., H. Geman. D. B. Madan and M. Yor (2002). The fine structure of asset returns: an empirical investigation. *Journal of Business*, 75, 305-32.

[12] CEIOPS (2006). Consultation Paper 20. CEIOPS-CP-09/06. November 2006.

[13] Devroye, L. (1986). *Non uniform random variate generation*. Springer-Verlag.

[14] Financial Service Authority (2006). Solvency II: a new framework for prudential regulation of insurance in the EU. A discussion paper. February 2006.

[15] FOPI (2004). White Paper of the Swiss Solvency Test.

[16] Gerber, H. U. and E. S. W. Shiu (1994). Option pricing by Esscher transforms (with discussion). *Transactions of the Society of Actuaries*, 46, 99-140; discussion: 141-91.

[17] Glasserman, P. (2004). *Monte Carlo methods in financial engineering*. Springer.

[18] Grosen, A. and P. L. Jørgensen (2000). Fair valuation of life insurance liabilities: the impact of interest rate guarantees, surrender options, and bonus policies. *Insurance: Mathematics and Economics*, 26, 37-57.

[19] Grosen, A. and P. L. Jørgensen (2002). Life insurance liabilities at market value: an analysis of investment risk, bonus policy and regulatory intervention rules in a barrier option framework. *Journal of Risk and Insurance*, 69, 63-91.

[20] Guillén M., P. L. Jørgensen and J. Perch-Nielsen (2004). Return smoothing mechanism in life and pension insurance: path-dependent contingent claims. *Insurance: Mathematics and Economics*.

[21] Hubalek, F. and C. Sgarra (2005). Esscher transforms and the minimal entropy martingale measure for exponential Lévy models. Working Paper N. 214, Centre for Analytical Finance, University of Aarhus.

[22] Kassberger, S., R. Kiesel and T. Liebmann (2005). Fair valuation of insurance contracts under Lévy process specifications. Working Paper.

[23] Keller, P. (2005). Swiss Solvency Test (powerpoint presentation), FOPI.

[24] Madan, D. B., P. Carr and E. Chang (1998). The variance gamma process and option pricing. *European Finance Review*, 2, 79-105.

[25] Madan, D. B. and F. Milne (1991). Option pricing with VG martingale components. *Mathematical Finance*, 1, 39-45.

[26] Madan, D. B. and E. Seneta (1990). The variance gamma (VG) model for share market returns. *Journal of Business*, 63, 511-24.

[27] Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 125-44.

[28] Mina, J. and J. Y. Xiao (2001). Return to RiskMetrics: the evolution of a standard. *RiskMetrics*, New York.

[29] Press, W. H., S. A. Teukolsky, W. T. Vetterling and B. P. Flannery (2002). *Numerical Recipes in C++: The art of Scientific Computing*. Cambridge University Press.

[30] Ribeiro, C. and N. Webber (2004). Valuing path dependent options in the variance-gamma model by Monte Carlo with a gamma bridge. *The Journal of Computational Finance*, 7, 2, 81-100.

[31] Siu T. K., J. W. Lou and H. Yang (2006). Pricing participating products under a generalized jump-diffusion with a Markov-switching compensator. 10^{th} International Congress on Insurance: Mathematics and Economics, Leuven, July 2006.

[32] Tanskanen, A. J. and J. Lukkarinen (2003). Fair valuation of path-dependent participating life insurance contracts. *Insurance: Mathematics and Economics*, 33, 595-609.