

ANALYTICAL FORMULAS  
FOR OPTIONS EMBEDDED IN LIFE INSURANCE POLICIES

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ABSTRACT

Analytical formulas for evaluating the integer moments of the present value of guaranteed benefits embedded in Unit Linked life insurance policies and their exercise probabilities are found. Such guarantees are given in the case of insured's death or in the case of surrender, with the surrender conditions contractually settled. Some results proposed in Carr (1998) are generalized in several directions and in particular the absence of arbitrage opportunity hypothesis is removed.

KEYWORDS

Unit Linked life insurance policies, surrender and death guarantees, stochastic maturity options.

## 1. INTRODUCTION

While for Black-Scholes European options formula the standard normal distribution tabulation is necessary, it is a well-known, surprising result that the price of European and American options is given by closed analytical formulas, without the necessity of any tabulation, if such options have exponential stochastic maturity in a Black-Scholes scenario with the absence of arbitrage opportunity.

Such formulas are derived in Carr (1998) which has the aim to obtain quasi analytical formulas for pricing classical American options (e.g with fixed deterministic maturity) by means of a recursive method thanks to the results for stochastic (exponential) maturity. Carr's method is based on a sequence of Erlang distributions that converges to a degenerate distribution with zero variance and with expected value equal to the deterministic maturity.

Other authors investigate the problem of pricing American options with stochastic maturity, but not for actuarial purposes: see El Karoui, Martellini (2001), Dorobantu, Pontier (2006).

A survey of applications of the stochastic maturity hypothesis is in Berrada (2005), in which it is described how to obtain the optimal exercise boundary and the expected value, for real options and employee stock options with stochastic (uniform, triangular) maturity.

We generalize Carr's results in the light of an actuarial point of view: in a life actuarial framework considering the stochastic maturity of the option as the stochastic death time is quite natural. The exponential maturity implies a constant death intensity, which is a reasonable hypothesis for a first approximation. The contents of this note have to be set in the area of the valuation of options embedded in life insurance policies, widely considered in actuarial literature both for European type options, as in Brennan, Schwartz (1976) and Bacinello, Ortu (1993), and for American type options, as in Grosen, Jorgensen (1997 e 2000), Bacinello (2002 e 2004) and Vannucci E. (1999, 2003a e 2003b).

Here we find analytical formulas, which do not need any tabulation, for evaluating (probability distributions, probabilities of exercise, moments of any order...) stochastic benefits of Unit Linked life policies with guarantees, which are given in case of death or in case of surrender. The surrender conditions are contractually settled and they force the holder to surrender his policy with a guarantee of minimum return rate, when the current value of the underlying asset hits a fixed barrier for the first time. Alternatively, if the insured dies before the barrier is hit, he receives at least a fixed amount when the underlying asset value is not sufficiently raised.

From an operative point of view, the analytical results allow to build policies which comply with the customer's requirements about levels and costs of guarantees.

An important feature is to have contractually settled the surrender conditions. If these conditions are left to the free decision of policy-holders the respective risk is hard to evaluate: for an attempt of modelling this aspect see Romagnoli (2007).

By the way the analysis of innovations in life insurance policies with flexible return guarantees has been the object of a MIUR (Italy) research (2003): for some contributions see Vannucci E. (2005), Vannucci L. (2005).

The paper is structured as follows. In section 2 we describe the financial scenario and the structure of the policy. In section 3 we obtain the analytical formulas for evaluating the options embedded in the policy. In section 4 we accommodate our results with the existing ones, reintroducing f.i. the absence of the arbitrage opportunity hypothesis. Some numerical examples are proposed in section 5. In section 6 we consider a first demographic improvement, which can bring to closed analytical formulas once more. The paper ends with a brief conclusion and two technical appendixes.

## 2. POLICY STRUCTURE

We consider a Unit Linked type policy issued at time 0. We assume that the underlying asset value  $S_t$ , with  $t \geq 0$  and initial value  $S_0 = s$ , follows a Geometric-Brownian motion, typical of the Black-Scholes scenario; such a policy provides two kinds of guarantees, one in case of death and one in case of surrender.

In details policy's benefits are:

- at the exponential distributed death time,  $T$ , a death-payment is made: this payment is equal to the maximum between the random underlying value,  $S_T$ , and a fixed guaranteed amount  $k_1$ ;
- alternatively, if at the random time  $U$ , with  $U < T$ , the underlying unit value hits for the first time the barrier  $b$ , with  $b < s$ , then the policy must be surrendered and the holder receives the payment  $k_2 \exp(gU)$ , with  $k_2 \geq b$  and  $g \geq 0$  is the guarantee interest intensity on the capital  $k_2$ .

A natural hypothesis it is to consider  $T$ , related to the demographic risk, and  $U$ , related to the financial risk, as independent random variables.

First, such a kind of policy would protect the holder (or more precisely his family) from the negative economic consequences of death event and, in second order, from too large losses in the underlying investment: hence, it may be fair to consider  $k_1$  much bigger than, not only of  $k_2$ , but also of  $s$ .

We remark that the cost of death guarantee without the possibility of surrender is equivalent to an European option with random (exponential) maturity and with fixed strike price  $k_1$ . Besides, if we assume  $g = 0$ ,  $k_1 = k_2$  and the absence of arbitrage opportunity hypothesis then we have, as a particular case of our model, Carr's model for the American option with stochastic (exponential) maturity.

Letting  $C_1$  be the random cost of death guarantee and  $C_2$  the random cost of surrender guarantee. The aim of this paper is to obtain the analytical solutions for

- the probability of exercise of each guarantee:  $P(C_1 > 0)$  and  $P(C_2 > 0)$

- the expected value and the variance, but also any integer moment, of the random cost of each guarantee:  $E [C_1^h]$  and  $E [C_2^h]$  for  $h = 1, 2, \dots$

For the total random cost  $C_1 + C_2$ , considering that the events  $C_1 > 0$  and  $C_2 > 0$  are mutually exclusive so that  $E [C_1^i C_2^j] = 0$  when  $i$  and  $j$  are positive integers, we have

$$P (C_1 + C_2 > 0) = P (C_1 > 0) + P (C_2 > 0)$$

and besides for  $h = 1, 2, \dots$

$$E [(C_1 + C_2)^h] = E [C_1^h] + E [C_2^h]$$

Obviously, by means of the  $h$ -moments, for  $h = 1, 2, \dots$ , of a generic random variable  $X$  other useful informations can be obtained, such as

$$\text{variance} \quad \sigma^2 [X] = E [X^2] - E^2 [X]$$

$$\text{skewness} \quad \gamma [X] = \frac{E [X^3] - 3E [X^2] E [X] + 2E^3 [X]}{\left( \sqrt{E [X^2] - E [X]^2} \right)^3}$$

$$\text{kurtosis} \quad \kappa [X] = \frac{E [X^4] - 4E [X^3] E [X] + 6E [X^2] E^2 [X] - 3E^4 [X]}{\left( \sqrt{E [X^2] - E^2 [X]} \right)^4}$$

We remark that the four parameters  $b$ ,  $k_1$ ,  $k_2$  and  $g$  can be freely used to comply with customer's requirements (and also those of the insurance company !) about levels and costs of guarantees.

Other parameters of the model describe the Black-Scholes scenario, related to the random motion of the underlying asset value, and they are

- $\mu$  the drift of the Brownian process of the random return of the underlying asset (its value,  $S_t$ , is lognormal distributed), viz. the drift of the process  $\ln \left( \frac{S_t}{s} \right)$ , with  $t \in [0, +\infty)$  and with  $S_0 = s$
- $\sigma > 0$  the volatility of the same Brownian process
- $r$  the valuation rate in a flat, here assumed, interest rate structure with discount factors  $\exp (-rt)$  for  $t \in [0, +\infty)$

If we assume the absence of arbitrage opportunity hypothesis then it will follow for these parameters

$$r = \mu + \frac{\sigma^2}{2}$$

Considering an actuarial framework,  $\mu$  and  $\sigma$  describe the investment skill of the insurance company and  $r$  the free risk opportunity of the policy-holder; so that here we assume

$$r \leq \mu + \frac{\sigma^2}{2}$$

but our results will work in any case, so also if  $r > \mu + \frac{\sigma^2}{2}$ .

Finally, let

- $a$  be the constant insured's death intensity with  $P(T > t) = \exp(-at)$

From an operative point of view we can consider the inverse of the expected further life of the policy-holder as a first approximation of the value of  $a$ . Obviously, with exponential assumption for  $T$ , death time can be seen as the first stopping event of a Poisson process of parameter  $a$ . In section 6 of this note we will propose a relaxation of the insured's death constant density hypothesis, that can bring to closed analytical formulas once more.

### 3. FORMULAS FOR EMBEDDED OPTIONS

For the actuarial purposes of this paper, we prefer to study the posed problem with classical methods, which allow immediate interpretations and very helpful informations, in comparison to those gettable by stochastic differential equations and Itô calculus, generally limited to the expected values of the random variables of interest.

To begin with we have to consider the two following probabilities.

First, the probability of the event  $U \in [t, t + dt)$  and  $T > t$  ( $dt$  infinitesimal), viz.

$$\begin{aligned} P(U \in [t, t + dt), T > t) &\equiv m(t) dt = \\ &= \frac{|z_b|}{\sigma\sqrt{2\pi t^3}} \exp\left(-\frac{(z_b - \mu t)^2}{2\sigma^2 t}\right) \cdot \exp(-at) dt \end{aligned}$$

with  $t \in (0, +\infty)$  and where  $z_b = \ln \frac{b}{s}$  represents the negative return, applied to the initial value  $s$  of the underlying asset, with which the barrier  $b$  (whatever  $t$ ) is hit. As above-mentioned, at the first barrier's hitting the holder must surrender his policy.

Secondly, the probability of the event  $T \in [t, t + dt)$  and  $\ln\left(\frac{S_t}{s}\right) \in [x, x + dx)$  and  $U \in (t, +\infty]$  ( $dt$  and  $dx$  infinitesimals), viz.

$$\begin{aligned} P\left(T \in [t, t + dt), \ln\left(\frac{S_t}{s}\right) \in [x, x + dx), U \in (t, +\infty]\right) &\equiv n(t, x) dt dx = \\ &= a \exp(-at) dt \cdot \\ &\cdot \left( \frac{\exp\left(-\frac{(x - \mu t)^2}{2\sigma^2 t}\right)}{\sigma\sqrt{2\pi t}} - \exp\left(\frac{2z_b\mu}{\sigma^2}\right) \cdot \frac{\exp\left(-\frac{(x - 2z_b - \mu t)^2}{2\sigma^2 t}\right)}{\sigma\sqrt{2\pi t}} \right) dx \end{aligned}$$

with  $t \in (0, +\infty)$  and  $x \in \left(\ln \frac{b}{s}, +\infty\right)$ .

The first result, relative to  $m(t) dt$ , is well-known for Brownian motion: see e.g. Bhattacharya, Waymire (1990). For completeness we remark that

$$P(U \in (t, +\infty)) = \frac{1}{\sqrt{2\pi}} \left| \frac{z_b}{\sigma} \right| \exp\left(\frac{\mu z_b}{\sigma^2}\right) \int_t^\infty \frac{\exp\left(-\frac{z_b^2}{2v\sigma^2} - \frac{\mu^2 v}{2\sigma^2}\right)}{\sqrt{v^3}} dv$$

and just differentiating this function with respect to  $t$  we can obtain the density probability function  $m(t)$ . Note that the value of  $P(U \in (t, +\infty))$  is less than 1 for each  $t \in [0, +\infty)$  if either  $\mu > 0$ ,  $z_b < 0$  or  $\mu < 0$ ,  $z_b > 0$ : in these cases  $1 - P(U \in (0, +\infty))$  gives the probability that the underlying asset value,  $s$  at time 0, never hits the barrier  $b$ , viz.  $P(U = +\infty)$ .

The second result, relative to  $n(t, x) dt dx$ , is less common: it can be found by the Reflection Principle for the trajectories of Brownian motions (see for an hint the Appendix A).

The following factor in  $n(t, x) dt dx$

$$\left( \frac{\exp\left(-\frac{(x - \mu t)^2}{2\sigma^2 t}\right)}{\sigma\sqrt{2\pi t}} - \exp\left(\frac{2z_b\mu}{\sigma^2}\right) \cdot \frac{\exp\left(-\frac{(x - 2z_b - \mu t)^2}{2\sigma^2 t}\right)}{\sigma\sqrt{2\pi t}} \right) dx$$

represents the probability  $P\left(\ln\left(\frac{S_t}{s}\right) \in [x, x + dx), U \in (t, +\infty]\right)$  for each given  $t \in (0, +\infty)$ : note that if  $U = +\infty$  then the barrier has never been hit. Analyzing the formula of this probability, we can point out how the event  $U \in (t, +\infty]$ , or equivalently  $S_\tau > b$  for each  $\tau \in [0, t]$ , "twists" the usual (when no barrier is considered) normal density (with parameters  $\mu t$  and  $\sigma^2 t$ ) of the random variable  $\ln\left(\frac{S_t}{s}\right)$ . It follows by integration that

$$\begin{aligned} P(U \in (t, +\infty]) &= \\ &= \int_{\ln\left(\frac{b}{s}\right)}^{+\infty} \left( \frac{\exp\left(-\frac{(x - \mu t)^2}{2\sigma^2 t}\right)}{\sigma\sqrt{2\pi t}} - \exp\left(\frac{2z_b\mu}{\sigma^2}\right) \cdot \frac{\exp\left(-\frac{(x - 2z_b - \mu t)^2}{2\sigma^2 t}\right)}{\sigma\sqrt{2\pi t}} \right) dx \end{aligned}$$

and obviously  $P(U \in (t, +\infty]) \geq P(U \in (t, +\infty))$ , being

$$P(U = +\infty) = P(U \in (t, +\infty]) - P(U \in (t, +\infty))$$

With the formulas of these two probabilities it is quite evident how to solve the posed problem. In details, we have to calculate the following integrals: for  $C_1$  with  $h = 1, 2, \dots$

$$P(C_1 > 0) = \int_0^{\infty} \int_{z_b}^{\ln \frac{k_1}{s}} n(t, x) dx dt$$

$$E[C_1^h] = \int_0^{\infty} \int_{z_b}^{\ln \frac{k_1}{s}} n(t, x) \cdot (k_1 - s \exp x)^h \cdot \exp(-hrt) dx dt$$

and for  $C_2$  with  $h = 1, 2, \dots$

$$P(C_2 > 0) = \int_0^{\infty} m(t) dt$$

$$E[C_2^h] = \int_0^{\infty} \left( m(t) \cdot (k_2 \exp(gt) - b)^h \cdot \exp(-hrt) \right) dt$$

All these integrals can be calculated by means of the results reported in Appendix B of this paper.

Considering the benefit  $C_1$ , by using the results of Appendix B, where we define  $G_h \equiv \sqrt{\mu^2 + 2\sigma^2(a + hr)}$  for  $h = 0, 1, 2, \dots$ , it is necessary to calculate integrals of the following type with  $h = 0, 1, 2, \dots$  and  $i = 0, 1, 2, \dots, h$

$$\begin{aligned} \psi_j(h, i) &\equiv \\ &\equiv \int_{z_b}^{\ln \frac{k_1}{s}} \left[ \frac{a}{G_h} \exp\left(-\frac{|x|}{\sigma^2} G_h\right) \cdot \exp\left(\frac{\mu x}{\sigma^2}\right) + \right. \\ &\quad \left. - \left( \frac{a}{G_h} \exp\left(-\frac{|x - 2z_b|}{\sigma^2} G_h\right) \cdot \exp\left(\frac{\mu x}{\sigma^2}\right) \right) \right] \cdot e^{ix} dx \end{aligned}$$

setting  $j = 1$  if  $k_1 \leq s$  and  $j = 2$  if  $k_1 > s$ .

Calculating these integrals is easy. In particular, with  $k_1 \leq s$  it follows ( $x$  is negative in the integration interval)

$$\psi_1(h, i) \equiv \frac{a}{G_h} \left( \frac{\left(\frac{k_1}{s}\right)^{\frac{\mu + G_h}{\sigma^2} + i} - \left(\frac{b}{s}\right)^{\frac{\mu + G_h}{\sigma^2} + i}}{\frac{\mu + G_h}{\sigma^2} + i} + \left(\frac{b}{s}\right)^{\frac{2G_h}{\sigma^2}} \cdot \frac{\left(\frac{k_1}{s}\right)^{\frac{\mu - G_h}{\sigma^2} + i} - \left(\frac{b}{s}\right)^{\frac{\mu - G_h}{\sigma^2} + i}}{\frac{\mu - G_h}{\sigma^2} + i} \right)$$

while, with  $k_1 > s$  it follows ( $x$  changes sign in the integration interval)

$$\psi_2(h, i) \equiv \frac{a}{G_h} \left( \begin{array}{c} \left( \frac{k_1}{s} \right)^{\frac{\mu - G_h}{\sigma^2} + i} - 1 \quad 1 - \left( \frac{b}{s} \right)^{\frac{\mu + G_h}{\sigma^2} + i} \\ \frac{\mu - G_h}{\sigma^2} + i \quad \frac{\mu + G_h}{\sigma^2} + i \\ - \left( \frac{b}{s} \right)^{\frac{2G_h}{\sigma^2}} \cdot \frac{\left( \frac{k_1}{s} \right)^{\frac{\mu - G_h}{\sigma^2} + i} - \left( \frac{b}{s} \right)^{\frac{\mu - G_h}{\sigma^2} + i}}{\frac{\mu - G_h}{\sigma^2} + i} \end{array} \right)$$

Given the two functions  $\psi_j(h, i)$ , where  $j = 1$  or  $j = 2$ , for the random benefit  $C_1$  we obtain

$$P(C_1 > 0) = \psi_j(0, 0)$$

$$E[C_1] = k_1 \cdot \psi_j(1, 0) - s \cdot \psi_j(1, 1)$$

$$E[C_1^2] = k_1^2 \cdot \psi_j(2, 0) + s^2 \cdot \psi_j(2, 2) - 2sk_1 \cdot \psi_j(2, 1)$$

and more generally for each  $h = 1, 2, \dots$

$$E[C_1^h] = \sum_{i=0}^h \binom{h}{i} k_1^{h-i} (-s)^i \psi_j(h, i)$$

For benefit  $C_2$  the calculus is easier. The introduction of the function (see Appendix B)

$$\varphi(\beta) \equiv \left( \frac{b}{s} \right)^{\frac{\mu + \sqrt{\mu^2 + 2\sigma^2(a+\beta)}}{\sigma^2}}$$

allows to obtain immediately

$$P(C_2 > 0) = \varphi(0)$$

$$E[C_2] = k_2 \cdot \varphi(r - g) - b \cdot \varphi(r)$$

$$E[C_2^2] = k_2^2 \cdot \varphi(2r - 2g) + b^2 \cdot \varphi(2r) - 2bk_2 \cdot \varphi(2r - g)$$

and more generally for each  $h = 1, 2, \dots$

$$E[C_2^h] = \sum_{i=0}^h \binom{h}{i} k_2^{h-i} (-b)^i \varphi(hr - (h-i)g)$$

#### 4. PARTICULAR CASES

To find some well-known results concerning American type options, e.g. Carr formula for  $E[C_1 + C_2]$ , starting from ours, we must consider

- $g = 0$
- the absence of arbitrage opportunity hypothesis



With these two assumptions we have a relevant simplification for the analytical solution of  $E[C_1 + C_2]$ . Indeed, being

$$\frac{\frac{a}{G_1}}{\frac{\mu + G_1}{\sigma^2} + 1} - \frac{\frac{a}{G_1}}{\frac{\mu - G_1}{\sigma^2} + 1} = \frac{a}{r - (\mu + \frac{\sigma^2}{2}) + a},$$

this difference is 1 with the absence of arbitrage opportunity hypothesis.

With these two assumptions, our results allow to obtain for  $k_1 \leq s$  a simple formula for  $E[C_1 + C_2]$  that points out the role of parameter  $b$  in the embedded option valuation. Note that the term

$$b \cdot \left( 1 - \frac{\frac{a}{G_1}}{\frac{\mu + G_1}{\sigma^2} + 1} + \frac{\frac{a}{G_1}}{\frac{\mu - G_1}{\sigma^2} + 1} \right)$$

is reported for completeness but it could be cancelled in the following expansion of  $E[C_1 + C_2]$

$$\begin{aligned} E[C_1 + C_2] &= \omega_1(b) \equiv \\ &= \left(\frac{b}{s}\right)^{\frac{\mu + G_1}{\sigma^2}} \left( \begin{aligned} &k_2 - \frac{\frac{a}{G_1} k_1}{\frac{\mu + G_1}{\sigma^2} + 1} + \frac{\frac{a}{G_1} k_1}{\frac{\mu - G_1}{\sigma^2} + 1} \\ &- b \cdot \left( 1 - \frac{\frac{a}{G_1}}{\frac{\mu + G_1}{\sigma^2} + 1} + \frac{\frac{a}{G_1}}{\frac{\mu - G_1}{\sigma^2} + 1} \right) \end{aligned} \right) + \\ &+ \left(\frac{b}{s}\right)^{\frac{2G_1}{\sigma^2}} \left(\frac{k_1}{s}\right)^{\frac{\mu - G_1}{\sigma^2}} \left( \frac{\frac{a}{G_1} k_1}{\frac{\mu - G_1}{\sigma^2} + 1} - \frac{\frac{a}{G_1} k_1}{\frac{\mu - G_1}{\sigma^2}} \right) + \\ &+ \left(\frac{k_1}{s}\right)^{\frac{\mu + G_1}{\sigma^2}} \left( \frac{\frac{a}{G_1} k_1}{\frac{\mu + G_1}{\sigma^2}} - \frac{\frac{a}{G_1} k_1}{\frac{\mu + G_1}{\sigma^2} + 1} \right) \end{aligned}$$

Analogously, for  $k_1 > s$  (again the above-mentioned term could be cancelled) we have

$$\begin{aligned} E[C_1 + C_2] &= \omega_2(b) \equiv \\ &= \left(\frac{b}{s}\right)^{\frac{\mu + G_1}{\sigma^2}} \left( \begin{aligned} &k_2 - \frac{\frac{a}{G_1} k_1}{\frac{\mu + G_1}{\sigma^2} + 1} + \frac{\frac{a}{G_1} k_1}{\frac{\mu - G_1}{\sigma^2} + 1} \\ &- b \cdot \left( 1 - \frac{\frac{a}{G_1}}{\frac{\mu + G_1}{\sigma^2} + 1} + \frac{\frac{a}{G_1}}{\frac{\mu - G_1}{\sigma^2} + 1} \right) \end{aligned} \right) + \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{b}{s}\right)^{\frac{2G_1}{\sigma^2}} \left(\frac{k_1}{s}\right)^{\frac{\mu-G_1}{\sigma^2}} \left( \frac{\frac{a}{G_1}k_1}{\frac{\mu-G_1}{\sigma^2} + 1} - \frac{\frac{a}{G_1}k_1}{\frac{\mu-G_1}{\sigma^2}} \right) + \\
& + \frac{ak_1}{G_1} \left( \frac{\left(\frac{k_1}{s}\right)^{\frac{\mu-G_1}{\sigma^2}} - 1}{\frac{\mu-G_1}{\sigma^2}} + \frac{1}{\frac{\mu+G_1}{\sigma^2}} \right) - \frac{sa}{G_1} \left( \frac{\left(\frac{k_1}{s}\right)^{\frac{\mu-G_1}{\sigma^2}+1} - 1}{\frac{\mu-G_1}{\sigma^2} + 1} + \frac{1}{\frac{\mu+G_1}{\sigma^2} + 1} \right)
\end{aligned}$$

With the aforesaid assumptions ( $g = 0$  and no arbitrage opportunity hypothesis) the maximization of  $\omega_1(b)$  or of  $\omega_2(b)$  brings to the same exact optimal barrier value

$$b_{ott} = k_1 \cdot \left( \frac{\frac{k_2(\mu+G_1)}{2ak_1\sigma^2} + \frac{1}{\mu-G_1}}{\frac{1}{\mu-G_1} - \frac{1}{\mu-G_1+\sigma^2}} \right)^{\frac{\sigma^2}{G_1-\mu}}$$

if  $b_{ott} \in (0, \min(k_1, k_2))$ , otherwise the optimal solution is a corner solution:  $b_{ott} = 0$  or  $b_{ott} = \min(k_1, k_2)$ . The condition  $b_{ott} \in (0, \min(k_1, k_2))$  is verified if  $k_1 = k_2$  as in Carr's model, but it may not hold for  $k_1$  much bigger than  $k_2$ : see the example relative to Tab. 1 in the section 5.

The above-mentioned value of  $b_{ott}$  and those of  $\omega_1(b_{ott})$  and  $\omega_2(b_{ott})$  when  $k_1 = k_2$ , as well as  $g = 0$  and no arbitrage opportunity hypothesis, coincide with those of Carr: he derives his formulas starting from the theory of stochastic differential equations and applying Itô Lemma, without giving the detailed results which we have here obtained applying more standard methods.

We remark that, with arbitrage opportunity hypothesis, viz.  $r \neq \mu + \frac{\sigma^2}{2}$ , and with  $g = 0$ , the maximization of  $E[C_1 + C_2]$  brings to an equation in  $b$ , which doesn't have exact explicit solutions, but only numerical approximations of them. In the maximization (of  $E[C_1 + C_2]$ ) problem, to search  $b_{ott}$  is equivalent to maximize respect to  $b$  functions as

$$b^A (Cb + 1) - Db^E$$

with the real parameters  $A, D, E$  positive and  $E > A$ . The real parameter  $C$  is "small" in absolute value, with the sign equal to the sign of  $\mu + \frac{\sigma^2}{2} - r$ . Note that it is  $C = 0$  if and only if we assume the no arbitrage opportunity hypothesis: in this case, with  $g = 0$ , the equivalent function to maximize becomes  $b^A - Db^E$ , which allows exact analytical treatments of the maximization problem.

Finally, with arbitrage opportunity hypothesis, viz.  $r \neq \mu + \frac{\sigma^2}{2}$ , and with  $g > 0$ , the above-posed problem becomes even more difficult: the maximization of  $E[C_1 + C_2]$  is in general equivalent to maximize respect to  $b$  functions as

$$b^F + b^A (Cb + H) - Db^E$$

with the real parameters  $A, D, E, F, H$  positive,  $E > A > F$  and with  $C$  as above-mentioned.

Lastly, we remark that if  $b = 0$  then no surrender guarantee is active: in this case the option embedded is of European type with stochastic (exponential) maturity. If  $a = 0$  then no death guarantee is active: in this case the option embedded is of Russian type with infinity maturity. In both cases the corresponding formulas would be considerably reduced.

## 5. NUMERICAL EVIDENCE

All the results given in this section are obtained considering a policy-holder with an expected further life of 40 years, so it is fair to assume  $a = 0.025$ .

The other parameters of the model are fixed as follows:  $s = 100$ ,  $k_1 = 400$ ,  $k_2 = 95$ ,  $g = 0.01$ ,  $\mu = 0.05$ ,  $\sigma = 0.2$ ,  $r = 0.03$ . In Tab. 1 we show the values of (1)  $P(C_1 > 0)$ , (2)  $E[C_1]$ , (3)  $\sigma[C_1]$ , (4)  $P(C_2 > 0)$ , (5)  $E[C_2]$ , (6)  $\sigma[C_2]$ , (7)  $P(C_1 + C_2 > 0)$ , (8)  $E[C_1 + C_2]$ , (9)  $\sigma[C_1 + C_2]$  for some increasing values of  $b$

Tab. 1  $s = 100$ ,  $k_1 = 400$ ,  $k_2 = 95$ ,  $g = 0.01$ ,  $\mu = 0.05$ ,  $\sigma = 0.2$ ,  $r = 0.03$

$b$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
5	0.517	81.395	101.604	0.000	0.006	0.526	0.517	81.401	101.600
15	0.514	81.128	101.637	0.004	0.191	3.154	0.518	81.319	101.534
25	0.505	80.026	101.627	0.017	0.873	6.707	0.522	80.898	101.160
35	0.485	77.507	101.341	0.046	2.228	10.242	0.531	79.735	100.147
45	0.454	73.063	100.416	0.097	4.205	12.960	0.550	77.268	98.167
55	0.408	66.273	98.309	0.174	6.476	14.176	0.582	72.748	94.906
65	0.348	56.808	94.190	0.283	8.402	13.377	0.631	65.210	89.979
75	0.270	44.434	86.703	0.431	9.007	10.364	0.701	53.440	82.610
85	0.176	29.002	73.240	0.621	6.951	5.554	0.797	35.952	70.652
95	0.063	10.447	46.112	0.861	0.511	1.395	0.924	10.959	46.018

Some remarks on the results proposed in Tab. 1. For instance, the figures of the row corresponding to  $b = 75$  tell that

- the policy-holder pays a premium of  $100 + 53.440 + l = 153.440 + l$  ( $l$  is the loading)
- with 0.270 probability he dies receiving a payment of 400, partially covered by the value of the underlying asset
- with 0.431 probability the policy is surrendered (before death)
- with  $1 - 0.701 = 0.299$  probability he dies receiving a payment of more than 400, fully covered by the value of the underlying asset.

We remark also that the value of  $E[C_1 + C_2]$  is "stable" from  $b = 5$  to  $b = 45$ .

In the case relative to Tab.1,  $k_1$  is much bigger than  $k_2$  so that the maximum value for  $E[C_1 + C_2]$  could be (or near to) the corner value,  $b_{ott} = 0$ : big values of  $b$  in the range  $[0, 95]$  considerably reduce  $E[C_1]$ , see (2), and the increasing-decreasing movement of  $E[C_2]$ , see (5) with maximum value "near"  $b = 75$ , doesn't change the decrease of  $E[C_1 + C_2]$ .

To find the expected value of the guarantees costs at current value we can consider our formulas with  $r = 0$ : for  $b = 75$  we obtain (same interpretation of Tab. 1)

Tab. 2  $s = 100, k_1 = 400, k_2 = 95, g = 0.01, \mu = 0.05, \sigma = 0.2, r = 0$

$b$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
75	0.270	56.650	103.025	0.431	10.500	12.953	0.701	67.150	97.941

We can deduce by Tab.1 (row  $b = 75$ ) and Tab. 2 that

- with 0.270 probability the payment of 400 is covered by the value of the underlying asset, in average  $400 - 56.650/0.270 = 190.185$ , and by the death guarantee benefit, in average  $56.650/0.270 = 209.815$ , with expected present value  $44.434/0.270 = 164.570$ : an estimation of the payment time of the death guarantee benefit is so obtainable by the solution of the equation  $209.815 = 164.570 \exp(0.03t)$ , viz.  $\hat{t}_T = 8.096$
- with 0.431 probability the policy is surrendered (before death) with an average payment of  $75 + 10.500/0.431 = 99.362$  with expected present value  $75 + 9.007/0.431 = 95.898$ : an estimation of the payment time of the surrender guarantee benefit is so obtainable by the solution of  $99.362 = 95.898 \exp(0.03t)$ , viz.  $\hat{t}_U = 1.183$

For completeness, we show in Tab. 3 (without comments) for  $b = 75$  some values of skewness and of kurtosis: (1)  $\gamma[C_1]$ , (2)  $\kappa[C_1]$ , (3)  $\gamma[C_2]$ , (4)  $\kappa[C_2]$ , (5)  $\gamma[C_1 + C_2]$ , (6)  $\kappa[C_1 + C_2]$

Tab. 3  $s = 100, k_1 = 400, k_2 = 95, g = 0.01, \mu = 0.05, \sigma = 0.2, r = 0.03$

$b$	(1)	(2)	(3)	(4)	(5)	(6)
75	1.786	4.725	0.286	1.092	1.793	4.846

With the same values for the parameters of the model, except now  $k_1 = 240$  and  $k_2 = 100$ , we show in the following Tab. 4 the values of (1)  $P(C_1 > 0)$ , (2)  $E[C_1]$ , (3)  $\sigma[C_1]$ , (4)  $P(C_2 > 0)$ , (5)  $E[C_2]$ , (6)  $\sigma[C_2]$ , (7)  $P(C_1 + C_2 > 0)$ , (8)  $E[C_1 + C_2]$ , (9)  $\sigma[C_1 + C_2]$  for some increasing values of  $b$ , as in Tab.1,

Tab. 4  $s = 100, k_1 = 240, k_2 = 100, g = 0.01, \mu = 0.05, \sigma = 0.2, r = 0.03$

$b$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
5	0.399	31.304	47.789	0.000	0.007	0.555	0.400	31.310	47.788
15	0.397	31.157	47.762	0.004	0.203	3.344	0.401	31.360	47.747
25	0.388	30.582	47.567	0.017	0.932	7.162	0.405	31.514	47.508
35	0.371	29.337	47.016	0.046	2.401	11.043	0.417	31.738	46.814
45	0.344	27.253	45.908	0.097	4.592	14.162	0.441	31.846	45.363
55	0.307	24.242	44.018	0.174	7.213	15.799	0.480	31.455	42.866
65	0.258	20.283	41.059	0.283	9.660	15.389	0.541	29.943	39.125
75	0.198	15.417	36.591	0.431	10.997	12.646	0.628	26.414	34.056
85	0.126	9.738	29.776	0.621	9.922	7.800	0.748	19.660	27.463
95	0.045	3.381	17.979	0.861	4.754	2.255	0.905	8.135	17.210

In this case  $k_1$  and  $k_2$  are "sufficiently" near so that  $b_{ott}$  (for the maximization of  $E[C_1 + C_2]$ ) is interior to  $(0, 100)$ . The maximum value of  $E[C_1 + C_2]$  in Tab. 4 is obtained with  $b = 45$ . We remark only that the value of  $E[C_1 + C_2]$  is here "stable" from  $b = 5$  to  $b = 55$ .

## 6. A DEMOGRAPHIC IMPROVEMENT

We are aware that our results are based on special assumptions, namely the Geometric-Brownian motion of the underlying asset value and the constant death intensity and that both ought to hold for very large periods of time considering the actuarial framework.

Of course, natural generalizations could be opportune and here we consider a first attempt for death time  $T$ . In the place of  $P(T > t) = e^{-at}$  we can consider for  $t \in [0, +\infty)$

$$P(T > t) = \sum_{i=1}^n \alpha_i \exp(-\gamma_i t)$$

with  $\gamma_i > 0$  and proper other parameters  $\alpha_i$  for  $i = 1, 2, \dots, n$ . For instance, with  $n = 2$  we can assume

$$1 - F_T(t) = P(T > t) = ce^{-at} + (1 - c)e^{-bt}$$

with  $a$  and  $b$  positive real parameters with  $a < b$  and  $c \in \left[0, \frac{b}{b-a}\right]$ . In this case death intensity is

$$\frac{F_T^0(t)}{1 - F_T(t)} = \frac{ace^{-at} + b(1 - c)e^{-bt}}{ce^{-at} + (1 - c)e^{-bt}}$$

This intensity varies from  $ac + b(1 - c)$  for  $t = 0$  to  $a$  for  $t \rightarrow +\infty$ : it decreases if  $c \in [0, 1)$ ; it is constant if  $c = 1$  and it increases (more realistically) if  $c \in \left(1, \frac{b}{b-a}\right]$ . Note that the first three moments of a realistic  $T$  could be well-matched even in this simple case with only three parameters  $a, b, c$ .

It is immediate to convince ourselves that analytical type formulas, that we have found with constant death intensity, could be easily generalized to this

case. Obviously, without constant death intensity the optimal surrender barrier doesn't correspond to the optimal exercise frontier of the embedded American option, even if  $g = 0$  and no arbitrage opportunity hypothesis are assumed.

## CONCLUSION

The analytical formulas we have derived easily allow some valuable sensitivity analyses in order to calibrate the premiums that have to be paid for guaranteed benefits.

Besides, these formulas can be used to control the efficiency of simulation methods. When they are tested with the hypotheses here assumed, they must give valuations near to the exact ones !

Finally, the authors together have discussed all the aspects of the work. Both have verified analytical procedures and the properness of results. If a division of the work must be made, then E. has been more engaged in the formulation of actuarial models and L. in the probabilistic features.

## APPENDIX A

In this appendix we will outline how to obtain the probability density of  $\ln\left(\frac{S_t}{s}\right)$  conditioning to the event " $S_t$  doesn't hit the barrier  $b$  before  $t$ " or equivalently " $\ln\left(\frac{S_t}{s}\right)$  doesn't hit the barrier  $z_b$  before  $t$ ".

For the Bernoullian random walk, starting in 0, with independent increments  $+1$  and  $-1$ , with the up probability  $p$ , and with a barrier in  $-B$  with  $B$  a fixed positive integer, by applying the Reflexion Principle, we have ( $S_k$  is the position of the random walk at time  $k$  with  $k = 0, 1, \dots, n$  and  $n + h$  even)

$$\begin{aligned} P(S_n = h \mid S_k > -B \text{ for } k = 0, 1, \dots, n) &= \\ &= \left( \binom{n}{\frac{n+h}{2}} - \binom{n}{\frac{n+h}{2} + B} \right) p^{\frac{n+h}{2}} (1-p)^{\frac{n-h}{2}} \end{aligned}$$

Basing on the Fundamental Correspondence Limit Theorem, to obtain the analogous result in  $t$  for a Wiener process with parameters  $\mu$  and  $\sigma > 0$ , we divide the time interval  $[0, t]$  in  $[nt]$  subintervals, each of length  $\frac{1}{n}$ , and we fix the increments  $\Delta_n > 0$  and  $-\Delta_n$  and the up probability  $p_n$  as follows

$$\begin{cases} p_n = \frac{1}{2} \left( \frac{\mu}{\sqrt{n\sigma^2 + \mu^2}} + 1 \right) \\ \Delta_n = \frac{\sqrt{n\sigma^2 + \mu^2}}{n} \end{cases}$$

With this parametrization if  $H_{[nt]}$  is the position at time  $t$  of the rescaled walk, with generic determination  $h_{[nt]}$ , it follows

$$E[H_{[nt]}] = \Delta_n (2p_n - 1) \cdot [nt] = \mu t$$

$$\sigma^2 [H_{[nt]}] = 4\Delta_n^2 p_n (1 - p_n) \cdot [nt] = \sigma^2 t$$

From (note  $z_b < 0$  in the model)

$$h_{[nt]} \simeq [nt] \cdot (2p_n - 1) + x \cdot \sqrt{p_n (1 - p_n)} [nt]$$

$$B_n \simeq \frac{-z_b \sqrt{[n]}}{\sigma}$$

the expected result can be obtained by the following limit

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \sqrt{p_n (1 - p_n)} [nt] \cdot \\ & \cdot \left( \left( \frac{[nt]}{[nt] + h_{[nt]}} \right) - \left( \frac{[nt]}{[nt] + h_{[nt]} + B_n} \right) \right) \cdot p_n^{\frac{[nt] + h_{bntc}}{2}} (1 - p_n)^{\frac{[nt] - h_{bntc}}{2}} = \\ & = \frac{\exp\left(-\frac{(x - \mu t)^2}{2\sigma^2 t}\right)}{\sigma \sqrt{2\pi t}} - \exp\left(\frac{2z_b \mu}{\sigma^2}\right) \cdot \frac{\exp\left(-\frac{(x - 2z_b - \mu t)^2}{2\sigma^2 t}\right)}{\sigma \sqrt{2\pi t}} \end{aligned}$$

## APPENDIX B

To obtain analytical solutions for the moments of  $C_2$  it is necessary to solve improper integrals of the type ( $A$  and  $B$  are two positive real constants)

$$\begin{aligned} & \int_0^{+\infty} \frac{1}{\sqrt{t^3}} \exp\left(-\left(A^2 t + \frac{B^2}{t}\right)\right) dt = \text{(given the substitution } t = \frac{B}{A} w^{-2}\text{)} \\ & = 2 \cdot \sqrt{\frac{A}{B}} \cdot \int_0^{+\infty} \left( \exp\left(-AB \left(w^2 + \frac{1}{w^2}\right)\right) \right) dw = \\ & = 2 \cdot \sqrt{\frac{A}{B}} \cdot \exp(2AB) \cdot \int_0^{+\infty} \left( \exp\left(-AB \left(w + \frac{1}{w}\right)^2\right) \right) dw = \\ & \text{(since } \int_0^{+\infty} = \int_0^1 + \int_1^{+\infty}\text{, with } w + \frac{1}{w} = z \text{ it is } w = \frac{z \pm \sqrt{z^2 - 4}}{2}\text{)} \\ & = 2 \cdot \sqrt{\frac{A}{B}} \cdot \exp(2AB) \cdot \left( \int_2^2 (\exp(-ABz^2)) d\frac{z - \sqrt{z^2 - 4}}{2} + \right. \\ & \quad \left. + \int_2^{+\infty} (\exp(-ABz^2)) d\frac{z + \sqrt{z^2 - 4}}{2} \right) = \\ & = 2 \cdot \sqrt{\frac{A}{B}} \cdot \exp(2AB) \cdot \int_2^{\infty} (\exp(-ABz^2)) d\sqrt{z^2 - 4} = \end{aligned}$$

(given  $z = 2 \cosh x$  and so  $\sqrt{z^2 - 4} = 2 \sinh x$ )

$$\begin{aligned}
&= 4 \cdot \sqrt{\frac{A}{B}} \cdot \exp(2AB) \cdot \int_0^\infty \left( \exp\left(-4AB (\cosh x)^2\right) \right) d \sinh x = \\
&= 4 \cdot \sqrt{\frac{A}{B}} \cdot \exp(2AB) \cdot \int_0^\infty \left( \exp\left(-4AB \left((\sinh x)^2 + 1\right)\right) \right) d \sinh x =
\end{aligned}$$

(finally, given  $\sinh x = \frac{y}{2\sqrt{2AB}}$ )

$$\begin{aligned}
&= 4 \sqrt{\frac{A}{B}} \cdot \frac{1}{2\sqrt{2AB}} \cdot \exp(-2AB) \cdot \int_0^\infty \left( \exp\left(-\frac{z^2}{2}\right) \right) dz = \\
&= 2 \cdot \sqrt{\frac{A}{B}} \cdot \frac{1}{\sqrt{2AB}} \cdot \exp(-2AB) \cdot \sqrt{\frac{\pi}{2}} = \frac{\sqrt{\pi}}{B} \cdot \exp(-2AB)
\end{aligned}$$

In our specific case  $z_b$  is negative and if we consider a generic real value  $\beta$  (with  $\beta = 0$  or  $\beta = r$  or  $\beta = r - g$  or  $\beta = 2r$  or  $\beta = 2r - 2g$  or  $\beta = 2r - g$  or ...) it is

$$\begin{aligned}
&\int_0^\infty \left( \frac{|z_b|}{\sigma\sqrt{2\pi t^3}} \exp\left(-\frac{(z_b - \mu t)^2}{2\sigma^2 t}\right) \cdot \exp(-at) \cdot \exp(-\beta t) \right) dt = \\
&= \frac{|z_b|}{\sigma\sqrt{2\pi}} \exp\left(\frac{z_b \mu}{\sigma^2}\right) \int_0^\infty \left( \frac{1}{\sqrt{t^3}} \exp - \left( \frac{z_b^2}{2\sigma^2} \cdot \frac{1}{t} + \left( \frac{\mu^2 + 2\sigma^2(a + \beta)}{2\sigma^2} \right) t \right) \right) dt \\
&= \exp\left(\frac{z_b \mu - |z_b| \sqrt{\mu^2 + 2\sigma^2(a + \beta)}}{\sigma^2}\right) = \\
&= \left(\frac{b}{s}\right) \frac{\mu + \sqrt{\mu^2 + 2\sigma^2(a + \beta)}}{\sigma^2} \equiv \varphi(\beta)
\end{aligned}$$

To obtain analytical solutions for the moments of  $C_1$  it is necessary to solve improper integrals of the type ( $A$  and  $B$  are two positive real constants)

$$\begin{aligned}
&\int_0^{+\infty} \frac{1}{\sqrt{t}} \exp\left(-\left(A^2 t + \frac{B^2}{t}\right)\right) dt = \text{(by the substitution } t = u^{-1}\text{)} \\
&= \int_0^{+\infty} \frac{1}{\sqrt{u^3}} \exp\left(-\left(B^2 u + \frac{A^2}{u}\right)\right) du = \frac{\sqrt{\pi}}{A} \cdot \exp(-2AB)
\end{aligned}$$

since this integral is taken back, by the aforesaid substitution, to the previous one.



In this specific case, given  $G_h = \sqrt{\mu^2 + 2\sigma^2(a + hr)}$  with  $h = 0, 1, 2, \dots$ ,

$$\begin{aligned} & \int_0^{\infty} a \exp(-at) \frac{\exp\left(-\frac{(x - \mu t)^2}{2 \cdot \sigma^2 \cdot t}\right)}{\sigma \cdot \sqrt{t} \cdot \sqrt{2\pi}} \cdot \exp(-hrt) dt = \\ & = \frac{a}{\sqrt{\mu^2 + 2\sigma^2(a + hr)}} \exp\left(-\frac{|x|}{\sigma^2} \sqrt{\mu^2 + 2\sigma^2(a + hr)}\right) \cdot \exp\left(\frac{\mu x}{\sigma^2}\right) = \\ & = \frac{a}{G_h} \exp\left(-\frac{|x|}{\sigma^2} G_h\right) \cdot \exp\left(\frac{\mu x}{\sigma^2}\right) \end{aligned}$$

and

$$\begin{aligned} & \int_0^{\infty} a \exp(-at) \cdot \exp\left(\frac{2z_b \mu}{\sigma^2}\right) \cdot \frac{\exp\left(-\frac{(x - 2z_b - \mu t)^2}{2 \cdot \sigma^2 \cdot t}\right)}{\sigma \cdot \sqrt{t} \cdot \sqrt{2\pi}} \cdot \exp(-hrt) dt = \\ & = \frac{a}{\sqrt{\mu^2 + 2\sigma^2(a + hr)}} \exp\left(-\frac{|x - 2z_b|}{\sigma^2} \sqrt{\mu^2 + 2\sigma^2(a + hr)}\right) \cdot \exp\left(\frac{\mu x}{\sigma^2}\right) = \\ & = \frac{a}{G_h} \exp\left(-\frac{|x - 2z_b|}{\sigma^2} G_h\right) \cdot \exp\left(\frac{\mu x}{\sigma^2}\right) \end{aligned}$$

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