Valuation of Participating Inflation Annuities with Stochastic Mortality, Interest and Inflation rates

Eduardo F. L. de Melo [1*]

[*] *COPPEAD Graduate School of Business, Federal University of Rio de Janeiro, Brazil*

SUSEP - Superintendência de Seguros Privados, Brazil

Abstract

In this paper, we evaluate a common annuity instrument present in the brazilian market, and maybe also in others high inflation markets: the participating guaranteed inflation annuity. Besides the monthly income, this sort of annuity pays the policyholder from time to time (for example: every 5 years) the positive difference between the reserve accrued by a specified fixed income fund return plus the portfolio observed "mortality return", and the reserve evaluated using a fixed interest rate plus a predetermined "mortality return" based on an annuity table. The monthly income grows according to the inflation rate. The actuarial factor used to calculate the initial income is based on the predetermined table and the fixed interest rate. The model makes use of multi-factor CIR and Vasicek frameworks for the term structures of interest and inflation rates, respectively. For the mortality rate modelling, we used the Lee-Carter approach estimated with brazilian population data. All experiments were carried out with Monte Carlo simulation with descriptive sampling. The results indicate a significant difference in the annuity value whether the way the excess of return is paid to the policyholder and the frequency this happens.

JEL Classification: G13, G22, G23, C22

Insurance branch codes: IB10, IM10, IE22, IE51

Keywords: Inflation Annuity; Multi-factor CIR and Vasicek models; Lee-Carter Mortality model; Interest-Mortality options.

[1] E-mail: eduardoflm@yahoo.com.br

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1 Introduction

The literature about the valuation of participating life insurance contracts dealt mainly with the pricing and hedging of embedding options in the accumulation period and with guaranteed annuity options. However, very few or none have explored a common annuity instrument present in the brazilian market, and maybe also in others high inflation markets: the participating guaranteed inflation annuity. Besides the monthly income, this sort of annuity pays the policyholder from time to time (for example: every 5 years) the positive difference between the reserve accrued by a specified fixed income fund return plus the portfolio observed "mortality return", and the reserve evaluated using a fixed interest rate plus a predetermined "mortality return" based on an annuity table.

The "mortality return" can de described as the monetary benefit caused by the mutualism effect inherent to a portfolio of annuities. Mathematically, the mortality return from the age x to the age $x+t$ on the time interval $[s, s+t]$ is denoted by $\frac{l_s(x)}{l_{s+t}(x+t)} - 1$, where $l_s(x)$ is the number of individuals alive with age x on time s.

As a characteristic of each policy, the payment of this option can be made in two different ways: (i) credited directly in the retiree current account (we will call this as "Cash-bonus" option), or (ii) reverted to the reserve and, thus, increasing the value of the monthly income (we will call this as "Benefit-bonus" option). The guarantee can be seen as a "put" option, which we will refer as "Guarantee". The actuarial factor used to calculate the initial monthly income is based on the predetermined annuity table and the fixed interest rate. The income is accrued by the inflation rate.

In a context which the bonds yields are decreasing as well as the population mortality rates, as a result of a increasing longevity, the modelling and valuation of participating annuities by insurance companies is crucial for internal risk management purposes, reserving, solvency requirements and also to measure the profitability of a annuity portfolio. Some examples of insolvency due to the level of annuity guarantees are known in the literature, as the case of Equitable Life in UK (Ballotta and Haberman, 2003). However, the correct valuation of such instruments is not an easy task and become even harder when the cash flows are periodical and also depend on inflation and mortality rates.

In this paper, we provide a framework to evaluate this liability using a risk neutral approach. For the interest and inflation rates modelling, we used the two-factor CIR (Cox, Ingersoll and Ross, 1985) and Vasicek (1977) models, respectively. The mortality rate is adjusted with the original Lee-Carter model (Lee and Carter, 1992). All the models for interest, inflation and mortality rates were formulated in the state space form and estimated by the Kalman filter.

According to the Feller condition, the rates modelled in the CIR approach are never negatives. This is a important feature for nominal interest rates. However, inflation rates can assume negative values. Thus, for the short inflation rate, we chose the bi-factor Vasicek model. Chan (1998) also models the behavior of the inflation rate directly by means of an Ornstein-Uhlenbeck diffusion process. The choice of a two-factor model is due to the fact that it explains a good proportion of the total variation in the yield curve, according to results presented by Rebonato (1998). Regarding the mortality rates, the model adjusted to our data was the Lee-Carter model (LC)[2], which presented an excellent fit.

We are dealing with cross-section and time-series data both for the mortality model and for the interest and inflation rates models. In this situation, the Kalman filter is very suitable to be applied. For this, we refer to some papers in the literature which the Kalman filter was used for the modelling of term structure of interest rates, Babbs and Nowman (1999), and for mortality rates, De Jong (2005) and Hari et al. (2007).

Over the last years, many authors have applied risk neutral pricing theory from financial economics to calculate the value of embedded options in open pension funds and life insurance contracts. Initially, the work was focused on valuing return guarantees embedded in equity-linked insurance policies, see for example, Brennan and Schwartz (1976), Hipp (1996) and Boyle and Hardy (1997). Among others papers in the literature that deal with the valuation of embedded return guarantees, or participating life contracts, we also refer to Aase and Persson (1997), Grosen and Jorgensen (1997), Miltersen and Persson (1999), Vellekoop et al. (2006), Coleman et al. (2006) and Bauer et al. (2006). Regarding the guaranteed annuity options, we refer to Boyle and Hardy (2003), Pelsser (2003) and Ballotta and Haberman (2003). Melo (2007) evaluated inflation minimum guarantees present in many brazilian life insurance contracts using Vasicek model and copulas for the dependence structure.

In section 2, we introduce the cash flows of the excess of return and of the minimum guarantee. In section 3, we present the models for the term structure of interest and

[2]We also fitted the Makeham (1860) and Thiele (1872) models to the data in affine versions following the approach of Schrager (2006). However, the results for the time series behavior of the mortality rates were not good as the ones produced by the Lee-Carter model fit.

inflation rates as well as the model for the mortality rates and the way they are going to be estimated. In section 4, we provide results, numerical examples based on simulation experiments as well as some sensitivity analysis, and, finally, in section 5, we conclude the paper.

2 The Excess of Return and the Minimum Guarantee

In this section, we characterize the pay-offs of the excess of return and of the minimum guarantee. Once the instruments treated in the paper are based on the values of the reserve and backing assets in the deccumulation period, we present their basic process without the effects of any options. For the backing assets:

$$dR(x)_t^A = r_t R(x)_t^A dt + \mu_{x+t}(t) R(x)_t^A dt - g_t dt \tag{1}$$

which in integral form is:

$$R(x)_u^A = R(x)_t^A e^{\int_t^u \mu_{x+s}(s)+r_s ds} - \int_t^u g_s e^{\int_s^u \mu_{x+a}(a)+r_a da} ds$$

where $u \geq t$. For the reserve:

$$dR(x)_t^L = (i_t + f) R(x)_t^A dt + \mu_{x+t}^* R(x)_t^L dt - g_t dt, \tag{2}$$

where μ_{x+t}^* is the deterministic mortality rate for the age $x + t$ determined by an annuity table. $R(x)_t^A$ is the asset value counterpart of the reserve $R(x)_t^L$, so, on time $t = 0$, $R(x)_0^A = R(x)_0^L$. r_t is the short interest rate and x is the retirement age, or the age when the annuity begins. $\mu_x(t)$ is the mortality intensity of a person aged x at time t. The dynamics of $R(x)_t^L$ is analogous to that of $R(x)_t^A$. However, for $R(x)_t^L$, the mortality force is not stochastic. It is totally determined by a life table, such as, for example, 1949 US A-1949 Male (AT-49Male), 1983 US IAM Male (AT-83Male) or 1996 US Annuity 2000 Basic Male (AT-2000BM), all of them downloadable from SOA - Society of Actuaries website (*www.soa.org*). The interest rate for $R(x)_t^L$ is given by the composition of the inflation rate i_t and a fixed return f, for example 6% per year. One should note that we only considered in our approach the systematic risk of mortality, we are thus assuming that the insurance company has a well diversified portfolio of lives. The unsystematic mortality risk is not being accounted. g_t is the income received by the policyholder on time t. Without the effects of the options, this income is denoted by:

$$g_u = g_t e^{\int_t^u i_s ds}$$

where i_t is the short inflation rate. g_0 is calculated according to the annuity table, the fixed return and the amount $R(x)_0^L$ available in the policyholder account on time $t = 0$. After these definitions, we can present the pay-off of the bonus ($Cash$ and $Benefit$) options ($C(x)$) at time T for a person with a retirement age x:

$$C(x)_T = \beta(R(x)_T^A - R(x)_T^L)^+ \tag{3}$$

where β is the percentage of the excess return the insurer gives to the policyholder ($0 \leq \beta \leq 1$). This option can be understood as the option of the excess of return. It is important to note that both the "Cash-bonus" an the "Benefit-bonus" options have the same pay-off, the difference is the way the pay-off is paid to the policyholder. The policyholders have the right to receive the pay-off of the bonus options only at the maturity. After the pay-off settlement, another bonus option is created for the next maturity, which is the next period that there will be the payment of the excess of return. The pay-off of the "Guarantee" option at time T would be:

$$(R(x)_T^L - R(x)_T^A)^+ \tag{4}$$

Nevertheless, according to regulatory requirements, during the whole life of the contract, every time the value of the assets falls below the value of the reserve, the company needs to aport the corresponding amount of capital in order to equalize their values. This sort of contingent cash flow is not fully characterized by (3). The maturity is a random variable and once exercised, another "Guarantee" option is created, so this can be considered a solvency constraint or the minimum guarantee itself. Formally, we can represent the process of this cash flow as:

$$P(x)_t = (R(x)_t^L - R(x)_t^A)^+ \tag{5}$$

where $t \in [0, \infty)$.

One should note that the exercise of the bonus options or the Guarantee option interferes in the dynamics of the reserve asset counterpart $R(x)_t^A$ and/or in the reserve $R(x)_t^L$ itself. If we want to price all the periodical options till de end of the policyholder's life, we have to redefine the reserves dynamics. Then, in the case of the "Cash-bonus" option, we have:

$$ {}^{Cash}dR(x)_t^A = r_t R(x)_t^A dt + \mu_{x+t}(t) R(x)_t^A dt - g_t dt - C(x)_t dI_{\{t=t_k\}} + P(x)_t dI_{\{R(x)_t^L > R(x)_t^A\}}, \tag{6} $$

where I is a indicator function, t_k are the periodical times when there is an excess of return payment. In the case of "Cash-bonus" options, the dynamics of $R(x)_t^L$ remains the same as in (2). However, for the "Benefit-bonus" option, we have:

$$ {}^{Benefit}dR(x)_t^A = r_t R(x)_t^A dt + \mu_{x+t}(t) R(x)_t^A dt - g_t^* dt + P(x)_t dI_{\{R(x)_t^L > R(x)_t^A\}}, \tag{7} $$

$$ {}^{Benefit}dR(x)_t^L = f R(x)_t^L dt + \mu_{x+t}^* R(x)_t^L dt - g_t^* dt + C(x)_t dI_{\{t=t_k\}}. \tag{8} $$

When the "Benefit-bonus" is exercised, the income g_t is increased by the annuitisation of the amount of the option intrinsic value. Thus, we need to define a new income g_t^* driven by the process:

$$ dg_t^* = i_t g_t^* dt + C(x)_t dI_{\{t=t_k\}} AF_{x+t_k}, $$

where AF_{x+t_k} is the annuitisation factor for the age $x + t_k$, based in the fixed return f and the pre-determined annuity table.

3 Interest, inflation and mortality rates modelling

Since we are modelling both the interest and inflation rates risks and the mortality risk, we must define a suitable probability space for our approach. In this setup, the time of death of a person is represented by a stopping time τ with respect to the filtration $\mathcal{G}_t^\tau$ generated by the stopping time. $\mathcal{H}_t$ is a filtration containing the financial information. In this context, $\mathcal{F}_t$ is the filtration containing all information (financial and actuarial) and $\mathcal{F}_t = \mathcal{G}_t^\tau \vee \mathcal{H}_t$. As a consequence of the Cox-process setup, the filtration $\mathcal{H}_t$ is conditionally independent on $\mathcal{G}_t^\tau$ (Jamshidian, 2004). Considering a probability space $(\Omega,\ \mathcal{F},\ (\mathcal{F}_t)_{t>0},\ \mathbb{P})$ where the filtration $(\mathcal{F}_t)$ satisfies the usual conditions, we define the following stochastic differential equations in a multi-factor CIR and Vasicek term structure for the interest and inflation short rates:

$$ r_t = \sum_{k=1}^{M} X_t^k $$

$$i_t = \sum_{j=1}^{N} Z_t^j$$

where r_t is the short interest rate and i_t is the short inflation rate. The dynamics of the processes X_t and Z_t are given by:

$$dX_t^k = \alpha^k(\eta^k - X_t^k)dt + \gamma^k.\sqrt{X_t^k}.dW_{r_t}^k \tag{9}$$

$$dZ_t^j = \kappa^j(\theta^j - Z_t^j)dt + \sigma^j.dW_{i_t}^j \tag{10}$$

If we consider a vector $\mathbb{W} = (dW_{r_t}^1, ..., dW_{r_t}^M, dW_{i_t}^1, ..., dW_{i_t}^N)$, then the matrix

$$\mathbb{W}.\mathbb{W}^T = \mathbf{\Sigma.dt}$$

represents the Pearson correlation matrix with dimension $M+N$ for the motions $dW_{r_t}^k$ and $dW_{i_t}^j$, for $k = 1, ..., M$ and $j = 1, ..., N$. t denotes the time, W Wiener processes. Once we are dealing with affine term structure models, according to Duffie and Kan (1996), the price of a zero-coupon bond is given by:

$$P(X,t,T) = A(t,T).e^{-B(t,T).X_t}, \tag{11}$$

where $A(t,T)$ is a scalar function and $B(t,T)$ is an 1xM vector, for the interest rate case, or 1xN for the inflation rate. The price of an "inflation zero-coupon bond" is analogous. The equation for the yields is therefore:

$$Y(X,t,T) = -\frac{lnP(X,t,T)}{T-t} = -\frac{lnA(t,T)}{T-t} + \frac{B(t,T).X_t}{T-t}.$$

Although the last equation dictates an exact relationship between the yield $Y(X,t,T)$ and the state variables X_t, in econometric estimation it is usually treated as an approximation giving rise to the measurement equation:

$$Y(X,t,T) = -\frac{lnA(t,T)}{T-t} + \frac{B(t,T).X_t}{T-t} + \epsilon(t,T) \tag{12}$$

$$\epsilon(t,T) \sim N(0, \sigma(t,T)),$$

where $\sigma(t,T)$ is a diagonal matrix with the standard deviations for each of the maturities $(T-t)$. To complete the state space representation, the transition equation for X_t

over a discrete time interval h needs to be specified. Defining $\Phi(Xt; h) = Var(X_{t+h} \mid X_t)$, Duan and Simonato (1999) show that the transition equation for X_t has the form:

$$X_{t+h} = a(h) + b(h)X_t + \Phi(X_t, h)^{1/2}.\xi_{t+h}, \tag{13}$$

where $\xi_t \overset{iid}{\sim} N(0, I_M)$, and $\Phi(X_t, h)^{1/2}$ represents the Cholesky factorization of $\Phi(X_t, h)$. For the CIR model, the state space model defined by (11) and (12) is non-Gaussian because the conditional variance of X_{t+h} in (12) depends on X_t. For the Vasicek model, the conditional variance term $\Phi(X_t, h)$ is not a function of X_t and the state space model is Gaussian. To estimate the non-Gaussian state space model, Duan and Simonato modify the Kalman filter recursions to incorporate the presence of $\Phi(X_t, h)$ in the conditional variance of ξ_{t+h}. The Kalman filter functions in S-Plus, the software used in the applications, were modified accordingly. Almeida and Vicente (2006) show that the functions $A(.)$ and $B(.)$ have the form:

$$A(t, T) = \prod_{k=1}^{M} A_k^r(t, T) \tag{14}$$

$$B(t, T) = [B_1(t, T), ..., B_M(t, T)]. \tag{15}$$

So the state space representation of the multi-factor CIR and Vasicek models has the following parameters:

Table 1: *Parameters for the state space representation of the multi-factor CIR and Vasicek models.*

Function	CIR	Vasicek
$A_n^{\cdot}(t,T)$	$[\frac{2\gamma_n.exp((\kappa_n+\lambda_n+\gamma_n)\tau/2)}{(\kappa_n+\lambda_n+\gamma_n)(exp(\gamma_n.\tau)-1)+2\gamma_n}]^{2\kappa_n\theta_n/\sigma_n^2}$	$exp(\gamma_n(B_n^{\cdot}(t,T)-\tau)-\frac{\sigma^2.B_n^{\cdot 2}(t,T)}{4\kappa_n})$
$B_n^{\cdot}(t,T)$	$\frac{2(exp(\gamma_n.\tau)-1)}{(\kappa_n+\lambda_n+\gamma_n)(exp(\gamma_n.\tau)-1)+2\gamma_n}$	$\frac{1-exp(-\kappa_n\tau)}{\kappa_n}$
γ_n	$\sqrt{(\kappa_n+\lambda_n)^2+2\sigma_n^2}$	$\theta_n+\frac{\sigma_n.\lambda_n}{\kappa_n}-\frac{\sigma_n^2}{2\kappa_n^2}$
$a_n^{\cdot}(h)$	$\theta_n(1-exp(-\kappa_n.h))$	$\theta_n(1-exp(-\kappa_n.h))$
$b_n^{\cdot}(h)$	$exp(-\kappa_n.h)$	$exp(-\kappa_n.h)$
$\Phi_n(.,h)$	$X_t^i\frac{\sigma_n^2}{\kappa_n}[exp(-\kappa_n h)-exp(-2\kappa_n h)]+$ $\theta_n\frac{\sigma_n^2}{2\kappa_n}[1-exp(-\kappa_n h)]^2$	$\frac{\sigma_n^2}{2\kappa_n}[1-exp(-2\kappa_n h)]$

λ_n is the risk premium parameter and $n = 1, ..., M$ or $n = 1, ..., N$. The parameters were estimated by maximum likelihood (Harvey, 1989). In this paper, we also assume the existence of a pricing measure $\mathbb{Q}$ under which the discounted bond prices are martingales. Once we are simulating the cash flows in the risk neutral measure, the following transformations apply for the CIR model:

$$\alpha_k^Q = \alpha_k + \lambda_k$$

$$\eta_k^Q = \frac{\alpha_k \eta_k}{\alpha_k^Q}$$

For the Vasicek model:

$$\kappa_j^Q = \kappa_j$$

$$\theta_j^Q = \theta_j + \frac{\sigma_j \lambda_j}{\kappa_j}$$

After the fit of the multi-factor models, we estimated the matrix $\mathbf{\Sigma}$ using the residuals of the state variables, or, more specifically, through the use of the sample correlation matrix of $\hat{\xi}_t$.

- The mortality rate model

We considered the Lee-Carter approach (Lee and Carter, 1992) for the modelling of the mortality rates stochastically. For others approaches on the modelling of stochastic mortality rates, we refer to Dahl (2004), Schrager (2006) and Biffis (2006). The main drawback using this approach is the impossibility to get a closed formula for the survival probabilities. In order to obtain these probabilities for the Lee-Carter model, we have to use simulation.

The market for mortality contingent claims is incomplete, so that for the applications, we assume that the actuarial probability measure is the same as the risk neutral probability measure. In this context, the Lee-Carter model (LC) has the following structure:

$$ln(m_x(t)) = a_x + b_x k(t) + e_x(t) \tag{16}$$

$$k(t+1) = k(t) + l + \zeta_t, \tag{17}$$

where $m_x(t)$ is the central death rate for age x at time t. The a_x coefficients describe the average shape of the age profile, the b_x coefficients describe the pattern of deviations from this age profile when the parameter k varies and l represents the drift in the k process. ζ_t is normal distributed with mean zero and standard deviation σ_k. Based on this, we approached the mortality rate in the following way:

$$\mu_x(t) \approx m_x(t) \tag{18}$$

$$1 - e^{-m_x(t)} \approx 1 - e^{-\mu_x(t)} = q_x(t)$$

where $q_x(t)$ is the probability that an individual aged x at time t will die between t and $t+1$. As pointed by Lee (2000), if we constrain the sum of b_x to be equal to 1.0 and that the sum of k_t to be equal to zero, a_x must be the average values over time of the $ln(m_x(t))$ for each x. Then, in our state space approach, besides the unobserved state variable $k(t)$, we also estimated the b_x coefficients.

4 Results and sensitivity analysis

For the modelling of the short interest and inflation rates, we collected swaps rates data from the site of the brazilian futures board of exchange (*www.bmf.com.br*). We interpolated the yields to the maturities of 1 month, 3 months, 6 months, 1 year and 2 years with *cubic spline*. The chosen inflation index is the IGP-M and the fixed rate of return is 6%. We worked with the AT-49M and AT-83M mortality tables. These were some of the most common guarantees sold in the market until recently. The short and risk-free interest rate is the DI (brazilian interbank offered rate), the most used benchmark for fixed income funds.

Due to the atypical high volatility behavior of the interest rates during the period from October/2001 to August/2003, we considered the sample period from September/2003 to February/2007 (42 months) for the time series of the term structures of interest rates and inflation rates. We have got the following estimates using the Kalman filter:

Table 2: *Estimates of the parameters for the short Interest and Inflation rates models under measure* $\mathbb{Q}$.

Parameter	Factor 1	Factor 2
α	3.394	0.365
η	0.020	0.158
γ	0.373	0.340
κ	3.055	0.010
θ	-0.038	0.093
σ	0.083	0.026

According to the estimated parameters, it is important to note that on average the real interest rate will revert to 12.3%. This value is more than enough to cover the fixed return 6% and also most of the mortality variability. Since the parameters were calibrated based in past observations, it is not considered the popular expectancy, or even desire, that this real interest rate will decrease in the medium term.

The smoothed inflation yield estimates are displayed in Figure 1. The model fits well on the medium and long end of the yield curve but poorly on the short end.

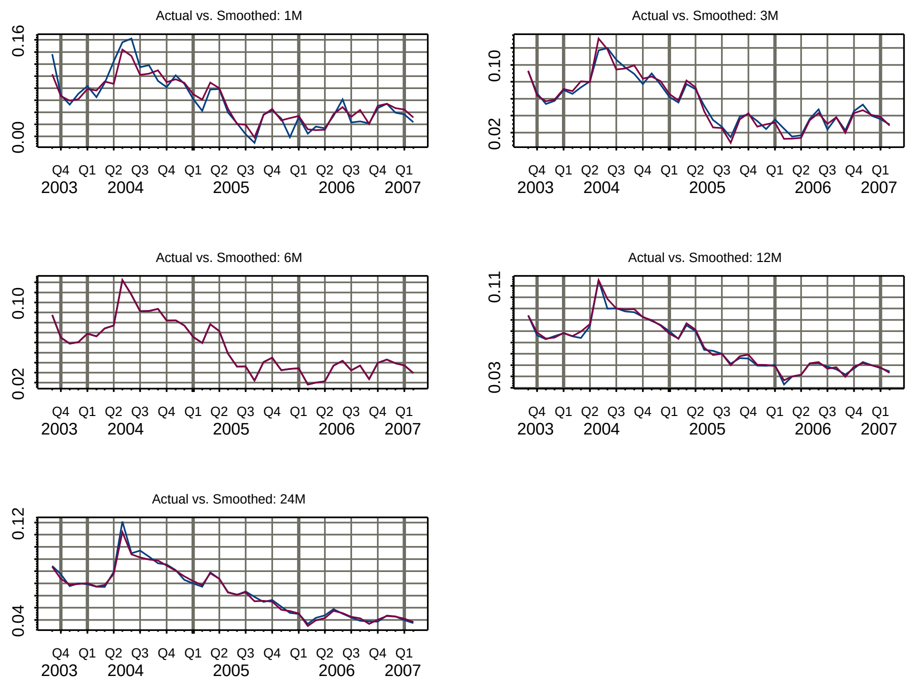

Figure 1. Actual x Smoothed estimated inflation yields from the bi-factor Vasicek term structure model - 1, 3, 6, 12 and 24 months.

In Figure 2, we display the smoothed interest yield estimates. Opposite to the inflation case, the model fits well on the short and medium end of the yield curve but poorly on the long end.

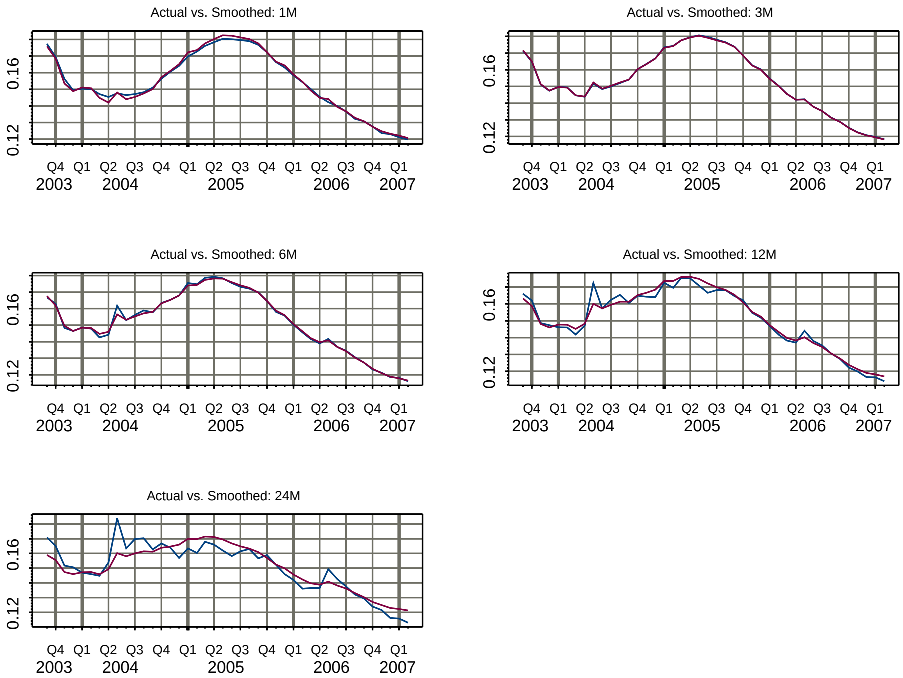

Figure 2. Actual x Smoothed estimated interest yields from the bi-factor CIR term structure model - 1, 3, 6, 12 and 24 months.

For the modelling of the mortality intensity, we used the deaths rates ($m_x(t)$) data provided by the Brazilian Institute of Geography and Statistics (IBGE - *www.ibge.gov.br*). We considered the sample period from 1998 to 2005 (8 years), and the ages 20 to 80. One should note that these rates are based in the entire population, not only in the group of people that actually buys life insurance or pension funds. Definitely, the mortality for this group is lower, since they have better financial conditions than the average population.

For the Lee-Carter model (LC), the estimates of the drift l and the standard deviation σ_k are -0.169 and 0.022, respectively. Figure 3 shows the estimated values of a_x and b_x coefficients for the brazilian population. As noted before, the a_x coefficients are just the average values of the logs of the death rates. The b_x coefficients describe the relative sensitivity of death rates to variation in the k parameter. It can be seen that, in fact, some ages are much more sensitive than others to the time change.

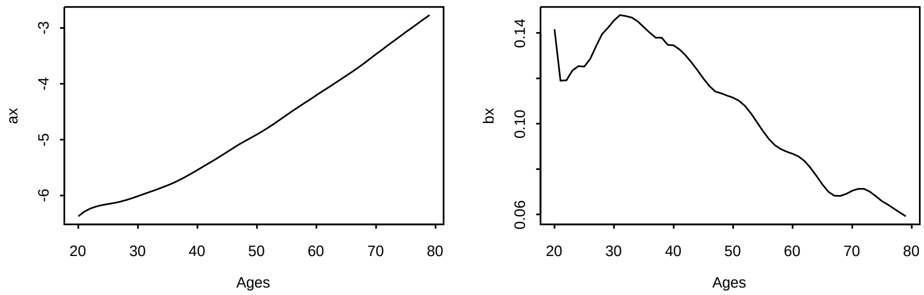

Figure 3. Estimated values of a_x and b_x coefficients for the brazilian population.

Generally speaking, based in the behavior of the b_x, the younger the age, the greater its sensitivity to variation in the k parameter. As Lee (2000) observed, the exponential rate of change of an age group's mortality is proportional to the b_x value: $dln(m_x, t)/dt = (dk_t/dt)b_x$. If k declines linearly with time, then dk_t/dt will be constant and each m_x will decline at its own constant exponential rate. In Figure 4, it is shown the time series of the estimated k_t.

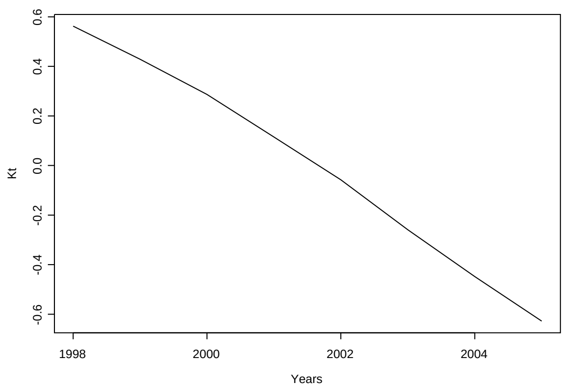

Figure 4. Estimated k_t for the sample period (1998 - 2005).

In Figure 5, we display the smoothed death central rates estimates. The model fits very well for almost all ages. For the older ages, for example 75 years, the fit is not as good as it is for the younger and medium ages.

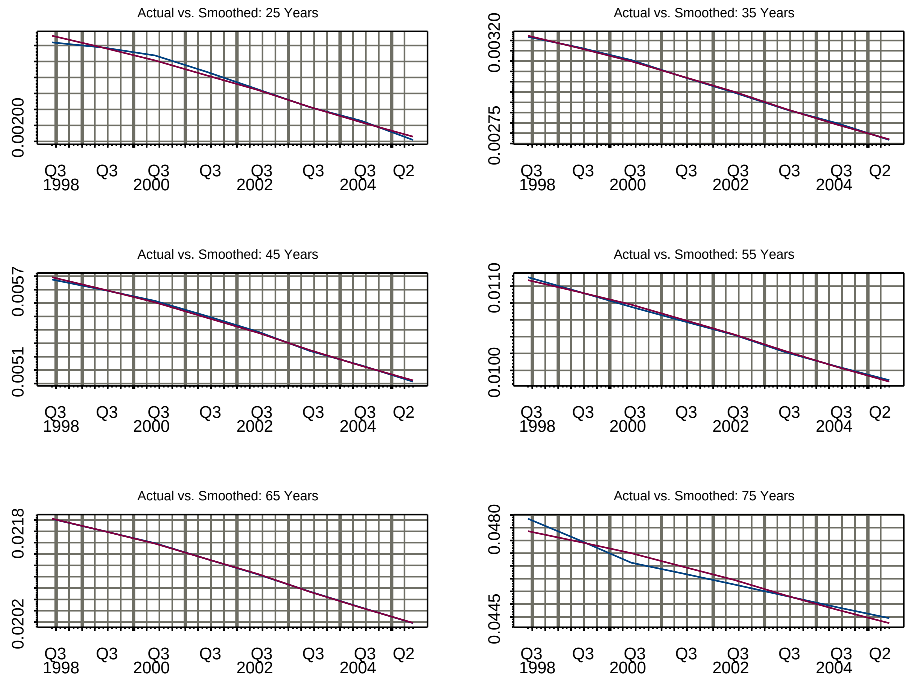

Figure 5. Actual x Smoothed estimated death central rates from the Lee-Carter model.

In Figure 6, we compare the static mortality table adjusted by the Lee-Carter model with the AT-49M and AT-83M. If the insurer portfolio follows the brazilian average mortality, the company that adopt the AT-83M shall have no problem with the longevity risk in the short term. However, it is clear that the AT-49M is not realistic anymore to the brazilian population, specially for the older ages, which are the most important for the annuity market. It is important to note that the curve shown in Figure 6 is based on estimates fitted for the last year of observations. In the next subsection, we simulated projected mortality behavior for the pricing of the options.

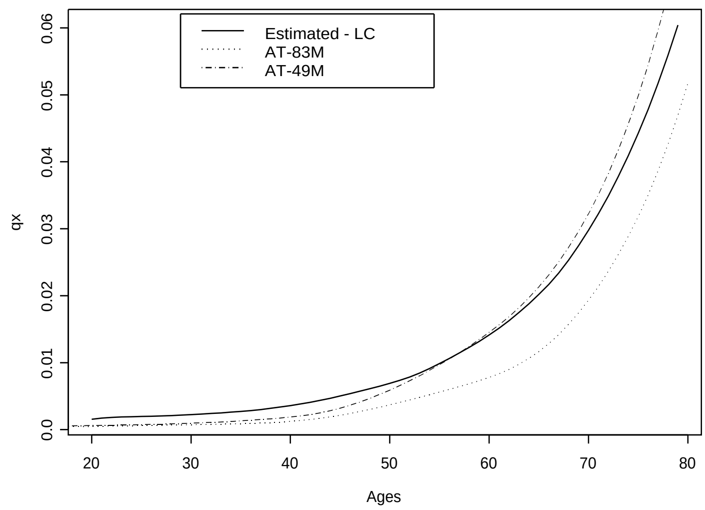

Figure 6. Comparison among the mortality table generated by the Lee-Carter model to the brazilian population and the AT-49M and AT-83M tables.

4.1 Sensitivity analysis

Now, we turn our attention to the values of the options. In order to price them, all experiments were carried out with Monte Carlo simulation. We considered a representative 60-year-old person with 1 monetary unit accumulated in his individual account in the moment of retirement. This amount is transformed in a irreversible life annuity with 12 payments per year. We simulated the variables every month for 20 years (240 months), according to equations (5), (6), (7), (8), (9), (15) and (16), considering the following time discretization for the factors of the short interest and inflation rates:

$$\Delta X_t^k = \alpha^k(\eta^k - X_t^k)dt + \gamma^k.\sqrt{X_t^k}w_r^k\sqrt{\Delta t}$$

$$\Delta Z_t^j = \kappa^j(\theta^j - Z_t^j)dt + \sigma^j w_i^j\sqrt{\Delta t}$$

where we considered $\Delta t = 1/12$, $k = 1, 2$, $j = 1, 2$. The random variables $(w_r^1, w_r^2, w_i^1, w_i^2)$ with unit variance are simulated from a joint Normal distribution with sample correlation matrix of $\hat{\xi}_t$. We simulated the mortality rates independently from r and i. Since we have four factors for the interest and inflation rates, besides the simulation of the mortality rates, we resorted to a variance reduction technique, the descriptive sampling (details in Saliby, 1990). Because of this, we simulated 1000 runs for each set.

As said before, the guarantee is composed of a 6% fixed return (f) plus the "mortality return" determined by the tables AT-49M, AT-83M or AT-2000BM, which define the μ_x^*. The percentage of excess of return is equal to 100% ($\beta = 1$). For the discount of the cash flows, we used the market interest rates and survival probabilities. The experiments were based on the two different kinds of bonus option, the "Cash" and the "Benefit". The sort of the option is determined in each policy. We also considered different frequencies for the payment of the excess of return (the bonus option): 1, 3 and 5 years. It is important to note that we evaluated the entire cash flows (the excesses of return and the guarantee) till the last considered age, which is 80 years. In other words, for the "Guarantee" option the values represent the entire commitment of the insurance company that the value of the assets will not fall below the value of the liabilities. In Table 3, we show the results for the "Cash-bonus" (first panel) and "Benefit-bonus" (second panel) options and the related "Guarantee" options:

Table 3: *Values of the options. Mortality tables considered: AT-49M, AT-83M and AT-2000BM. Periodicity when the excess of return is reverted to the policyholder: 1, 3 or 5 years. Table A: fixed return (f) = 3%. Table B: fixed return (f) = 6%. First panel: Cash-bonus option(Guarantee). Second panel: Benefit-bonus option(Guarantee).*

Table A ($f = 3\%$)	Periodicity		
1st Panel	1	3	5
AT-49M	0.719(0.005)	0.766(0.004)	0.822(0.004)
AT-83M	0.794(0.004)	0.842(0.004)	0.930(0.003)
AT-2000BM	0.815(0.003)	0.866(0.002)	0.962(0.002)
2nd Panel			
AT-49M	1.041(0.026)	0.949(0.018)	0.919(0.016)
AT-83M	1.023(0.004)	0.966(0.003)	0.910(0.003)
AT-2000BM	1.051(0.013)	0.985(0.010)	0.927(0.010)

Table B ($f = 6\%$)	Periodicity		
1st Panel	1	3	5
AT-49M	0.597(0.011)	0.625(0.010)	0.693(0.009)
AT-83M	0.659(0.006)	0.694(0.005)	0.756(0.004)
AT-2000BM	0.653(0.005)	0.695(0.003)	0.773(0.002)
2nd Panel			
AT-49M	0.944(0.031)	0.942(0.027)	0.937(0.024)
AT-83M	0.971(0.016)	0.963(0.013)	0.951(0.011)
AT-2000BM	0.895(0.013)	0.867(0.011)	0.834(0.006)

Table 3 provides us with some interesting insights. The most obvious is the comparison among the results with AT-49M, AT-83M and AT-2000BM. Once the AT-49M table has higher death probabilities (see Figure 6), the "Guarantee" option becomes more expensive and the bonus options cheaper.

The "Benefit-bonus" options are more expensive than the "Cash" ones, because the excess of return is reverted to the reserve and is accrued by the same rates of the reserve dynamic and discounted by a stochastic survival probability which has an exponentially positive trend on time. In the "Cash-bonus" option the excess is paid to the policyholder just once in each period and he decides what to do with the money. In the case which the excess of return is reverted to the reserve, increasing the monthly income, the "Guarantee" options are also more expensive. This can be attributed to the fact that more money is in risk when there is such a reversion, caused exactly by the increase in the liability.

Regarding the frequency when the excess of return is reverted to the policyholder, it is observed a different behavior for the prices of the bonus options whether the excess of return is paid as a cash payment or as an increase in the monthly income. This should be analyzed bearing in mind the whole portfolio and not the retiree individually.

For the "Cash-bonus", there is a decrease in the value of the bonus options if the payments become more frequent (periodicity from 5 years to 1 year). This can be explained by the fact that we considered a representative agent and only the systematic mortality risk is being accounted. Because of this, for the entire portfolio of policyholders, the bonus option is more valuable the less frequent the payment of the excess of return happens. If we consider the idiosyncratic risk, it may be better for the retiree to receive the payments as soon as possible, since there is no bequest. For the "Benefit-bonus", there is an increase in the value of the bonus options if the payments become more frequent (periodicity from 5 years to 1 year), mainly because the excess of return is reverted to the reserve and paid monthly to the policyholder as an amount being accrued by inflation till the rest of his life. The others excesses of return are added on each time according to the periodicity and are again accrued by inflation.

In the case of the "Guarantee" prices, the increasing behavior for more frequent excess of return payments is due to the extra risk a more frequent reversion brings to the insurance company. Once the excess of return is paid to the policyholder, the reserve and its asset counter-part become equal, and this widen the risk of deficit in the following periods.

5 Concluding remarks

Very few works in the literature have explored a common annuity instrument present in the brazilian market, and maybe also in others high inflation markets: the participating guaranteed inflation annuity. Besides the monthly income, this sort of annuity pays the policyholder from time to time (for example: every 5 years) the positive difference between the reserve accrued by a specified fixed income fund return plus the portfolio observed mortality rate, and the reserve evaluated using a fixed interest rate plus a predetermined annuity table. As a characteristic of each policy, the payment of this option can be made in two different ways: (i) credited directly in the retiree current account (we called this as "Cash-bonus" option), or (ii) reverted to the reserve and, thus, increasing the value of the monthly income ("Benefit-bonus" option). The guarantee can be seen as a "put" option, which we referred as "Guarantee".

In a context which the bonds yields are decreasing as well as the population mortality rates, as a result of a increasing longevity, the modelling and valuation of participating annuities by the insurance companies is crucial for internal risk management purposes, reserving, solvency requirements and also to measure the profitability of a annuity portfolio.

Following a risk neutral approach, we modelled the short interest (r_t) and inflation (i_t) rates by means of the two-factor CIR and Vasicek models, respectively. The mortality rate ($\mu_x(t)$) was fitted by the Lee-Carter model. All the models for interest, inflation and mortality rates were formulated in the state space form and estimated by the Kalman filter.

As main results, we could observe a significant difference in the value of the options whether the excess of return is reverted to the reserve or paid directly to the policyholder. According to the interests of the insurance company, this result should be taken in account in the design of new products. Another interesting feature is regarding the frequency when the excess of return is reverted to the policyholder. It is observed an increase in the value of the “Guarantee” options if the payments become more frequent (for example: from 5 years to 1 year). This is due to the extra risk a more frequent reversion brings to the insurance company. Once the excess of return is paid to the policyholder, the reserve and its asset counter-part become equal, and this widen the risk of deficit in the following periods.

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